SYDNEY BOYS HIGH SCHOOL

2019

YEAR 12 TRIAL HSC ASSESSMENT TASK

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using **black** pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–16, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Examiner: **S.G.**

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. The relation $(z+2)(\overline{z}+2)=4$, when graphed on an Argand diagram, would be a:

   (A) circle of radius 4 with centre at (–2,0)

   (B) circle of radius 2 with centre at (–2,0)

   (C) circle of radius 4 with centre at (2,0)

   (D) circle of radius 2 with centre at (2,0)

2. The cubic equation $2x^3-8x^2+px-12=0$ has roots α, β and γ.
   Given that $\alpha^2+\beta^2+\gamma^2=13$, what is the value of p?

   (A) 3

   (B) –3

   (C) $\frac{3}{2}$

   (D) $-\frac{3}{2}$

3. Which of the following is an expression for $\int xe^{-x}\,dx$?

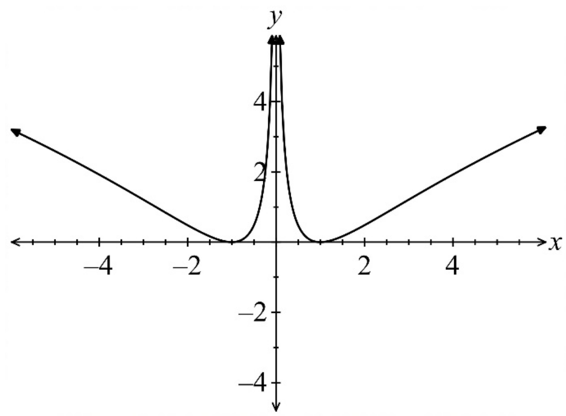
   (A) $-xe^{-x}-\int e^{-x}\,dx$

   (B) $-xe^{-x}+\int e^{-x}\,dx$

   (C) $xe^{-x}-\int e^{-x}\,dx$

   (D) $xe^{-x}+\int e^{-x}\,dx$

4. Which of the following could be the equation of this curve?



(A) $y = \log_e(x^2)$

(B) $y = \left[\log_e(x)\right]^2$

(C) $y = \left|\log_e(x)\right|$

(D) $y = \left(\log_e|x|\right)^2$

5. The numbers x, y and z are purely imaginary.

Which of the following must always be true?

(A) $x^2 + y^2 + z^2 \geq 0$

(B) $x^5 y^{10} z^{15} \geq 0$

(C) $x^2 y^2 + x^2 z^2 + y^2 z^2 \geq 0$

(D) $x^6 y^{12} z^{16} \geq 0$

6. $\displaystyle\int \frac{2x}{(x+1)(x+3)}\,dx$ is equal to:

(A) $\displaystyle\int \left(\frac{3}{x+3} - \frac{1}{x+1}\right) dx$

(B) $\displaystyle\int \left(\frac{3}{x+3} - \frac{3}{x+1}\right) dx$

(C) $\displaystyle\int \left(\frac{3}{x+3} + \frac{1}{x+1}\right) dx$

(D) $\displaystyle\int \left(\frac{3}{x+3} + \frac{3}{x+1}\right) dx$

7. An archer finds that on average he hits the bullseye four times out of five.
If he fires four arrows, what is the probability that he will miss the bullseye at least 3 times?

(A) 0.0016

(B) 0.0272

(C) 0.512

(D) 0.8192

8. The horizontal line $y = -k$, where k is a positive integer, intersects the curve $y = \log_2 x$ at different points according to the values of k.
What is the limiting sum of the x-coordinates of all the points of intersection?

(A) $\frac{1}{2}$

(B) 1

(C) 2

(D) 4

9. The equation $P(x)=3x^3-4x^2+2x-7$ has zeroes α, β and γ.
Which equation has roots $\alpha+2$, $\beta+2$ and $\gamma+2$?

(A) $3x^3-22x^2+54x-51=0$

(B) $3x^3-10x^2+30x+51=0$

(C) $3x^3-8x^2+8x-56=0$

(D) $24x^3-16x^2+4x-7=0$

10. If $x=\tan\theta$ and $y=\frac{1}{2}\sin 2\theta$, which of the following is an expression for $\frac{dy}{dx}$?

(A) $\cos^2\theta-2\sin^2 2\theta$

(B) $\cos^2\theta-\sin^2 2\theta$

(C) $\cos^2\theta-\frac{1}{2}\sin^2 2\theta$

(D) $\cos^2\theta-\frac{1}{4}\sin^2 2\theta$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section.

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) It is given that $z = 3 + 3i$ and w is a complex number such that $|zw| = 12$.

i) Show that $|w| = 2\sqrt{2}$. **2**

ii) Given that $\arg(w) = -\dfrac{\pi}{6}$, find w in exact Cartesian form. **2**

b)

i) Find real numbers a and b such that $(a + ib)^2 = -3 + 4i$. **3**

ii) Hence solve the equation $z^2 - 3z + (3 - i) = 0$. **2**

c) Define the polynomial $H(x)$ as $H(x) = ax^3 - 3x^2 - 6x + b$, where a and b are real numbers. **3**

Find the values of a and b, given that $(x-1)^2$ is a factor of $H(x)$.

d) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{3\sec^2 x}{9 + \tan^2 x}\, dx$ **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

a) Let $y = \dfrac{x^2 - x + 2}{x + 1}$.

i) What are the equations of the asymptotes of the curve? **2**

ii) Find the coordinates of all the turning points. **2**

iii) Hence draw a third-page sketch of the curve $y = \dfrac{x^2 - x + 2}{x + 1}$, showing the above information. **2**

b) Let $z = \cos\theta + i\sin\theta$.

i) Show that $z^n + z^{-n} = 2\cos n\theta,\ n = 1, 2, 3, \ldots$ **1**

ii) Hence show that $4\cos\theta\cos 2\theta\cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$. **3**

iii) Hence, or otherwise, find $\int \cos x \cos 2x \cos 3x \, dx$. **1**

c)

i) Show that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ for all $f(x)$. **2**

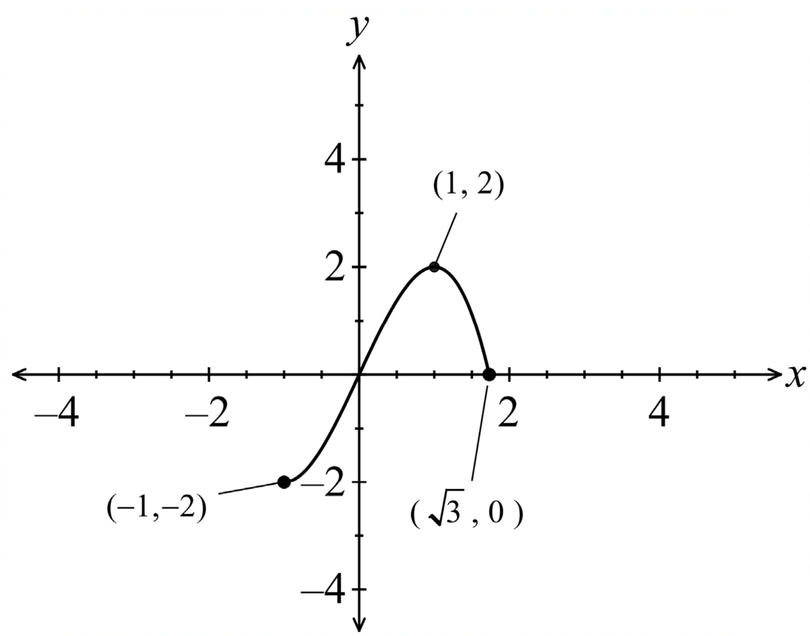
ii) Hence, or otherwise, evaluate $\int_0^2 x\,(2-x)^n\ dx$, where $n > 0$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

a) Find the gradient of the tangent to the curve $y^2 + xy - 1 = 0$ at the point (0,1). **2**

b) The graph of $y = f(x)$ is shown below.



