SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

VERITATE ET VIRTUTE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2000

MATHEMATICS

4 UNIT ADDITIONAL

Time allowed — 3 Hours
(plus 5 minutes reading time)

Examiner: C. Kourtesis

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start **each** section in a new booklet. Section A (questions 1, 2, 3), Section B (questions 4, 5, 6) and Section C (questions 7, 8).
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

SECTION A

Question 1. (Start a new booklet) **15 Marks**

(a) If $z = (1 - i)^{-1}$ **5**

(i) Express $\bar{z}$ in modulus-argument form,

(ii) If $(\bar{z})^{13} = a + ib$ where a and b are real numbers, find the values of a and b.

(b) Find the cartesian equation of the locus of a point P which represents the complex number z where **2**

$$|z - i| = |z|$$

(c) Sketch the region in the complex plane where **3**

$$\text{Re}\left[(2 - 3i)z\right] < 12$$

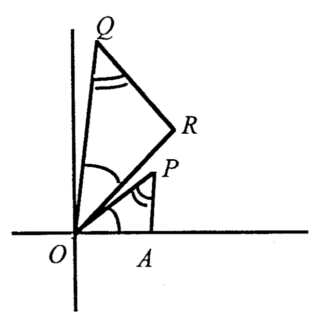
(d) (i) On an Argand diagram sketch the locus of a point P, corresponding to the complex number z, where **3**

$$|z - 3| = 3$$

(ii) Use your diagram in (i) to explain why

$$\arg(z - 3) = \arg z^2$$

(e) **2**



The points A, P and R in the complex plane correspond to the complex numbers 1, $\frac{3}{2} + i$ and $2 + 2i$ respectively. Triangles OAP and ORQ are similar with corresponding angles as indicated.

Find the complex number represented by Q.

Question 2. 15 Marks

(a) Find $\int \frac{dx}{x^2 - 4x + 9}$ 2

(b) (i) Express $\frac{4x-2}{(x^2-1)(x-2)}$ in the form $\frac{Ax+B}{x^2-1} + \frac{C}{x-2}$, where A, B and C are constants. 5

(ii) Hence evaluate

$$\int_3^6 \frac{4x-2}{(x^2-1)(x-2)}\,dx$$

(c) Find $\int \frac{e^{2x}}{e^x - 1}\,dx$ by using the substitution $u = e^x$. 3

(d) (i) If $u_n = \int_0^{\frac{\pi}{2}} \theta \sin^n\theta \, d\theta$ where $n \geq 1$, use integration by parts to prove that 5

$$u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}$$

(ii) Hence show that $u_5 = \frac{149}{225}$

Question 3. **15 Marks**

(a) The polynomial $P(z)$ has equation **3**

$$P(z) = z^4 - 2z^3 - z^2 + 2z + 10$$

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of two real quadratic factors.

(b) The remainder when $x^4 + ax + b$ is divided by $(x-2)(x+1)$ is $x+2$. Find the values of a and b. **2**

(c) (i) Show that **10**

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(ii) Find the general solution of the equation $\tan 3\theta = \sqrt{3}$

(iii) Using the substitution $x = \tan\theta$, express the equation in (ii) as a polynomial equation in terms of x.

(iv) Hence show that $\tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9} = 3\sqrt{3}$

(v) Find the polynomial of least degree that has zeros

$$\left(\cot\frac{\pi}{9}\right)^2, \left(\cot\frac{4\pi}{9}\right)^2, \left(\cot\frac{7\pi}{9}\right)^2$$

SECTION B

Question 4. (Start a new booklet) **15 Marks**

(a) Consider the function 9

$$F(x)=\left(\frac{x+4}{x}\right)^2,\ x\neq 0$$

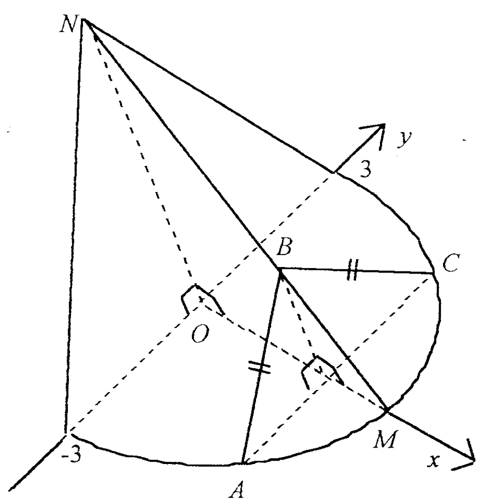
(i) Find all the turning points of $y=F(x)$,

(i) Determine the coordinates of the point of inflexion,

(iii) Find the equations of any asymptotes,

(iv) Sketch the curve $y=F(x)$ for all points in its domain.

