SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Class:

2020

YEAR 12
TERM 3
TRIAL HSC - ASSESSMENT TASK 3

Extension 2 Mathematics

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2 – 4)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 15)

- Attempt Questions 11 to 16
- Allow about 2 hour and 45 minutes for this section

Examiner: JJ

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1 A particle is moving along a straight line.
Initially its displacement is $x = 2$, its velocity is $v = 3$, and its acceleration is $a = 3$.

Which of the following is a possible equation that describes the motion of the particle?

A. $v = 3 + \ln(x - 1)$

B. $v^2 = 6(x - 3)$

C. $v = 3 + 2\cos\dfrac{\pi x}{2}$

D. $v = e^{-x+2} + 2$

2 Which of the following is the primitive of $\displaystyle\int \tan^4 x \, dx$?

You may assume that $\displaystyle\int \tan^4 x \, dx = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$.

A. $\tan^3 x - \tan x + x + C$

B. $\dfrac{\tan^3 x}{3} - \dfrac{\tan x}{2} + x + C$

C. $\tan^3 x - \dfrac{\tan x}{2} + x + C$

D. $\dfrac{\tan^3 x}{3} - \tan x + x + C$

3 A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$, where x is its displacement from the origin in metres and v is its velocity in m/s.
Given that the motion is simple harmonic, what is the amplitude?

A. 2π metres

B. 3 metres

C. 8 metres

D. 9 metres

4 A mass of m kilograms falls from a stationary balloon at height h metres above the ground. It experiences air resistance during its fall equal to mkv^2 where v m/s is its speed and k is a positive constant. The distance, in metres, of the mass to the ground as it falls is x.
The acceleration due to gravity is given by g and the positive direction is taken to be upwards.
What is the equation of motion?

A. $\ddot{x} = g - kv^2$

B. $\ddot{x} = g + kv^2$

C. $\ddot{x} = -g + kv^2$

D. $\ddot{x} = -g - kv^2$

5 Which of the following is an expression for $\int \frac{\sin x \cos x}{5 + \sin x}\, dx$?

A. $-5\ln\left|\, 5 + \sin x \,\right| + C$

B. $5\ln\left|\, 5 + \sin x \,\right| + C$

C. $-\sin x - 5\ln\left|\, 5 + \sin x \,\right| + C$

D. $\sin x - 5\ln\left|\, 5 + \sin x \,\right| + C$

6 Which of the following is the complex number $4\sqrt{3} - 4i$?

A. $4e^{-\frac{i\pi}{6}}$

B. $4e^{\frac{5\pi}{6}}$

C. $8e^{-\frac{i\pi}{6}}$

D. $8e^{\frac{5\pi}{6}}$

7 A student used a substitution in order to solve the definite integral $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$.

Which of the following shows the new integral, after the substitution is complete?

A. $\int_1^e \frac{u}{(1+u)^2}\,du$

B. $\int_0^1 \frac{u}{(1+u)^2}\,du$

C. $\int_0^1 \frac{1}{(1+u)^2}\,du$

D. $\int_1^e \frac{1}{(1+u)^2}\,du$

8 Let $I = \int_0^\pi x\sin x\,dx$. Given also that $I = \int_0^\pi (\pi - x)\sin(\pi - x)\,dx$, which of the following is the value of I?

A. π

B. $\frac{\pi}{2}$

C. 2

D. $\frac{\pi}{4}$

9 Without evaluating the integrals, which one the following integrals is greater than zero?

A. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos x}{x^3}\,dx$

B. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^{-1}(x^5)\,dx$

C. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^3 x}{x^5}\,dx$

D. $\int_{-1}^{1} (x^2-4)e^{-x^2}\,dx$

10 Let P be a polynomial with real coefficients. If $P(\overline{z}) = 2 - i$, what is the value of $P(z)$?

A. $2 - i$

B. $1 - 2i$

C. $2 + i$

D. $1 + 2i$

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Section II

90 marks
Attempt Questions 11–16
Allow 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet.

(a) $z_1 = 2+3i$, $z_2 = 3+2i$ and $z_3 = a+bi$, where a and b are real numbers.

(i) Find the exact value of $|z_1 + z_2|$. **1**

(ii) Find the exact value of $\mathrm{Re}(z_1\overline{z}_2)$ **1**

(iii) Given that $w = \dfrac{z_1 z_3}{z_2}$, find w in terms of a and b, giving your answer in the form $x + iy$, where x and y are real numbers. **2**

(b) Sketch on separate Argand diagrams

(i) $|z+2i| = 2$ **2**

(ii) $\arg(z) = \dfrac{\pi}{6}$ **2**

(iii) $|z+3i| = |z-i|$ **2**

(c) Let $z^2 = 4 + 4i$.

(i) Find z and express it in the form $a + ib$, where a and b are real numbers. **2**

(ii) If $z^2 w = z^2 - \overline{z^2}\sqrt{3}$, find w and express it in modulus-argument form. **2**

(iii) If iz is a solution to $x^n + k = 0$, where k and n are positive integers, find a possible value of n. **1**

Question 12 (14 marks) Use a SEPARATE Writing Booklet.