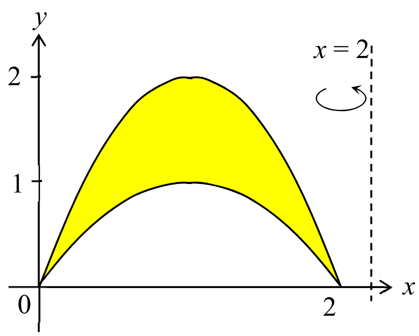
Draw separate half page diagrams of the graphs of each of the following on the insert provided:

i) $y = f\left(\left|x-1\right|\right)$. **2**

ii) $y = \left[f(x)\right]^2$. **2**

Question 13 continues on page 9

c) The shaded region bounded by the parabolas $y = 2x - x^2$ and $y = 4x - 2x^2$ between $x = 0$ and $x = 2$ is as shown in the diagram. **3**



Use the cylindrical shells method to find the volume of the solid formed when the shaded area is rotated about the line $x = 2$.

d) Evaluate $\displaystyle\int_1^{49} \frac{dx}{2+\sqrt{x}}$, leaving your answer in simplest exact form. **3**

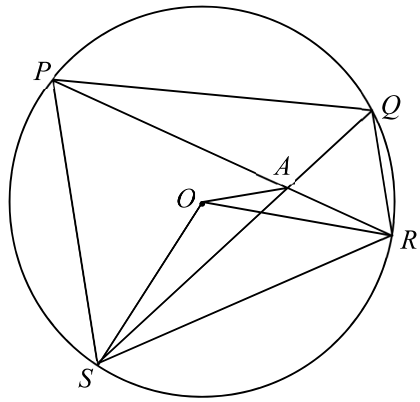
e) Use calculus to show that the following inequality is true for $x > -1$. **3**

$$x \geq \log_e(1+x)$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

a) P, Q, R and S are four points on a circle with centre O. **3**
PR and SQ meet at the point A such that the quadrilateral $OARS$ is cyclic.



Show that PS is parallel to QR.

b) A particle is fired vertically upwards with initial velocity u metres per second, and is subject to both gravity, g, and air resistance, which is proportional to the square of the velocity v.

i) Show that the motion of the particle can be represented by $\ddot{x} = -g - kv^2$ where k is a positive constant. **1**

ii) Find the greatest height H reached by the particle. **2**

iii) By considering a suitable equation of motion, show that the velocity w with which it returns to the point of projection is given by $w^2 = \dfrac{g}{k}\left(1 - e^{-2kH}\right)$. **3**

Question 14 continues on page 11

Question 14 (continued)

c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \; dx$, where n is a non-negative integer.

i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \; dx$, for $n \geq 2$. **2**

ii) Deduce that $I_n = \dfrac{n-1}{n} I_{n-2}$. **2**

iii) Evaluate I_4. **2**

End of Question 14

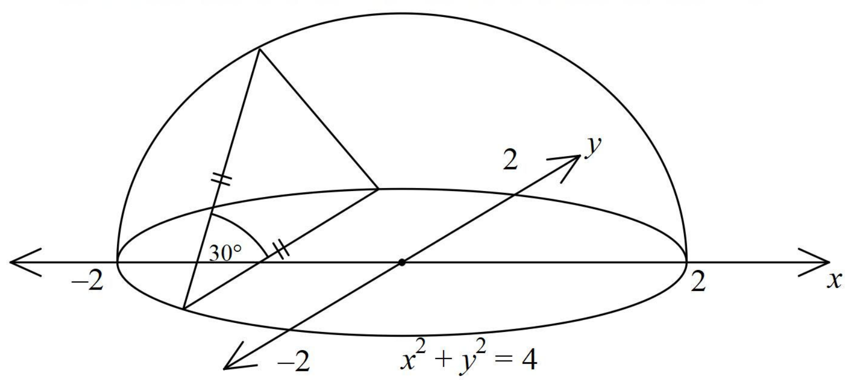
Question 15 (15 marks) Use a SEPARATE writing booklet

a) Use mathematical induction to prove that **3**

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

for any integer $n \geq 2$.

b) A solid has the region bounded by the circle $x^2+y^2=4$ as its base. **3**



The cross section of the solid above, taken perpendicular to the x-axis, is an isosceles triangle with one of the equal sides lying in the base of the solid and an angle of 30° between the equal sides, as shown.

Find the volume of the solid.

c) Six friends go to a restaurant that serves six different main courses. **3**
If each of the friends randomly chooses which meal they have, what is the probability that exactly two of the main course options are not chosen?

Question 15 continues on page 13

Question 15 (continued)

d)

i) Prove that, for any positive integers n and r, **2**

$$\frac{1}{{}^{n+r}C_{r+1}} = \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right).$$

ii) Hence, by expressing $\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}}$ as a simplified expression in terms of r, **4**

evaluate $\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_3}$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

a) If a, b and c are positive and unequal, prove that

i) $a+b\geq 2\sqrt{ab}$ **1**

ii) $(a+b)(b+c)(c+a)>8abc$ **2**

Question 16 continues on page 15

Question 16 (continued)

b)

i) Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then **2**

$$\frac{(\cot\theta + i)^{2n+1} - (\cot\theta - i)^{2n+1}}{2i} = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta},$$

for any positive integer n.

ii) Deduce that the solutions of the equation **3**

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \ldots + (-1)^n = 0$$

are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right).$$

where $m = 1, 2, \ldots, n$.

iii) Hence show that **2**

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}$$

iv) Given that $0 < \sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{\pi}{2}$, show that **2**

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta.$$

v) Hence show that **3**

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

End of paper

SYDNEY
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2019

YEAR 12 TRIAL HSC
ASSESSMENT TASK

Mathematics Extension 2

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1. B
2. A
3. B
4. D
5. C
6. A
7. B
8. B
9. A
10. C

X2 Y12 THSC 2019 Multiple choice solutions

Mean (out of 10): 8.72

1. $(z+2)(\bar{z}+2)=4$

$z\bar{z}+2z+2\bar{z}+4=4$

$x^2+y^2+4x+4=4$

$(x+2)^2+y^2=4$

Circle, centre $(-2,0)$, radius 2

B

A	5
B	111
C	1
D	5

2. $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$13=4^2-2\times\frac{p}{2}$

$13=16-p$

$p=3$

A

A	98
B	9
C	15
D	0

3. $\int xe^{-x}\,dx$ $\quad u=x \quad v=-e^{-x}$

$\quad u'=1 \quad v'=e^{-x}$

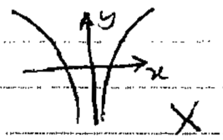
$=-xe^{-x}-\int 1.(-e^{-x})\,dx$

$=-xe^{-x}+\int e^{-x}\,dx$

B

A	3
B	119
C	0
D	0

4. A $y=2\log_e|x|$



✗

B. Domain $x>0$ ✗

C. Domain $x>0$ ✗

D. Only option left. Graph is consistent with expected features: Discontinuity at $x=0$, Zeroes at $x=-1$ and $x=1$, Range ≥ 0