(b) 6



A solid figure has a semi circular base of radius 3 cm. Cross sections taken perpendicular to the x axis are isosceles triangles. The vertical cross section containing the radius OM of the base of the solid is a right isosceles triangle OMN, where $OM = ON$.

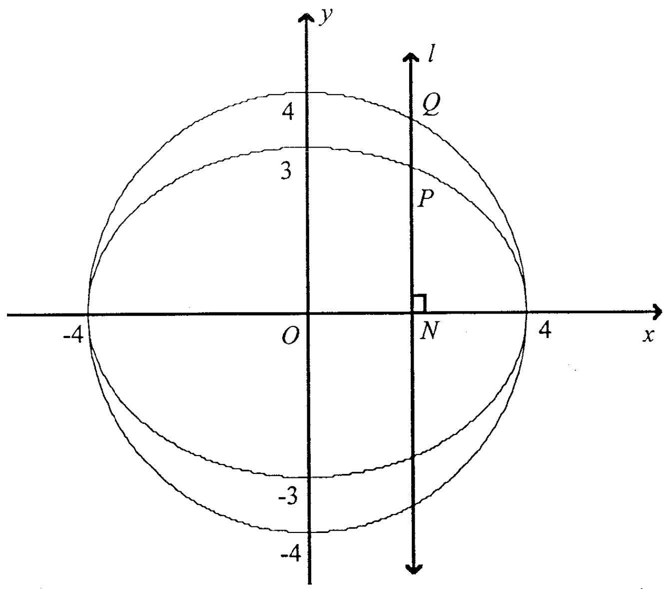
(i) Show that the area, A, of triangle ABC (where $AB = BC$) is given by

$$A=(3-x)(9-x^2)^{\frac{1}{2}}$$

(ii) Find the volume of the solid.

Question 5. 15 Marks

(a) 13



The diagram shows the ellipse, E, with equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ and its auxiliary circle C.
The coordinates of a point P on E are $(4\cos\theta, 3\sin\theta)$.

A straight line, l, parallel to the y axis intersects the x axis at N and the curves E and C at the points P and Q respectively.

(i) Find the eccentricity of E,

(ii) Write down the coordinates of N and Q,

(iii) Find the equations of the tangents at P and Q to the curves E and C respectively,

(iv) The tangents at P and Q intersect at a point R. Show that R lies on the x axis,

(v) Prove that $ON.OR$ is independent of the positions of P and Q.

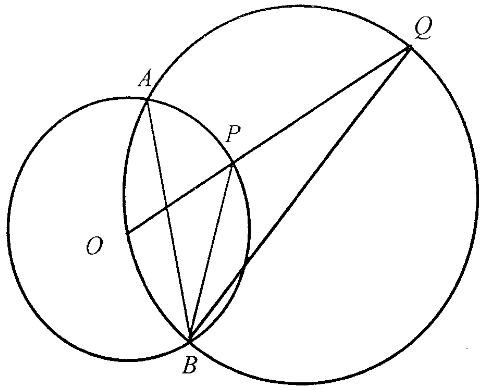
(b) State whether the following is True or False. Give brief reasons. 2

Note: You are NOT required to find the primitive function.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^9\theta \, d\theta > 0$$

Question 6. **15 Marks**

(a) **5**



In the diagram above, the centre O of the small circle APB lies on the circumference of the larger circle AQB. The points O, P and Q are collinear.

Prove that BP bisects $\angle ABQ$

(b) (i) Sketch the region in the number plane that contains all points satisfying simultaneously the inequalities **6**

$$x \le 1,\ y \ge 1 \text{ and } y \le e^x$$

(ii) This region is rotated through one complete revolution about the x axis. Use the method of cylindrical shells to show that the volume of the resulting solid is

$$\frac{\pi}{2}(e^3 - 3)$$

(c) If a function $f(x)$ is continuous for $a \le x \le b$ **4**

(i) Show that $\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$

(ii) Hence prove that

$$\left| \int_0^{\pi} 4^x \cos x\, dx \right| \le \frac{2^{2\pi} - 1}{2 \ln 2}$$

SECTION C

Question 7. (Start a new booklet) **15 Marks**

(a) A particle of mass m is projected vertically upwards in a medium where it experiences a resistance of magnitude mkv^2 where k is a positive constant and v is the velocity of the particle. 11

During the downward motion the terminal velocity of the particle is V. It's initial velocity of projection is $\frac{1}{3}$ of this terminal velocity.

