Sydney Boys High School

4 unit mathematics

Trial HSC Examination 1991

1. **(a)** Find $|5i+12|$
(b) Show that if z_1 and z_2 are complex numbers and $\overline{z}$ represents the conjugate of z, $\overline{z_1+z_2}=\overline{z}_1+\overline{z}_2$.
(c) Illustrate on separate Argand diagrams the loci represented by
(i) $|z-1-i|\leq 1$
(ii) $\arg(z-1)=\frac{\pi}{3}$
(iii) $\Im(z^2)=4$
(iv) $|z-i|=|z-1|$
(d) Find in modulus-argument form all complex numbers z such that $z^3=-1$ and plot them on an Argand diagram.

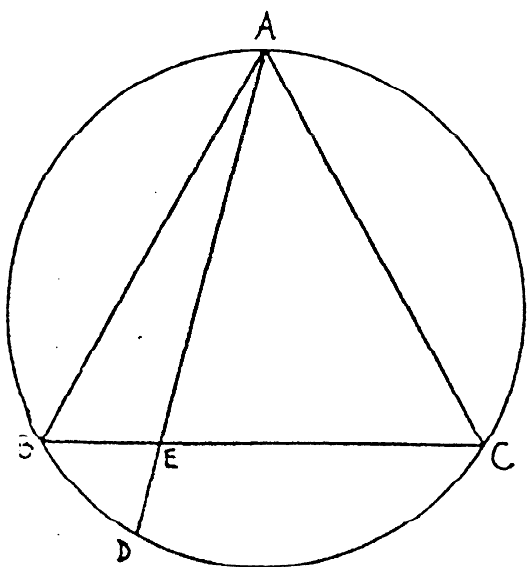
2. **(i)** Evaluate $\int_0^{\frac{2}{3}}\frac{dx}{4+9x^2}$.
(ii) By using partial fractions show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{x(\pi-2x)}=\frac{2}{\pi}\ln 2$.
(iii) Use the substitution $x=\tan y$ to evaluate $\int_0^1\frac{dx}{(x^2+1)^2}$.
(iv) If $U_n=\int_0^{\frac{\pi}{2}}x^n\sin x\,dx$, $n\geq 0$, prove that $U_n+n(n-1)U_{n-2}=n(\frac{\pi}{2})^{n-1}$, $n\geq 2$. Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}}x^4\sin x\,dx$.

3. **(a)** If $f(x)=2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}(1-x^2)$, find $f'(x)$. Hence, or otherwise, show that $f(x)=\frac{\pi}{2}$.
(b) The graph of $f(x)=\frac{ax^2}{x^2+bx+c}$ has the lines $x=1$, $x=-1$ and $y=2$ as asymptotes.
(i) Use this information to show that $f(x)=\frac{2x^2}{x^2-1}$.
(ii) Sketch the graph of $y=f(x)$ showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points and the equations of any asymptotes. (There is no needto investigate points of inflexion).
(c) **(i)** Find the equation of the tangent to the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ at the point $P(x_1,y_1)$ on the curve.
(ii) This tangent meets the coordinates axes at Q and R. Show that $OQ+OR=a$.

4. **(i)** Reduce $(x^2+2x)^2-9$ to irreducible factors over the complex numbers.
(ii) $P(x)$ is a real polynomial of least degree such that $P(i)=P(\frac{1}{2})=0$. Express $P(x)$ in general polynomial form.
(iii) If the equation $x^3+2x-1=0$ has roots $x=\alpha,\beta,\gamma$, find the value of $\alpha^3+\beta^3+\gamma^3$.

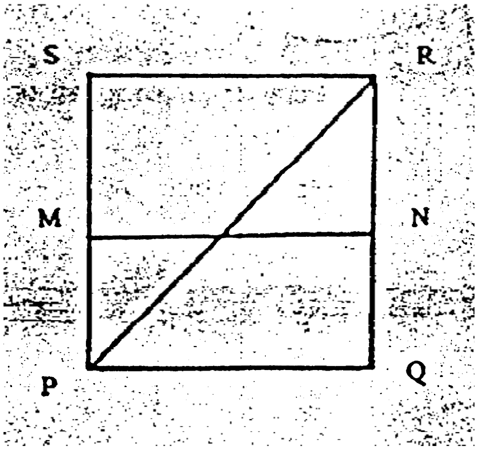
(iv) By considering the stationary values of $f(x) = x^3 - 3px^2 + 4q$, where p and q are positive real constants, show that the equation $f(x) = 0$ has three real, distinct roots if $p^3 > q$.
(v) If $G(x)$ is an odd function, then $G(-x) = -G(x)$. Use this definition to show that $\ln(x + \sqrt{x^2+1}\ )$ is an odd function.

5. **(a)** Sketch the graphs of each of the following:
(i) $y = \frac{1}{x} - 2$
(ii) $y = |-\frac{1}{x}|$.
(iii) $|y| = |\cos x|$ for $-2\pi \leq x \leq 2\pi$
(iv) $y = \sqrt{x} + \sqrt{1-x}$.
(b)



An isosceles triangle ABC is inscribed in a circle. $AB = AC$ and chord AD intersects BC at E.
(i) Copy this diagram.
(ii) Prove that $AB^2 - AE^2 = BE.CE$.

6. **(a)** Let $F(x) = e^{x^2}$ for all $x \geq 0$.
(i) Find $F^{-1}(x)$, the inverse function of $F(x)$ and state its domain and range.
bf(ii) On the same set of axes, indicate the regions represented by $\int_0^1 F(x)\ dx$ and $\int_1^e F^{-1}(x)\ dx$.
(iii) Evaluate $\int_0^1 F(x)\ dx + \int_1^e F^{-1}(x)\ dx$.
(b) The Anti-Bureaucracy Organisation wants to instal a rectangular notice board $PQRS$ of fixed area A square metres in each of its offices. The notice board is to be subdivided by two thin strips of red tape PR and MN (where MN is parallel to PQ) as shown below.



Find, in terms of A, the dimensions of the notice board so that the length of red tape used is a minimum.

7. **(i)** Find the general solution of $\tan A = -\cot 2A$.
(ii) Given that $z = \cos\theta + i\sin\theta$, use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$. Hence, or otherwise, solve the equation $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$.
(ii) If the terms of a series satisfy $q_n = \frac{q_{n-1}}{1+q_{n-1}}$ for all $n \geq 1$ prove by mathematical induction that $q_n = \frac{q_0}{1+nq_0}$ for all $n \geq 1$.

8. A particle P is projected from a point O with a speed V at an angle of elevation α° above the horizontal. At the same instant a second particle Q is projected horizontally with speed U from a point at a height h metres vertically above O, so that the particles move in the same vertical plane. Write down the coordinates of the particles at time t, relative to horizontal and vertical axes at O. (Air resistance is neglected for both particles).
(a) Show that if the particles collide then $V > U$ and find, in terms of h, V and U, the time at which collision takes place.
(b) Show that if the particles collide on the same horizontal level as O then $V^2 = U^2 + \frac{1}{2}gh$.
(c) If the highest point of the trajectory of P has coordinates (C, H) relative to horizontal and vertical axes at O, show that:
(i) The angle of projection is $\tan^{-1}\frac{2H}{C}$.
(ii) The speed of projection is given by $V^2 = \frac{g}{2H}(4H^2 + C^2)$ where g is the acceleration due to gravity, assumed constant.