D

A	4
B	0
C	1
D	117

5. A x^2, y^2, z^2 would be negative numbers, ∴ sum negative ✗

B. $x^5y^{10}z^{15}$ would involve $i^{30}=-1$

If $x=y=z=i$, $x^5y^{10}z^{15}=i^{30}=-1$ ✗

C. x^2y^2, x^2z^2, y^2z^2 are each the product of negative numbers, therefore each is positive ∴ sum is positive. ✓

D Similar argument to B ✗

C

A	3
B	8
C	108
D	3

6. $\frac{2x}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{x+3}$

$\therefore 2x = A(x+3) + B(x+1)$

Let $x=-3$: $-6 = -2B$

$B = 3$

Let $x=-1$: $-2 = 2A$

$A = -1$

$\therefore \frac{2x}{(x+1)(x+3)} = \frac{-1}{x+1} + \frac{3}{x+3}$

A

A	118
B	1
C	2
D	1

7. P(miss at least 3 times)

$= P(\text{miss 3 times}) + P(\text{miss 4 times})$

$= {}^4C_3 \times 0.8^1 \times 0.2^3 + {}^4C_4\ 0.2^4$

$= 0.0272$

B

A	2
B	113
C	2
D	5

8. $\log_2 x = -k$

$\therefore x = 2^{-k}$

$= \frac{1}{2^k}$

$\text{Sum} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}}$

$= 1$

B

A	7
B	75
C	30
D	10

9. $u = x+2$

$x = u-2$

$\therefore 3(u-2)^3 - 4(u-2)^2 + 2(u-2) - 7 = 0$

$\therefore 3(u^3 - 3u^2.2 + 3u.4 - 8) - 4(u^2 - 4u + 4) + 2u - 4 - 7 = 0$

$\therefore 3u^3 - 18u^2 + 36u - 24 - 4u^2 + 16u - 16 + 2u - 11 = 0$

$\therefore 3u^3 - 22u^2 + 54u - 51 = 0$

A

A	116
B	5
C	1
D	0

10. $x = \tan\theta$ $\qquad y = \frac{1}{2}\sin 2\theta$

$\frac{dx}{d\theta} = \sec^2\theta$ $\qquad \frac{dy}{d\theta} = \frac{1}{2} \times 2\cos 2\theta$

$= \cos 2\theta$

$\therefore \frac{dy}{dx} = \frac{\cos 2\theta}{\sec^2\theta}$

$= \cos^2\theta \cos 2\theta$

Using substitutions:

θ	0	$\frac{\pi}{2}$	$\frac{\pi}{4}$
$\cos^2\theta \cos 2\theta$	1	0	0
A $\cos^2\theta - 2\sin^2 2\theta$	1	0	$-\frac{3}{2}$
B $\cos^2\theta - \sin^2 2\theta$	1	0	$-\frac{1}{2}$
C $\cos^2\theta - \frac{1}{2}\sin^2 2\theta$	1	0	0
D $\cos^2\theta - \frac{1}{4}\sin^2 2\theta$	1	0	$\frac{1}{4}$

C

Confirming:

$\cos^2\theta - \frac{1}{2}\sin^2 2\theta$

$= \cos^2\theta - \frac{1}{2}(2\sin\theta\cos\theta)^2$

$= \cos^2\theta - \frac{1}{2} \times 4\sin^2\theta\cos^2\theta$

$= \cos^2\theta(1 - 2\sin^2\theta)$

$= \cos^2\theta \cos 2\theta$

A	7
B	8
C	89
D	18

Question 11

a) i) $z = 3+3i$

$|z| = \sqrt{(3)^2+(3)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

$|zw| = 12$

$|z|.|w| = 12$

$3\sqrt{2} \times 2\sqrt{2} = 12$

$\therefore |w| = 2\sqrt{2}$

ii) $\arg w = -\frac{\pi}{6}$

$w = |w|\left(\cos(\arg w) + i\sin(\arg w)\right)$

$= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\sqrt{2}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$

$= \sqrt{6} - \sqrt{2}i$

b) i) $(a+ib)^2 = -3+4i$

$a^2 + 2abi - b^2 = -3+4i$

equate real & imaginary parts

$a^2 - b^2 = -3$ ——— 1

$2ab = 4$ ——— 2

$(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$= (-3)^2 + (4)^2$

$= 25$

$a^2+b^2=5$ — (3) since a, b are real.

(1) + (3)

$2a^2 = 2$

$a^2 = 1$

$a = \pm 1$

sub into (2)

$2(\pm 1)b = 4$

$b = \pm 2$

$a=1, b=2$ or $a=-1, b=-2$

ii) $z^2 - 3z + (3-i) = 0$

$$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(3-i)}}{2(1)}$$

$$z = \frac{3 \pm \sqrt{9-12+4i}}{2}$$

$$z = \frac{3 \pm \sqrt{-3+4i}}{2}$$

$$z = \frac{3 \pm (1+2i)}{2}$$

$$z = \frac{3+1+2i}{2} \quad \text{or} \quad \frac{3-(1+2i)}{2}$$

$z = 2+i$ or $1-i$

c) $H(x) = ax^3 - 3x^2 - 6x + b$

$H'(x) = 3ax^2 - 6x - 6$

since $(x-1)^2$ is a factor of $H(x)$,
$x=1$ is a zero of $H(x)$ & $H'(x)$

$H'(1) = 3a(1)^2 - 6(1) - 6 = 0$

$3a - 12 = 0$

$\underline{a = 4}$

$H(1) = a(1)^3 - 3(1)^2 - 6(1) + b = 0$

$a + b - 9 = 0$

$b = 9 - a$

$= 9 - (4)$

$\underline{b = 5}$

d) $I = \int_0^{\frac{\pi}{3}} \frac{3\sec^2 x}{9 + \tan^2 x} \, dx$

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

limit change

when $x = 0$ $\quad$ $x = \frac{\pi}{3}$

$u = 0$ $\quad$ $u = \sqrt{3}$.

$I = \int_0^{\sqrt{3}} \frac{3\cancel{\sec^2 x}}{9 + u^2} \cdot \frac{du}{\cancel{\sec^2 x}}$

$= 3 \int_0^{\sqrt{3}} \frac{du}{9 + u^2}$

$= \cancel{3} \cdot \frac{1}{\cancel{3}} \left[\tan^{-1}\left(\frac{u}{3}\right)\right]_0^{\sqrt{3}}$

$$= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \cancel{\tan^{-1}(0)}$$

$$= \frac{\pi}{6}$$

COMMENTS:

This question was done well by the majority of students. the modal mark was 15.

Part (c) could have been done using the sum and product of roots.

Ext 2 Y12 THSC 2019 Q12 solutions

Mean (out of 15): 12.29

12(a) (i) $y = \dfrac{x^2-x+2}{x+1}$

$= x-2+\dfrac{4}{x+1}$

$$\begin{array}{r l} & x-2 \\ x+1 \,\big) & x^2-x+2 \\ & x^2+x \\ & -2x+2 \\ & -2x-2 \\ & 4 \end{array}$$

Asymptotes: $x=-1$

$y = x-2$

(2)

Most handled this question well. The $x = -1$ asymptote was well done. Those who didn't identify the $y = x - 2$ asymptote usually had not performed the division to assist the identification.