(i) By considering the forces on the particle during its downward motion, show that

$$kV^2 = g$$

(where g is the acceleration due to gravity)

(ii) Show that during its upward motion the acceleration of the particle $\ddot{x}$ is given by

$$\ddot{x} = -g\left(1 + \frac{v^2}{V^2}\right)$$

(iii) If the distance travelled by the particle in its upward motion is x when its velocity is v, show that the maximum height H reached is given by

$$H = \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$$

(iv) The velocity of the particle is v when it is has fallen a distance y from its maximum height. Show that

$$y = \frac{V^2}{2g}\ln\left[\frac{V^2}{V^2 - v^2}\right]$$

(v) The velocity of the particle is U when it returns to its point of projection. Show that

$$\frac{V}{U} = 10^{\frac{1}{2}}$$

(b) (i) From 11 distinct consonants and 5 distinct vowels, how many words can be formed, each containing 5 distinct consonants and 3 distinct vowels? 4

(ii) In how many ways is it possible to allocate 6 people to 3 different courts in a singles tennis tournament ?

Question 8. 15 Marks

(a) (i) Show that $\frac{a+b}{2} \geq \sqrt{ab}$ for all positive numbers a and b. 4

(ii) If a, b, c and d are positive numbers prove that

$$4(ab+bc+cd+ad) \leq (a+b+c+d)^2$$

(b) If u and v are real numbers such that $u+v \neq 0$ and $v \neq 0$, 5

(i) Show that if there is only one real root of the equation $x^2+ux+v=0$ (where $0<x<1$) then

$$v(1+u+v)<0$$

(ii) Hence, or otherwise, prove that the equation

$$\frac{1}{x+2}+\frac{u}{x+1}+\frac{v}{x}=0$$

has only one positive root.

(c) Given the function $f(x)=x^n e^{-x}$ where n is a positive integer and $x>0$: 6

(i) Prove that there is only one turning point and that this occurs at $x=n$. Deduce that it is a maximum turning point.

(ii) Sketch the graph of $y=f(x)$,

(iii) By considering the values of $f(n)$, $f(n-1)$ and $f(n+1)$ prove that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1-\frac{1}{n}\right)^{-n}$$

THIS IS THE END OF THE PAPER.

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2000

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Sample Solutions

SHS THSC 2000 4U

Question 1

(a) $z = \frac{1}{1-i}$

$= \frac{1}{2}(1+i)$ $\quad \frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{4}\right)$ (1/2)

(i) $\bar{z} = \frac{1}{2}(1-i)$

$= \frac{1}{\sqrt{2}}\text{cis}\left(-\frac{\pi}{4}\right)$ (2)

(ii) $(\bar{z})^{13} = \left(\frac{1}{\sqrt{2}}\right)^{13}\text{cis}\left(-\frac{13\pi}{4}\right)$

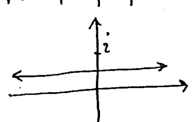
$= \frac{1}{64\sqrt{2}}\text{cis}\,\frac{3\pi}{4}$ ← (2)

$= \frac{1}{64\sqrt{2}}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ (1)

$\therefore a = \frac{-1}{64\sqrt{2}\sqrt{2}}$ & $b = \frac{1}{64\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$ (3)

$= -1/128$ $\quad = 1/128$ ←

(b) $|z - i| = |z|$



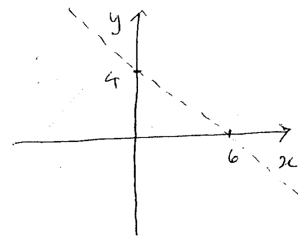
$y = \frac{1}{2}$ (2)

(c) $\text{Re}[(2-3i)z] < 12$

$\therefore \text{Re}[(2-3i)(x+iy)] < 12$

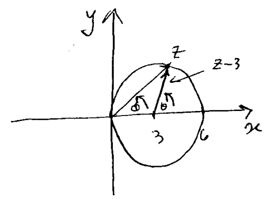
$\text{Re}[(2x+3y) + i(-3x+2y)] < 12$

$\therefore 2x + 3y < 12$ ← (2)



(3)

(d)(i) $|z - 3| = 3$



(1)

(ii) RTP $\arg(z-3) = \arg z^2$

In the diagram let $\theta = \arg(z-3)$ and $\phi = \arg z$

Now $\theta = 2\phi$ (angle at centre is double angle at circumference)

$\therefore \arg(z-3) = 2\arg z$

$= \arg z^2$ (2)

QED.