0	0.5	1	1.5	2	Mean
1	0	21	4	96	1.80

(ii) $y' = \dfrac{(x+1)(2x-1)-(x^2-x+2).1}{(x+1)^2}$

$= \dfrac{2x^2-x+2x-1-x^2+x-2}{(x+1)^2}$

$= \dfrac{x^2+2x-3}{(x+1)^2}$

$= \dfrac{(x+3)(x-1)}{(x+1)^2}$

$\therefore$ Stationary points at $(-3,-7)$ and $(1,1)$

x	-4	-3	-2	0	1	2
y'	$\frac{5}{9}$	0	-3	-3	0	$\frac{5}{9}$
Slope	/	—	\	\	—	/

$\therefore$ $(-3,-7)$ is a maximum turning pt

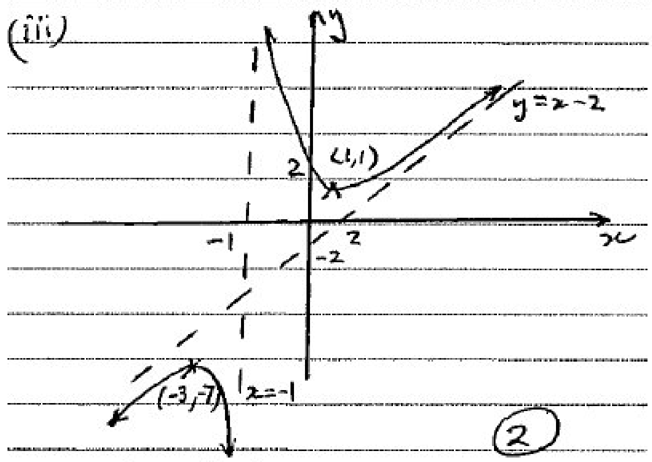
$(1,1)$ is a minimum turning point

(2)

Most students found the stationary points. However, many did not take the extra step and decide if they were turning points. Some of those who tested the gradient on either side of the stationary point did not take into account the existence of a discontinuity at $x = -1$ – their testing points should not have spanned the discontinuity. Students are expected to supply a value (or a detailed justification) to indicate whether the gradient id positive or negative, not just write + or -.

0	0.5	1	1.5	2	Mean
0	4	68	16	34	1.33

(iii)



(2)

Expected features:

Asymptotes – and curve approaching the asymptote.

Turning points

y intercept.

Many sketches were done well. A common error was not stating the y intercept. When sketching curves with asymptotes, students should draw the asymptotes first and then ensure that their graphs approach the asymptote.

0	0.5	1	1.5	2	Mean
2	2	11	31	76	1.73

(b) (i) $z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ (1)

Well done.

0	0.5	1	Mean
3	4	115	0.96

(ii) $4\cos\theta\cos 2\theta\cos 3\theta$

$= 4\left(\frac{z+z^{-1}}{2}\times\frac{z^2+z^{-2}}{2}\times\frac{z^3+z^{-3}}{2}\right)$

$= \frac{1}{2}\left(z^3+z^{-1}+z^1+z^{-3}\right)\left(z^3+z^{-3}\right)$

$= \frac{1}{2}\left(z^6+1+z^2+z^{-4}+z^4+z^{-2}+1+z^{-6}\right)$

$= \frac{1}{2}\left(2+(z^2+z^{-2})+(z^4+z^{-4})+(z^6+z^{-6})\right)$

$= \frac{1}{2}\left(2+2\cos 2\theta+2\cos 4\theta+2\cos 6\theta\right)$

$= 1+\cos 2\theta+\cos 4\theta+\cos 6\theta$

3

An interesting question which makes use of the result from part (i) in both directions.

0	0.5	1	1.5	2	2.5	3	Mean
23	11	6	0	2	1	79	2.09

(iii) $\int \cos x\cos 2x\cos 3x\,dx$

$= \frac{1}{4}\int(1+\cos 2x+\cos 4x+\cos 6x)\,dx$

$= \frac{1}{4}\left(x+\frac{1}{2}\sin 2x+\frac{1}{4}\sin 4x+\frac{1}{6}\sin 6x\right)+c$

1

Well done. Some didn't include the multiplier of $\frac{1}{4}$.

0	0.5	1	Mean
4	9	109	0.93

(c)(i) $\int_0^a f(a-x)\,dx$ Let $u=a-x$

$du=-dx$

$= \int_a^0 f(u)(-du)$ If $x=0$, $u=a$

If $x=a$, $u=0$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

2

Most who started with the RHS were able to get to the LHS comfortably.