(e) Let $A\hat{O}P = \theta$ $\therefore R\hat{O}Q = \theta$ (given)

Let $A\hat{O}R = \phi$

Now $\arg \vec{OQ} = \phi + \theta$

$= \arg \vec{OR} + \arg \vec{OP}$

$= \arg(\vec{OR} \times \vec{OP})$

Also $\frac{OQ}{OR} = \frac{OP}{1}$

$\therefore OQ = OR.OP$

ie $|\vec{OQ}| = |\vec{OR}|.|\vec{OP}|$

$\therefore \vec{OQ} = \vec{OR}.\vec{OP}$

$= (2+2i)\left(\frac{3}{2}+i\right)$

$= 3 + 2i + 3i - 2$

$= 1 + 5i$ (2)

Question 2

(a) $\int \frac{dx}{x^2 - 4x + 9}$

$= \int \frac{dx}{(x-2)^2 + 5}$

$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-2}{\sqrt{5}}\right) + C$ (2)

(b) $\frac{4x-2}{(x^2-1)(x-2)} \equiv \frac{Ax+B}{x^2-1} + \frac{C}{x-2}$

$\therefore 4x - 2 \equiv (x-2)(Ax+B) + (x^2-1)C$

$\equiv Ax^2 + Bx - 2Ax - 2B + Cx^2 - C$

$\equiv (A+C)x^2 + (B-2A)x - 2B - C$

Equating Coefficients:

x^2: $0 = A + C$ — ①

x: $4 = -2A + B$ — ②

x^0: $-2 = -2B - C$ — ③

①+③: $-2 = A - 2B$ ④

②+2×④: $0 = -3B$

$\therefore B = 0$ $\quad \therefore A = -2,\ C = 2$ (3)

$\therefore \frac{4x-2}{(x^2-1)(x-2)} \equiv \frac{-2x}{x^2-1} + \frac{2}{x-2}$

$\therefore \int_3^6 \frac{4x-2}{(x^2-1)(x-2)}dx = \int_3^6\left(\frac{2}{x-2} - \frac{2x}{x^2-1}\right)dx$

$= \left[2\ln(x-2) - \ln(x^2-1)\right]_3^6$

$= \left[(2\ln 4 - \ln 35) - (2\ln 1 - \ln 8)\right]$

$= 2\ln 4 - \ln 35 + \ln 8$

$= 7\ln 2 - \ln 35$ (2)

(c) $I = \int \frac{e^{2x}}{e^x - 1}dx$ $\quad$ let $u = e^x$, $du = e^x dx$

$= \int \frac{u\,du}{u-1}$ ← ①

$= \int \frac{u - 1 + 1}{u-1}du$

$= \int\left(1 + \frac{1}{u-1}\right)du$ ← ②

$= u + \ln(u-1) + C$ (3)

$= e^x + \ln(e^x - 1) + C$

(d)(i) $u_n = \int_0^{\pi/2} \theta \sin^n\theta\, d\theta$

$= \int_0^{\pi/2} \theta.\sin^{n-1}\theta.\sin\theta.\,d\theta$

Let $u = \theta\sin^{n-1}\theta$ $\quad v' = \sin\theta$

$= \left[-\cos\theta.\theta.\sin^{n-1}\theta\right]_0^{\pi/2} + \int_0^{\pi/2}\cos\theta\left(\theta(n-1)\sin^{n-2}\theta\cos\theta + \sin^{n-1}\theta\right)d\theta$

$= 0 + \int_0^{\pi/2}\cos\theta\sin^{n-1}\theta\,d\theta + \int_0^{\pi/2}(n-1)\theta.\sin^{n-2}\theta.\cos^2\theta$

$= \frac{1}{n}\left[\sin^n\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2}(1-\sin^2\theta)\theta.\sin^{n-2}\theta\,d\theta$

$= \frac{1}{n} + (n-1)u_{n-2} - (n-1)u_n$ (3)

$\therefore u_n + (n-1)u_n = (n-1)u_{n-2} + \frac{1}{n}$

$\therefore u_n = \frac{n-1}{n}u_{n-2} + \frac{1}{n^2}$

Q2 (Contd)

(d) Contd

(ii) $u_5 = \frac{1}{25} + \frac{4}{5}u_3$

$= \frac{1}{25} + \frac{4}{5}\left(\frac{1}{9} + \frac{2}{3}(u_1)\right)$ (*)

$u_1 = \int_0^{\pi/2} \theta \sin\theta \, d\theta$

$= \int_0^{\pi/2} \theta \cdot \frac{d}{d\theta}(-\cos\theta)\, d\theta$

$= \left[\theta.(-\cos\theta)\right]_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \, d\theta$

$= 0 + [\sin\theta]_0^{\pi/2}$

$= 1$

Substituting into (*)

$u_5 = \frac{1}{25} + \frac{4}{5}\left(\frac{1}{9} + \frac{2}{3}\right)$

$= \frac{149}{225}$

(2)

Question 3

(a) $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$

$z - 2 + i$ is a factor

ie $z = 2 - i$ is root

$\therefore$ $z = 2 + i$ is also a root (conjugate root theorem)

$\therefore (z-2+i)(z-2-i)$ is a factor

ie $z^2 - 4z + 5$ is a factor

By long division

$P(z) = (z^2 - 4z + 5)(z^2 + 2z + 2)$

(b) By long division, the remainder

$x + 2 \equiv (a+5)x + (b+6)$

$\therefore$ $1 = a + 5$ $\quad 2 = b + 6$

$a = -4$ $\quad b = -4$

(c)(i) $\tan 3\theta = \tan(2\theta + \theta)$

$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$

$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan^2\theta}{1-\tan^2\theta}}$

$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(ii) $\tan 3\theta = \sqrt{3}$

$3\theta = \frac{\pi}{3} + k\pi$ $\quad k = 0, \pm 1, \pm 2$

$\therefore$ $\theta = \frac{\pi}{9} + \frac{k\pi}{3}$

$= \frac{\pi}{9}(3k+1)$

$\theta = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$, etc

(iii) $\tan 3\theta = \sqrt{3}$

$\therefore$ $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

Put $x = \tan\theta$

Q3 (c)(iii) Contd

$\therefore 3x - x^3 = \sqrt{3} - 3\sqrt{3}x^2$

ie $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$ (*)

(iv) $\tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}$ are all distinct roots of (*) since $x = \tan\theta$

Hence Sum of roots

$\frac{-b}{a} = 3\sqrt{3}$

$= \tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9}$

(v) Let roots of $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$ are α, β, γ then the req'd roots are $\frac{1}{\alpha^2} = \cot^2\frac{\pi}{3}$

$\frac{1}{\beta^2} = \cot^2\frac{4\pi}{3}$

$\frac{1}{\gamma^2} = \cot^2\frac{7\pi}{3}$

Now let $y = \frac{1}{\alpha^2}$

$\therefore \alpha = \sqrt{\frac{1}{y}}$, but since α is root of the above eqn it follows that

$\sqrt{\frac{1}{y}}^3 - 3\sqrt{3}\left(\sqrt{\frac{1}{y}}^2\right) - 3\sqrt{\frac{1}{y}} + \sqrt{3} = 0$

$\therefore \sqrt{\frac{1}{y}}^3 - 3\sqrt{\frac{1}{y}} = 3\sqrt{3}\cdot\frac{1}{y} - \sqrt{3}$

Square both sides

$\left(\frac{1}{y}\right)^3 - 6\left(\frac{1}{y}\right)^2 + \frac{9}{y} = \frac{27}{y^2} - \frac{18}{y} + 3$

Multiply both sides by y^3:

$1 - 6y + 9y^2 = 27y - 18y^2 + 3y^3$

$\therefore$ Required eqn is:

$3y^3 - 27y^2 + 33y - 1 = 0$

Put x for y:

$3x^3 - 27x^2 + 33x - 1 = 0$

Section B.

4.) a) $y = \frac{(x+4)^2}{x^2} = \left(1+\frac{4}{x}\right)^2$

$= \frac{x^2+8x+16}{x^2}$

$= 1 + \frac{8}{x} + \frac{16}{x^2}$

(i) $y' = -\frac{8}{x^2} - \frac{32}{x^3}$

$= -\frac{(8x+32)}{x^3} = -\frac{8(x+4)}{x^3}$

x	-5	-4	-3
y'	−	0	+

$y'=0$ when $x=-4$ $(-4, 0)$ ✓

∴ minimum ✓

(ii) $y'' = \frac{16}{x^3} + \frac{96}{x^4} = \frac{16x+96}{x^4}$ ✓

$y''=0 \Rightarrow x=-6.$ $\therefore y = \frac{4}{36} = \frac{1}{9}$

∴ POI $(-6, \frac{1}{9})$ ✓

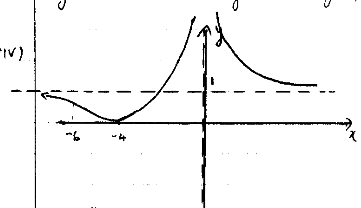
[Note the wording: "Find the P.O.I ....." so if $y''=0$ it must be the P.O.I.]