0	0.5	1	1.5	2	Mean
13	4	5	1	99	1.69

(ii) $\int_0^2 x(2-x)^n\,dx$

$= \int_0^2 (2-x)x^n\,dx$

$= \int_0^2 \left(2x^n-x^{n+1}\right)dx$

$= \left[\frac{2}{n+1}x^{n+1}-\frac{1}{n+2}x^{n+2}\right]_0^2$

$= \left[\frac{2}{n+1}\times 2^{n+1}-\frac{1}{n+2}2^{n+2}\right]-\left[0\right]$

$= \left[\frac{1}{n+1}-\frac{1}{n+2}\right]2^{n+2}$

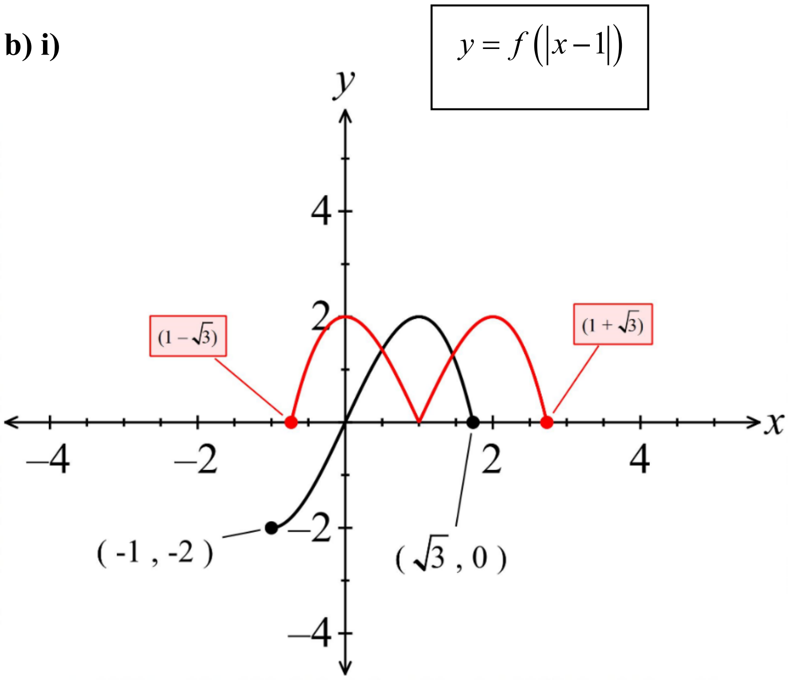
$= \frac{2^{n+2}}{(n+1)(n+2)}$

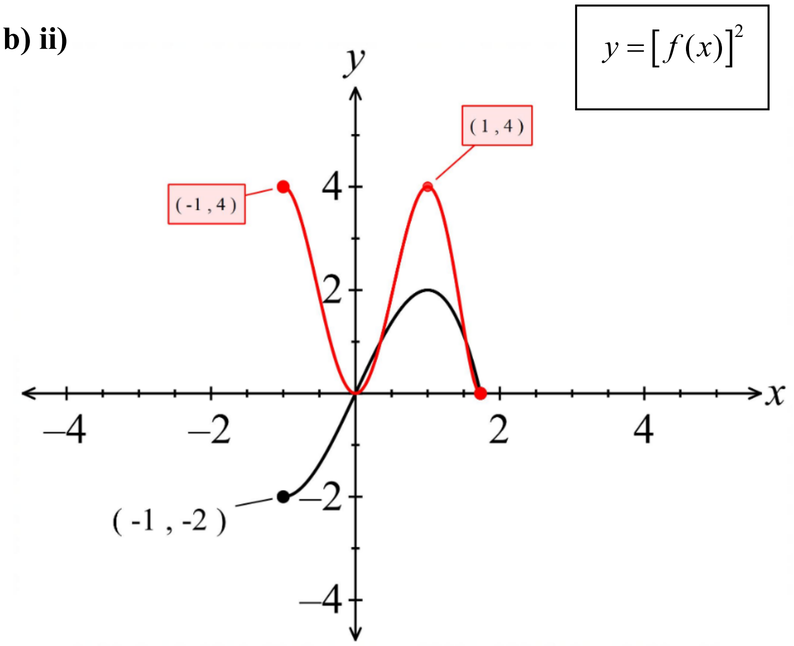
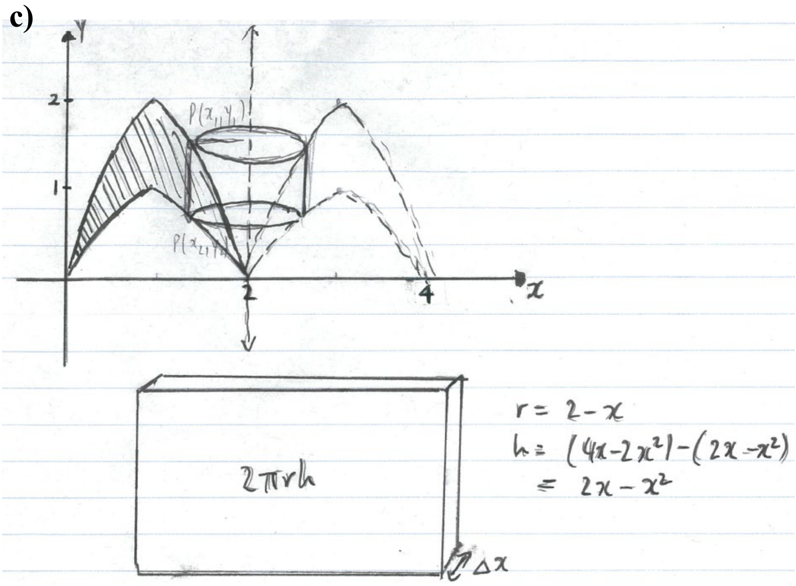
2

Generally done well.

0	0.5	1	1.5	2	Mean
6	6	2	11	97	1.77

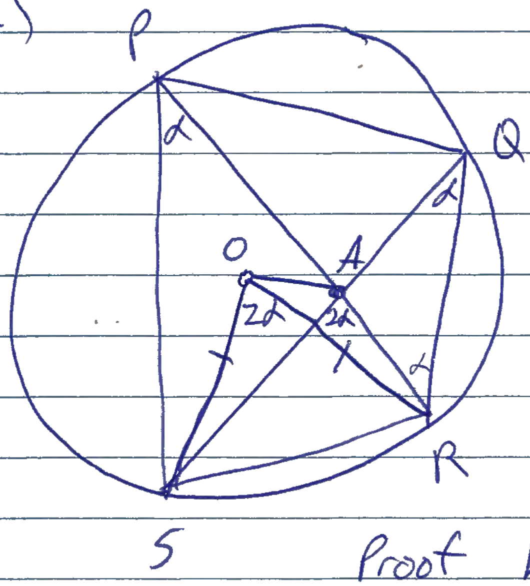
Question 13

Solution	Marking Criteria	Marker's comments
a) $y^2 + xy - 1 = 0$ $\frac{d(y^2)}{dy}\frac{dy}{dx} + x\frac{dy}{dx} + \frac{dx}{dx}y + \frac{d(-1)}{dx} = \frac{d(0)}{dx}$ $2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$ $\frac{dy}{dx}(2y + x) = -y$ $\frac{dy}{dx} = \frac{-y}{2y + x}$ At (0, 1) $m = -\frac{1}{2+0}$ $= -\frac{1}{2}$	**1 mark** for finding the correct $\frac{dy}{dx}$. **1 mark** for finding the correct gradient.	- Most candidates did well in this question as they know to implicit differentiate. - Few candidates did not read the question properly and went on to find the equation of the line. This is not necessary.
b) i) 	**1.5 marks** for the correct graph. **0.5 mark** for the critical values labelled.	- This was not done well by many candidates. - Many candidates shifted the graph to the right and reflected on the y-axis. Candidates did not take the account of the absolute value. - Only 1 mark was awarded if candidates did the above with correct labelling.

b) ii)  	**1.5 mark** for the correct graph and clear indication of critical values. **0.5 mark** showing the 2 intersections of $y=[f(x)]^2$ and $y=f(x)$ due to the steepness created by the “squared”.	This question was done really well by majority of the candidates.
c)  $\Delta V \approx 2\pi rh\delta x$ $= 2\pi(2-x)(2x-x^2)\delta x$ $V = \lim\limits_{\delta x\to 0}\sum\limits_{x=0}^{2} 2\pi(x^3-4x^2+4x)\delta x$ $= 2\pi\int_0^2 x^3-4x^2+4x\,dx$ $= 2\pi\left[\frac{x^4}{4}-\frac{4x^3}{3}+2x^2\right]_0^2$ $= 2\pi\left(\frac{16}{4}-\frac{32}{3}+8\right)$ $= \frac{8\pi}{3}$ units^3	**1 mark** for the correct expression for one shell. **1 mark** for the correct expression of the volume integral. **1 mark** for the correct answer.	Substantial number of candidates lost marks due to: • Not explicitly writing down the correct expression for 1 shell. • Careless errors in expanding $(2-x)(2x-x^2)$

d) $\int_1^{49} \frac{dx}{2+\sqrt{x}}$ $= \int_1^7 \frac{2u}{2+u} du$ $= 2\int_1^7 \frac{u+2-2}{u+2} du$ $= 2\int_1^7 1 - \frac{2}{u+2} du$ $= 2\left[u - 2\ln\lvert u+2\rvert\right]_1^7$ $= 2[(7-2\ln(9)) - (1-2\ln(3))]$ $= 2(6-2\ln(3))$ $= 12 - 4\ln(3)$ Let $u = \sqrt{x}$ $u^2 = x$ $\frac{du}{dx} = 2u$ $dx = 2u\ du$ $x = 1 \sim 49$ $u = 1 \sim 7$	**1 mark** for correct substitution. **1 mark** for correct integral after substitution. **1 mark** for the correct answer.	The substitution shown in the solution is not the only acceptable substitution, but it made problem easier to work out. Significant number of candidates rationalized the denominator but were not successful in getting the correct answer. Some candidates used $x = \tan^4(\theta)$ as their substitution with no success.
e) Let $y = x - \log_e(1+x)$ for $x > -1$ y is continuous for $x > -1$ $\frac{dy}{dx} = 1 - \frac{1}{1+x}$ $= \frac{x}{1+x}$ Let $\frac{dy}{dx} = 0$ to find stationary point(s) $0 = \frac{x}{1+x}$ $\therefore\ x = 0$ *When* $x = 0,\ y = 0 - \log_e(1+0)$ $= 0$ $\therefore (0,0)$ is a stationary pt. $\frac{d^2y}{dx^2} = \frac{d\left(1 - \frac{1}{1+x}\right)}{dx}$ $= \frac{1}{(1+x)^2}$ > 0 for $x > -1$ $\therefore (0,0)$ is a global minimum turning point as the 2nd derivative is positive, y is continous and only 1 stationary point. $\therefore\ x - \log_e(1+x)\ \geq\ 0$ for $x > -1$ $\therefore\ x\ \geq\ \log_e(1+x)$	**1 mark** for finding stationary point (0,0) using calculus. **1 mark** for using 2nd derivative or table to justify that (0,0) is a minimum turning point and more importantly global minimum. **1 mark** for a conclusion of why the inequality is true. **OR** **3 marks for** Alternative solution of graphing $y = x$ and $y = \log_e(x)$ must also be accompanied with explanation from calculus that $y = x$ is a tangent to $y = \log_e(x)$ at (0,0). Also students <u>must</u> use calculus to justify why $y = \log_e(x)$ is below $y = x$.	Candidates whom used the solution provided were more successful in proving the inequality. Candidates whom used the alternative solution listed in the marking criteria were generally less successful, as they relied on the graph rather than calculus (as stated in the question) to prove inequality or provided incorrect results on part of the domain. As the question states to use calculus, candidates did lose marks if not enough was used and/or justified.