(iii) $x=0$ is the vertical asymptote. ✓

$y=1$ is the horizontal asymptote. ✓

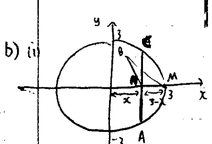
For $x>0$ $y'<0$

(iv)



✓✓✓

b) (i)



$AC = 2y = 2(9-x^2)^{\frac{1}{2}}$

[BN = BM = 3-x]

$\Delta ABC = \frac{1}{2}\times(3-x)\times 2(9-x^2)^{\frac{1}{2}}$

$= (3-x)(9-x^2)^{\frac{1}{2}}$ ✓✓

$V = \int_0^3 (3-x)(9-x^2)^{\frac{1}{2}}\,dx$ ✓

(ii) $= 3\int_0^3 \sqrt{9-x^2}\,dx + \int_0^3 -x\sqrt{9-x^2}\,dx$ ✓

$V = 3\times\frac{9\pi}{4} + \frac{1}{2}\int_0^3 -2x\sqrt{9-x^2}\,dx$

$u = 9-x^2$
$du = -2x\,dx$
$x=0,\ u=$
$x=3,\ u=0$

$= \frac{27\pi}{4} + \frac{1}{2}\int_9^0 \sqrt{u}\,du$

$= \frac{27\pi}{4} + \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^0$

$\int u^{\frac{1}{2}}du = \frac{2}{3}u^{3/2}$

$= \frac{27\pi}{4} + \frac{1}{3}(0-27)$

$= \frac{27\pi}{4} - 9$ ✓✓

5)

a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$a = 4, b = 3$

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{9}{16} = \frac{7}{16}$

$\therefore e = \frac{\sqrt{7}}{4}$ ✓

(ii) $N(4\cos\theta, 0)$ ✓

$Q(4\cos\theta, 4\sin\theta)$ ✓

(iii) P: $\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-9x}{16y} = \frac{-9(4\cos\theta)}{16(3\sin\theta)} = \frac{-3\cos\theta}{4\sin\theta}$ ✓

$\therefore y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$ ✓

$\therefore 4y\sin\theta - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$

$\therefore 3x\cos\theta + 4y\sin\theta = 12$

$\therefore \boxed{\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1}$ ✓

Q: $x^2 + y^2 = 16$

$2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-x}{y} = \frac{-4\cos\theta}{4\sin\theta} = \frac{-\cos\theta}{\sin\theta}$ ✓

$y - 4\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - 4\cos\theta)$ ✓

$y\sin\theta - 4\sin^2\theta = -x\cos\theta + 4\cos^2\theta$

$\boxed{x\cos\theta + y\sin\theta = 4}$ ✓

(iv) $3x\cos\theta + 4y\sin\theta = 12$ — (1)

$x\cos\theta + y\sin\theta = 4$ — (2)

(2) × 3

$\therefore 3x\cos\theta + 4y\sin\theta = 12$

$3x\cos\theta + 3y\sin\theta = 12$] −

$\therefore y\sin\theta = 0$

$y = 0 \quad (\sin\theta \neq 0)$ ✓

N.B. If $\theta = n\pi$ then P and Q co-incide on the x-axis anyway

$\therefore 3x\cos\theta = 12$

$\therefore x\cos\theta = 4$

$\therefore R(4\sec\theta, 0)$ ✓ which lies on the x-axis.

(v) $ON = |4\cos\theta|$

$OR = |4\sec\theta|$ ✓

$\therefore ON \cdot OR = 16$. which is independent of θ, thus independent of P and Q. ✓

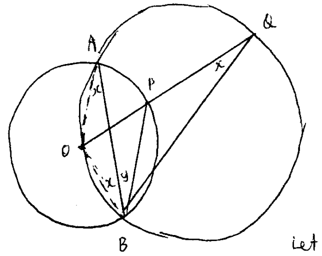
(b) $\int_{-\pi/4}^{\pi/4} \tan^9\theta \, d\theta > 0$

False ✓: $\tan\theta$ is odd, so $\tan^9\theta$ is odd. and continuous for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ ✓

$\boxed{\int_{-a}^{a} f(x)\,dx = 0 \quad \text{for odd } f(x)}$

$\int_{-\pi/4}^{\pi/4} \tan^9\theta \, d\theta = 0$

6 a)



let $x = O\hat{A}B$

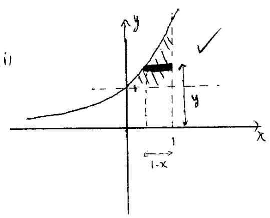
In circle AQB
$O\hat{Q}B = x$ ✓ (same segment as $O\hat{A}B$)

$OA = OB$
∴ ΔOAB is isosceles
∴ $O\hat{B}A = x$ ✓

let $y = A\hat{B}P$
∴ $O\hat{P}B = x+y$ ✓ (ΔOPB is isosceles)
∴ $B\hat{P}Q = 180 - (x+y)$ ✓
∴ $PBQ = y$ (angle sum of ΔPBQ) ✓

∴ PB bisects $A\hat{B}Q$

b) (i)