Q 14(a)



Given: OARS is cyclic.

Prove $PS \parallel QR$

Proof Let $\angle SOR = 2\alpha$

Then $\angle SPR = \angle SQR = \alpha$ ($\angle$ at circumference is half $\angle$ at centre standing on same arc)

Also $\angle SAR = 2\alpha$ ($\angle$s in same segment, OARS is cyclic)

Then $\angle ARQ + \angle RQA = \angle RAS$ (ext. $\angle$ of Δ = sum of 2 interior opp. $\angle$s)

$\Rightarrow \angle ARQ + \alpha = 2\alpha$

$\Rightarrow \angle ARQ = \alpha$.

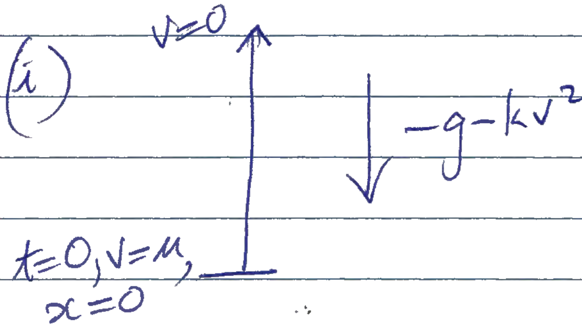
Then $\angle SPR = \angle ARQ = \alpha$ (alternate $\angle$s)

$\Rightarrow PS \parallel QR$ (alternate angles are equal, $\therefore$ lines are parallel)

3

Most students solved this quite efficiently.

14(b) (i)



Resistance is opposite the direction of motion.

$$\Rightarrow \ddot{x} = -g-kv^2$$ 1

(ii)

$$v\frac{dv}{dx} = -g-kv^2$$

$$\frac{dv}{dx} = \frac{-g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}$$

$$x = -\frac{1}{2k}\int \frac{2kv}{g+kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln\left(g+kv^2\right)+c$$

When $x=0$, $v=u$ $\Rightarrow 0 = -\frac{1}{2k}\ln\left(g+ku^2\right)+c$ 2

$$\Rightarrow c = \frac{1}{2k}\ln\left(g+ku^2\right)$$

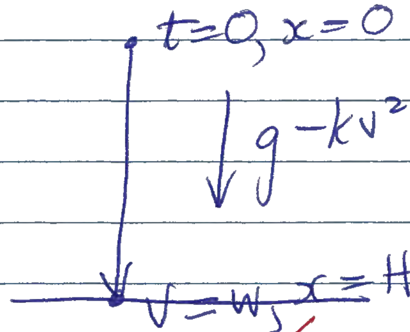
Then $x = -\frac{1}{2k}\ln\left(g+kv^2\right) + \frac{1}{2k}\ln\left(g+ku^2\right)$

$$x = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right)$$

When $v=0$, $x=H$ $\Rightarrow H = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$

Students generally had no problem with part (i) and (ii).

14 (b) (iii) Going down



$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$x = -\frac{1}{2k}\int \frac{-2kv}{g-kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln(g-kv^2)+c$$

$x=0, v=0 \Rightarrow 0 = -\frac{1}{2k}\ln g + c$

$$\Rightarrow c = \frac{1}{2k}\ln g.$$

So, $x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$

$$\Rightarrow 2kx = \ln\left(\frac{g}{g-kv^2}\right)$$

When $x=H, v=w \Rightarrow 2kH = \ln\left(\frac{g}{g-kw^2}\right)$

$$e^{2kH} = \frac{g}{g-kw^2}$$

$$e^{-2kH} = \frac{g-kw^2}{g}$$

$$e^{-2kH} = 1-\frac{k}{g}w^2$$

$$\frac{k}{g}w^2 = 1-e^{-2kH}$$

$$w^2 = \frac{g}{k}\left(1-e^{-2kH}\right) \quad \#$$

Quite a number of students had the downward equation of motion with the incorrect signs. The process was generally good.

3

Q14 (c) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

(i) $I_n = \int \sin^{n-1} x . \sin x \, dx$

$u = \sin^{n-1} x$ | $dv = \sin x$

$du = (n-1)\sin^{n-1} x . \cos x \, dx$ | $v = -\cos x$

$$I_n = \left[\sin^{n-1} x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos^2 x (n-1) \sin^{n-2} x \, dx$$

$$= \left[-\sin^{n-1} x . \cos x\right]_0^{\pi/2} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x . \cos^2 x \, dx.$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$$

2

$$\Rightarrow I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx.$$

The error some made was to split the integral into a square of sinx and a power of n-2. This did not resolve.

(ii) Deduce $I_n = \frac{n-1}{n} I_{n-2}$.

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx.$$

$$= (n-1)\left[\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx\right]$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$(n-1) I_n + I_n = (n-1) I_{n-2}$$

2

$$n I_n = (n-1) I_{n-2}$$

This part was done well.

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \#.$$

(iii) $I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$.

where $I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$

Then $I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$

2

and $I_4 = \frac{3}{4} I_2$

$= \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}$

This part was also done well.

Ext 2 Y12 THSC 2019 Q15 solutions

Mean (out of 15): 8.58

Q15(a)

$S(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$

Step 1: Show $S(2)$ is true

i.e. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} < 2\sqrt{2} - 1$

$LHS = 1 + \frac{1}{\sqrt{2}} = 1.7071\dots$

$RHS = 2\sqrt{2} - 1 = 1.8284\dots$

$\therefore S(2)$ is true

Step 2: Assume $S(k)$ is true

i.e. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} < 2\sqrt{k} - 1$

Show $S(k+1)$ is true

ie $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 1$

Technique 1:

$LHS < 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$

Aim: Show $2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 1$

i.e $2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$

i.e $\left(2\sqrt{k} + \frac{1}{\sqrt{k+1}}\right)^2 < \left(2\sqrt{k+1}\right)^2$

i.e $4k + \frac{4\sqrt{k}}{\sqrt{k+1}} + \frac{1}{k+1} < 4(k+1)$

i.e $\frac{4\sqrt{k}}{\sqrt{k+1}} + \frac{1}{k+1} < 4$

i.e $4\sqrt{k}\sqrt{k+1} + 1 < 4k + 4$

i.e $4\sqrt{k}\sqrt{k+1} < 4k + 3$

i.e $16k(k+1) < 16k^2 + 24k + 9$

i.e $16k^2 + 16k < 16k^2 + 24k + 9$

i.e $0 < 8k + 9$

This is a true statement for $k \geq 2$

$\therefore LHS < RHS$

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true

Technique 2:

$LHS < 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$

$4k^2 + 4k < 4k^2 + 4k + 1$

$2\sqrt{k(k+1)} < 2k + 1$

$\therefore 2\sqrt{k} < \frac{2k+1}{\sqrt{k+1}} = \frac{2(k+1) - 1}{\sqrt{k+1}}$

$= 2\sqrt{k+1} - \frac{1}{\sqrt{k+1}}$

$\therefore 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 1$

$\therefore LHS < RHS$

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true

Technique 3:

Consider $2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} - 2\sqrt{k+1} + 1$

$= 2\sqrt{k} + \frac{1 - 2(k+1)}{\sqrt{k+1}}$

$= 2\sqrt{k} + \frac{-2k-1}{\sqrt{k+1}}$

$= 2\sqrt{k} - \frac{2k+1}{\sqrt{k+1}}$

$= \frac{\sqrt{4k(k+1)} - (2k+1)}{\sqrt{k+1}}$

$= \frac{\sqrt{4k^2+4k} - (2k+1)}{\sqrt{k+1}}$

$< \frac{\sqrt{4k^2+4k+1} - (2k+1)}{\sqrt{k+1}}$

$$= \frac{2k+1-(2k+1)}{\sqrt{k+1}}$$

$$= 0$$

$\therefore$ LHS < RHS

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true.

$\boxed{\text{Technique 4:}}$

$$2\sqrt{k+1} - 1 - \left[2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}\right]$$

$$= 2\sqrt{k+1} - 2\sqrt{k} - \frac{1}{\sqrt{k+1}}$$

$$= \frac{2k+2 - 2\sqrt{k^2+k} - 1}{\sqrt{k+1}}$$

$$> \frac{2k+2 - 2\sqrt{k^2+k+\frac{1}{4}} - 1}{\sqrt{k+1}}$$

$$= \frac{2k+2 - 2\sqrt{(k+\frac{1}{2})^2} - 1}{\sqrt{k+1}}$$

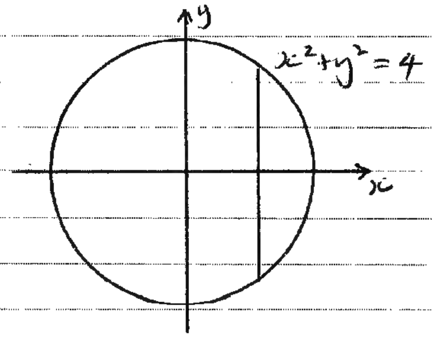
$$= \frac{2k+2-2k-1-1}{\sqrt{k+1}}$$

Step 3 $S(2)$ is true, and, if $S(k)$ is true, $S(k+1)$ is true. Therefore, by the process of Mathematical Induction, $S(n)$ is true for all integral $n \geq 2$.

Students found this question quite challenging. Most were able to show that the statement was true for $n = 2$. However, some seemed to feel that it was obvious that $1 + \frac{1}{\sqrt{2}} < 2\sqrt{2} - 1$. Many struggled to demonstrate that the inequality was true in general.

0	0.5	1	1.5	2	2.5	3	Mean
1	8	26	46	15	6	20	1.67

(b)



side length $= 2y$
$= 2\sqrt{4-x^2}$

Area of typical cross-section

$$= \frac{1}{2} \times \left(2\sqrt{4-x^2}\right)^2 \sin 30°$$

$$= 4 - x^2$$

Volume of a typical slice

$$= (4-x^2)\,\delta x$$

$$\text{Volume} = \lim_{\delta x \to 0} \sum_{x=-2}^{2} (4-x^2)\,\delta x$$

$$= \int_{-2}^{2} (4-x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left[8 - \frac{8}{3}\right] \times 2$$

$$= \frac{32}{3} \text{ units}^3$$

In general, this was done quite well. A common error was to, having determined that the area of a typical cross-section was y^2, integrate with respect to y.

0	0.5	1	1.5	2	2.5	3	Mean
3	2	2	3	13	9	90	2.67

(c) Total number of arrangements
$= 6^6$

Probability exactly 4 courses chosen:

Consider courses as ABCDEF
E and F not to be used.

For the six people, 3|1|1|1

or 2|2|1|1

are the possible arrangements.

3|1|1|1:

$\underbrace{{}^6C_3}_{\text{Form a group of 3}} \times \underbrace{4!}_{\text{Place the 4 groups}}$

2|2|1|1:

$\dfrac{{}^6C_2 \times {}^4C_2}{2!} \times 4!$

Form the groups of 2

Remove multiple counts by noting 2 groups of 2

Place the 4 groups.

No of arrangements
$= 20 \times 4! + 45 \times 4!$
$= 65 \times 4!$

Total arrangements of 4 courses
$= 65 \times 4! \times {}^6C_4$
$= 65 \times 4! \times 15$

$\therefore$ Probability $= \dfrac{65 \times 4! \times 15}{6^6}$
$= \dfrac{23400}{46656}$
$= \dfrac{325}{648}$

A challenging question. Students had problems tying together multiple ideas such as choosing 4 out of the 6 courses, that one course was chosen by 3 people or 2 courses were chosen by 2 people and that there were 6^6 possible outcomes if there were no restrictions.

0	0.5	1	1.5	2	2.5	3	Mean
10	12	31	56	3	10	0	1.25

(d)(i) $\dfrac{r+1}{r}\left(\dfrac{1}{{}^{n+r-1}C_r} - \dfrac{1}{{}^{n+r}C_r}\right)$

$= \dfrac{r+1}{r}\left(\dfrac{(n-1)!\,r!}{(n+r-1)!} - \dfrac{n!\,r!}{(n+r)!}\right)$

$= \dfrac{r+1}{r}\left(\dfrac{(n-1)!\,r!\,(n+r) - n!\,r!}{(n+r)!}\right)$

$= \dfrac{r+1}{r} \times \dfrac{r!\,(n-1)!\,(n+r-n)}{(n+r)!}$

$= \dfrac{r+1}{r} \times \dfrac{r!\,(n-1)!\,r}{(n+r)!}$

$= \dfrac{(r+1)\,r!\,(n-1)!}{(n+r)!}$

$= \dfrac{(r+1)!\,(n-1)!}{(n+r)!}$

$= \dfrac{1}{{}^{n+r}C_{r+1}}$

Most students were able to demonstrate this result.

0	0.5	1	1.5	2	Mean
13	11	16	7	75	1.49

(c)(ii) $\sum_{n=1}^{\infty} \frac{1}{^{n+r}C_{r+1}}$

$$= \frac{1}{^{r+1}C_{r+1}} + \frac{1}{^{r+2}C_{r+1}} + \frac{1}{^{r+3}C_{r+1}} + \cdots$$

$$= \frac{r+1}{r}\left\{\left(\frac{1}{^{r}C_r} - \cancel{\frac{1}{^{r+1}C_r}}\right) + \left(\cancel{\frac{1}{^{r+1}C_r}} - \cancel{\frac{1}{^{r+2}C_r}}\right) + \left(\cancel{\frac{1}{^{r+2}C_r}} - \frac{1}{^{r+3}C_r}\right) + \cdots \right\}$$

$$= \frac{r+1}{r}\left\{\frac{1}{^{r}C_r} \cdots - \frac{1}{^{r+n}C_r} + \frac{1}{^{r+n}C_r} \cdots \right\}$$

As $n \to \infty$

$^{r+n}C_r \to \infty$

$\therefore \frac{1}{^{r+n}C_r} \to 0$

$$= \frac{r+1}{r}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{^{n+2}C_3} = \frac{3}{2}$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{^{n+2}C_3} = \frac{3}{2} - 1$$

$$= \frac{1}{2}$$

Students who listed the terms of the series and then used the result from part (i) were able to supply the simplified expression in terms of r. A common error for the specific term was not noticing that this expression started at 2 rather than 1.

0	0.5	1	1.5	2	2.5	3	3.5	4	Mean
58	2	10	2	4	5	14	3	24	1.5

Question 16

a) i) a, b, c are positive & unequal

$$(\sqrt{a} - \sqrt{b})^2 > 0$$

$$a - 2\sqrt{ab} + b > 0$$

$$a + b > 2\sqrt{ab}$$

$$\therefore\ a + b \geq 2\sqrt{ab}$$

ii) Similarly $b + c > 2\sqrt{bc}$

$$c + a > 2\sqrt{ca}$$

$$(a+b)(b+c)(c+a) > 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ca}$$

$$= 8abc$$

$$\therefore\ (a+b)(b+c)(c+a) > 8abc.$$

COMMENT:

If $a > k$

it can be said that $a \geq k$.