✓

$\int y \ln y \, dy$ $\quad u' = y \quad u = \frac{1}{2}y^2$
$\quad v = \ln y \quad v' = \frac{1}{y}$

$= \frac{1}{2}y^2 \ln y \Big]_1^e - \int_1^e \frac{1}{2} y \, dy$

$= \frac{e^2}{2} - \left[\frac{y^2}{4}\right]_1^e$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$

$= \boxed{\frac{e^2}{4} + \frac{1}{4}}$ ✓

(ii) $\Delta A \doteqdot 2\pi y (1-x)$ ✓

$\Delta V \doteqdot 2\pi y (1-x) \Delta y$

$V = 2\pi \int_1^e y(1-x) \, dy$ ✓

$= 2\pi \int_1^e y(1-\ln y) \, dy$ ✓

$= 2\pi \left[\int_1^e y \, dy - \int_1^e y \ln y \, dy\right]$

$= 2\pi \left[\frac{y^2}{2}\Big]_1^e - \left(\frac{e^2}{4} + \frac{1}{4}\right) \right]$

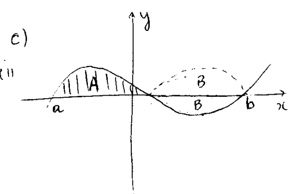
$= 2\pi \left[\frac{e^2}{2} - \frac{1}{2} - \frac{e^2}{4} - \frac{1}{4} \right]$

$= 2\pi \left[\frac{e^2}{4} - \frac{3}{4} \right] = 2\pi \left[\frac{e^2 - 3}{4} \right]$ ✓

$= \frac{\pi}{2} \left[e^2 - 3 \right]$

QED

c) (i)



(ii) B, A = areas (> 0)

$LHS = |A - B| = \left| \int_a^b f(x) \, dx \right|$

$RHS = A + B = \int_a^b |f(x)| \, dx$

$|f(x)|$ is always positive so $\int_a^b |f(x)| dx$ is a positive value.

Given that $f(x)$ can cross the x-axis for $a \le x \le b$ then $\int_a^b f(x) \, dx$ may involve subtraction ✓

Equality if $f(x) \ge 0$ for $a \le x \le b$.

∴ $\left| \int_a^b f(x) \, dx \right| \le \int_a^b |f(x)| dx$.

(ii) $|\cos x| \le 1$ $\quad \left| \int_0^\pi 4^x \cos x \, dx \right| \le \int_0^\pi |4^x \cos x| \, dx$ from (i)

$\le \int_0^\pi 4^x \, dx$ ✓

$= \frac{4^x}{\ln 4}\Big]_0^\pi$ ✓

$= \frac{2^{2x}}{2\ln 2}\Big]_0^\pi$

$= \frac{2^{2\pi} - 1}{2\ln 2}$ QED ✓

$\boxed{\int a^x \, dx = \frac{a^x}{\ln a} + C}$

7(a) $\uparrow mkv^2$

$\downarrow mg$

+ve direction $\downarrow$

(i) $\ddot{x} = g - kv^2$

At terminal velocity, $\ddot{x} = 0$

$\therefore g - kV^2 = 0$

$\therefore kV^2 = g$ 2

(ii) +ve direction $\uparrow$, $mg \downarrow \downarrow mkv^2$

$\ddot{x} = -(g + kv^2)$

$= -(g + \frac{g}{V^2}v^2)$

$= -g(1 + \frac{v^2}{V^2})$ 1

(iii) $v\frac{dv}{dx} = -g(1 + \frac{v^2}{V^2})$

$v\frac{dv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$

$\int \frac{v\,dv}{V^2 + v^2} = -\int \frac{g}{V^2}\,dx$

$\therefore \frac{1}{2}\ln(V^2 + v^2) = -\frac{g}{V^2}x + c$

When $x = 0$: $\frac{1}{2}\ln(V^2 + \frac{V^2}{9}) = c$

$\therefore \frac{g}{V^2}x = \frac{1}{2}\ln(\frac{10V^2}{9}) - \frac{1}{2}\ln(V^2 + v^2)$

At max ht, $v = 0$

$\therefore \frac{g}{V^2}H = \frac{1}{2}\ln\left(\frac{\frac{10V^2}{9}}{V^2}\right)$

$\therefore H = \frac{V^2}{2g}\ln\left(\frac{10}{9}\right)$ 3

(iv) $\ddot{y} = g - kv^2$

$\therefore v\frac{dv}{dy} = g - kv^2$

$\therefore \int \frac{v\,dv}{g - kv^2} = \int dy$

$\therefore -\frac{1}{2k}\ln(g - kv^2) = y + c$

When $y = 0$ $v = 0$

$\therefore -\frac{1}{2k}\ln g = c$

$\therefore y = \frac{1}{2k}\ln\frac{g}{g - kv^2}$

$= \frac{1}{2\frac{g}{V^2}}\ln\frac{g}{g - \frac{g}{V^2}v^2}$

$= \frac{V^2}{2g}\ln\frac{V^2}{V^2 - v^2}$ 3

(v) When $y = \frac{V^2}{2g}\ln(\frac{10}{9})$:

$\frac{V^2}{2g}\ln(\frac{10}{9}) = \frac{V^2}{2g}\ln\frac{V^2}{V^2 - U^2}$

$\therefore \frac{10}{9} = \frac{V^2}{V^2 - U^2}$

$\therefore 10V^2 - 10U^2 = 9V^2$

$\therefore V^2 = 10U^2$

$\therefore \frac{V^2}{U^2} = 10$ 2

$\therefore \frac{V}{U} = 10^{\frac{1}{2}}$

(b) (i) No of words

$= {}^{11}C_5 \times {}^{5}C_3 \times 8!$

(${}^{11}C_5$: choose 5 consonants; ${}^{5}C_3$: choose 3 vowels; $8!$: mix them up)

$(= 186\ 278\ 400)$ 2

(ii) No of ways

$= {}^{6}C_2 \times {}^{4}C_2$

(${}^{6}C_2$: choose 2 for the first court; ${}^{4}C_2$: choose 2 for the 2nd court)

$= 90.$ 2

8 (a) (i) $(\sqrt{a} - \sqrt{b})^2 \geq 0$

$a - 2\sqrt{ab} + b \geq 0$

$\therefore a + b \geq 2\sqrt{ab}$

$\therefore \frac{a+b}{2} \geq \sqrt{ab}$ 2

(ii) From (i)

$\frac{(a+c)+(b+d)}{2} \geq \sqrt{(a+c)(b+d)}$

$((a+c)+(b+d))^2 \geq 4(a+c)(b+d)$

$(a+b+c+d)^2 \geq 4(ab+bc+cd+da)$ 2

(b)(i) $f(x) = x^2 + ux + v = 0$ has one root between 0 and 1.

$f(0) = v$

$f(1) = 1 + u + v$

As there is 1 root between 0 and 1, $f(x)$ changes sign between 0 and 1.

$\therefore f(0)$ and $f(1)$ are of opposite sign

$\therefore f(0).f(1) < 0$

$\therefore v(1 + u + v) < 0.$ 2

(ii) $\frac{1}{x+2} + \frac{u}{x+1} + \frac{v}{x} = 0$

$x(x+1) + u(x+2)x + v(x+2)(x+1) = 0$

$\therefore x^2 + x + ux^2 + 2ux + vx^2 + 3vx + 2v = 0$

$\therefore x^2(1+u+v) + x(1+2u+3v) + 2v = 0$

For 2 distinct roots $\Delta > 0$

$\Delta = (1+2u+3v)^2 - 4(1+u+v).2v$

$=$ +ve $- 8 \times$ -ve

$=$ +ve $+ 8 \times$ +ve

> 0

$\therefore$ 2 distinct real roots

Product of roots $= 2v(1+u+v)$

< 0 3

$\therefore$ Roots are of opposite sign

$\therefore$ Only one root is positive.

(c) $f(x) = x^n e^{-x}$

(i) $f'(x) = e^{-x}.nx^{n-1} + x^n.-e^{-x}$

$= x^{n-1}e^{-x}(n - x)$

For st pt $f'(x) = 0$

$\therefore x = 0$ or $x = n$.

As $x > 0$, st pt at $x = n$.

$f'(n - \varepsilon) =$ +ve.+ve, +ve

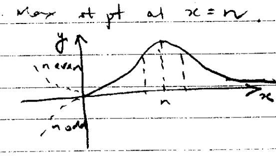
> 0

$f'(n + \varepsilon) =$ +ve.+ve.-ve 3

< 0

x	$n-\varepsilon$	n	$n+\varepsilon$
$f'(x)$	+ve	0	-ve
Graph	/	—	\

$\therefore$ Max st pt at $x = n$.

(ii) 

1

(iii) $f(n-1) < f(n)$

$\therefore (n-1)^n e^{-n+1} < n^n e^{-n}$

$\therefore e < (\frac{n}{n-1})^n = (\frac{n-1}{n})^{-n} = (1 - \frac{1}{n})^{-n}$

$f(n) > f(n+1)$

$\therefore n^n e^{-n} > (n+1)^n e^{-n-1}$

$\therefore e > (\frac{n+1}{n})^n = (1 + \frac{1}{n})^n$ 2

$\therefore (1 + \frac{1}{n})^n < e < (1 - \frac{1}{n})^{-n}$