Students were not penalised for not distinguishing between $>$ and $\geq$.

b) i) $LHS = \dfrac{(\cot\theta + i)^{2n+1} - (\cot\theta - i)^{2n+1}}{2i}$

$$= \frac{\left(\frac{\cos\theta}{\sin\theta} + i\right)^{2n+1} - \left(\frac{\cos\theta}{\sin\theta} - i\right)^{2n+1}}{2i}$$

$$= \frac{\left(\frac{\cos\theta + i\sin\theta}{\sin\theta}\right)^{2n+1} - \left(\frac{\cos\theta - i\sin\theta}{\sin\theta}\right)^{2n+1}}{2i}$$

$$= \frac{(\cos\theta + i\sin\theta)^{2n+1} - (\cos(-\theta) + i\sin(-\theta))^{2n+1}}{2i\sin^{2n+1}\theta}$$

$$= \frac{\cos(2n+1)\theta + i\sin(2n+1)\theta - (\cos(2n+1)(-\theta) + i\sin(2n+1)(-\theta))}{2i\sin^{2n+1}\theta}$$

$$= \frac{\cancel{\cos(2n+1)\theta} + i\sin(2n+1)\theta - (\cancel{\cos(2n+1)\theta} - i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}$$

$$= \frac{\cancel{2i}\sin(2n+1)\theta}{\cancel{2i}\sin^{2n+1}\theta}$$

$$= \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

$= RHS$

COMMENT:

Mostly done well.

Don't skip steps in a <u>show that</u>

Use De Moivre's theorem.

ii) consider $\dfrac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}=0$

$$LHS=\frac{\cancel{\binom{2n+1}{0}\cot^{2n+1}\theta}+\binom{2n+1}{1}\cot^{2n}\theta . i+\cancel{\binom{2n+1}{2}\cot^{2n-1}\theta .(i)^2}+\binom{2n+1}{3}\cot^{2n-2}\theta .(i)^3+\cancel{\binom{2n+1}{4}\cot^{2n-3}\theta .(i)^4}+\dots}{2i}$$

$$-\left[\frac{\cancel{\binom{2n+1}{0}\cot^{2n+1}\theta}+\binom{2n+1}{1}\cot^{2n}(-i)+\cancel{\binom{2n+1}{2}\cot^{2n-1}\theta .(-i)^2}+\binom{2n+1}{3}\cot^{2n-2}\theta(-i)^3+\cancel{\binom{2n+1}{4}\cot^{2n-3}\theta(-i)^4}+\dots}{2i}\right]$$

$$=\frac{\cancel{2i}\left[\binom{2n+1}{1}\cot^{2n}\theta-\binom{2n+1}{3}\cot^{2n-2}\theta+\binom{2n+1}{5}\cot^{2n-4}\theta+\dots+(-1)^n\right]}{\cancel{2i}}$$

$$=\binom{2n+1}{1}(\cot^2\theta)^n-\binom{2n+1}{3}(\cot^2\theta)^{n-1}+\binom{2n+1}{5}(\cot^2\theta)^{n-2}+\dots+(-1)^n$$

let $x=\cot^2\theta$

$$\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\dots+(-1)^n=0$$

from (i)

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}=0$$

$$\sin(2n+1)\theta=0$$

$(2n+1)\theta=m\pi$, where m is an integer.

$$\theta=\frac{m\pi}{2n+1}$$

$$\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\dots+(-1)^n=0$$

is a polynomial equation of degree n.

Therefore, there are n solutions.

$m=1,2,3,\dots,n$ Note: $m\neq0$ as $\sin\theta\neq0$.

The solutions are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right), \text{ where } m = 1, 2, 3, \ldots, n.$$

COMMENT:

This question was answered poorly.

The binomial coefficients suggest a binomial expansion.

Deduce requires us to use part (i)

This should given us an idea as to what to try.

iii)

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = -\frac{b}{a} \quad \text{(SUM OF ROOTS)}$$

$$= - - \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}}$$

$$= \frac{\cancel{(2n+1)}!}{3!\,(2n+1-3)!} \times \frac{1!\,(2n+1-1)!}{\cancel{(2n+1)}!}$$

$$= \frac{(2n)!}{3!\,(2n-2)!}$$

$$= \frac{2n(2n-1)\cancel{(2n-2)!}}{6\,\cancel{(2n-2)!}}$$

$$= \frac{n(2n-1)}{3}$$

COMMENT:

This was done quite well by the majority of students.

iv) $\sin\theta < \theta < \tan\theta$

$\sin^2\theta < \theta^2 < \tan^2\theta$ $\quad f(x)=x^2$ is increasing for $x>0$

$\frac{1}{\sin^2\theta} > \frac{1}{\theta^2} > \frac{1}{\tan^2\theta}$ $\quad f(x)=\frac{1}{x}$ is decreasing for $x>0$

$\operatorname{cosec}^2\theta > \frac{1}{\theta^2} > \cot^2\theta$

$1+\cot^2\theta > \frac{1}{\theta^2} > \cot^2\theta$

$\therefore \cot^2\theta < \frac{1}{\theta^2} < 1+\cot^2\theta$

COMMENT:

This was done quite well.
Some students forgot to change the direction of the sign when taking the reciprocal.

v) $\cot^2\left(\frac{m\pi}{2n+1}\right) < \frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < 1+\cot^2\left(\frac{m\pi}{2n+1}\right)$

$$\sum_{m=1}^{n}\cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n}\frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < \sum_{m=1}^{n}\left(1+\cot^2\left(\frac{m\pi}{2n+1}\right)\right)$$

$$\frac{\pi^2}{(2n+1)^2}\sum_{m=1}^{n}\cot^2\left(\frac{m\pi}{2n+1}\right) < \frac{\pi^2}{(2n+1)^2}\sum_{m=1}^{n}\frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < \frac{\pi^2}{(2n+1)^2}\sum_{m=1}^{n}\left(1+\cot^2\left(\frac{m\pi}{2n+1}\right)\right)$$

$$\frac{\pi^2}{(2n+1)^2}\cdot\frac{n(2n-1)}{3} < \sum_{m=1}^{n}\frac{1}{m^2} < \frac{\pi^2}{(2n+1)^2}\left[n+\frac{n(2n-1)}{3}\right]$$

$$\frac{\pi^2}{3}\cdot\frac{2n^2-n}{4n^2+4n+1} < \sum_{m=1}^{n}\frac{1}{m^2} < \frac{\pi^2}{3}\cdot\frac{2n^2+2n}{4n^2+4n+1}$$

$$\lim_{n\to\infty}\frac{\pi^2}{3}\cdot\frac{2n^2-n}{4n^2+4n+1} < \lim_{n\to\infty}\sum_{m=1}^{n}\frac{1}{m^2} < \lim_{n\to\infty}\frac{\pi^2}{3}\cdot\frac{2n^2+2n}{4n^2+4n+1}$$

$$\frac{\pi^2}{3}\cdot\frac{2}{4} < \sum_{m=1}^{\infty}\frac{1}{m^2} < \frac{\pi^2}{3}\cdot\frac{2}{4}$$

$$\frac{\pi^2}{6} < \sum_{m=1}^{\infty}\frac{1}{m^2} < \frac{\pi^2}{6}$$

$$\therefore \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{1}{6}$$

COMMENT:

Not many students made progress with this question.

- The result is an equality despite the inequalities in the previous part which suggests sandwiching between the same value.
- In order to match part (iii) & (iv) $\theta = \frac{m\pi}{2n+1}$.
- since $n \to \infty$ we should expect work with limits

A common mistake for students that did make progress was to forget $\sum_{m=1}^{n} 1 = n$.

OVERALL:

b) ii) & b) v) were the hardest questions.

The other 9 MARKS were quite obtainable.