(a) Evaluate the following integrals:

(i) $\displaystyle\int_0^3 \frac{9}{9+x^2}\,dx$ **2**

(ii) $\displaystyle\int_1^2 \frac{1}{x+\sqrt{x}}\,dx$ **2**

(iii) $\displaystyle\int_2^5 \frac{1+x}{\sqrt{x-1}}\,dx$ **2**

(b) Find the primitive for the following real functions:

(i) $x\ln x$ **2**

(ii) $\left(9-x^2\right)^{\frac{1}{2}}$ **3**

(c) Find $\displaystyle\int \frac{13-4x}{(2x+1)^2(x+3)}\,dx$. **4**

You may use $\displaystyle\frac{13-4x}{\left(2x+1\right)^2\left(x+3\right)} = \frac{6}{(2x+1)^2} + \frac{A}{(2x+1)} + \frac{B}{(x+3)}$ (Do NOT prove this.)

Question 13 (14 marks) Use a SEPARATE Writing Booklet.

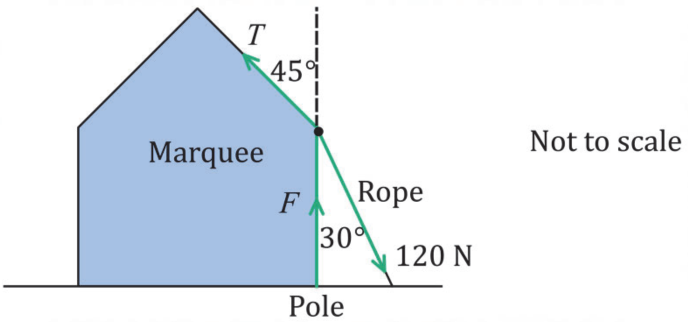
(a) A particle moves in simple harmonic motion. **2**

Its equation is $x = a\cos\sqrt{2}t - 2\sin\left(\sqrt{2}t + \frac{\pi}{4}\right)$, where $a \in \mathbb{R}$.

This could be written as $A\cos\left(\sqrt{2}t + \varepsilon\right)$, where $\varepsilon, A \in \mathbb{R}$ and $A > 0$ (Do NOT prove this).

The particle's maximum velocity is $2\sqrt{5}$ m/s. Find A.

(b) The side of a marquee is supported by a vertical pole supplying a force of F newtons and a rope with a tension of 120 newtons. **3**
The tension in the marquee fabric is T newtons as shown below.



Find the exact values of T and F.

(c) The polynomial P is given by $P(z) = z^4 - 2z^3 + 126z^2 - 242z + 605$.

(i) Given that $(z - ki)$ is a factor of P, find k, where k is a real number. **1**

(ii) Hence, or otherwise, state all the zeroes of P in cartesian form. **3**

(d) A particle A is projected horizontally at a velocity of 50 m/s from the top of a tower which is 100 metres high. **3**
At the same instant, another particle B is projected from the bottom of the tower, in the same vertical plane at a velocity of 100 m/s with an angle of elevation of 60°.

Using the base of the tower as the origin, show that the particles will collide and find the co-ordinates of the point where they do so.
The position vector of the particle, $\underset{\sim}{r}$, from an initial point (h, k), at time t is

$$\underset{\sim}{r}(t) = \left(Vt\cos\phi + h\right)\underset{\sim}{i} + \left(-\frac{gt^2}{2} + Vt\sin\phi + k\right)\underset{\sim}{j}, \qquad \text{(Do NOT prove this)}$$

where V is the magnitude of the initial velocity and ϕ is the angle, with respect to the horizontal, of the initial velocity.

Use $g = 10$ and give your answer to the nearest metre.

Question 13 continues on page 9

Question 13 (continued)

(e) A Formula 1 testing vehicle of mass M kg is capable of a top speed of 360 km/h.

After it reaches this top speed, two different retarding forces combine to bring it to rest:

First, a constant breaking force of $\dfrac{2}{5}M$ newtons due to the application of the brakes.

Second, due to a parachute released from the back of the vehicle, a resistive force of $\dfrac{Mv^2}{200}$ newtons, where v is the speed of the car in metres per second.

After the vehicle has reached top speed both of the above forces are applied and the vehicle eventually comes to a stop.

Show that the distance travelled, x metres, after the application of the two forces, is given by $x = 100\ln\left(\dfrac{80+100^2}{80+v^2}\right)$. **2**

End of Question 13

Question 14 (17 marks) Use a SEPARATE Writing Booklet.

(a) Integrate $e^{2\sqrt{x+1}}$ with respect to x. **3**

(b) Using mathematical induction, prove that $1+\frac{1}{2}+\ldots+\frac{1}{n}>\frac{2n}{n+1}$, **3**

for $n \geq 2$ and n is an integer.

(c) A particle is moving along the x-axis so that its acceleration after t seconds is given by

$$\ddot{x}=4x(x^2-3)$$

The particle starts at the origin with an initial velocity of $\sqrt{10}$ cm/s.

(i) If v is the velocity of the particle, find v^2 as a function of x. **2**

(ii) Show that the particle remains at all times within the interval $-1 \leq x \leq 1$. **2**

(d) Consider the integral $I=\int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)}$.

(i) Using an appropriate substitution, show that $I=\int_a^b \frac{du}{\tan u}$, **2**

stating the values for a and b.

(ii) Hence show that $I=\frac{1}{2}(\ln 3-\ln 2)$ **2**

Question 14 continues on page 11

(e) In the Argand diagram below, two concentric circles have radii r and R, where $r < R$.

The centre, C, of the two circles is represented by the complex number ci.

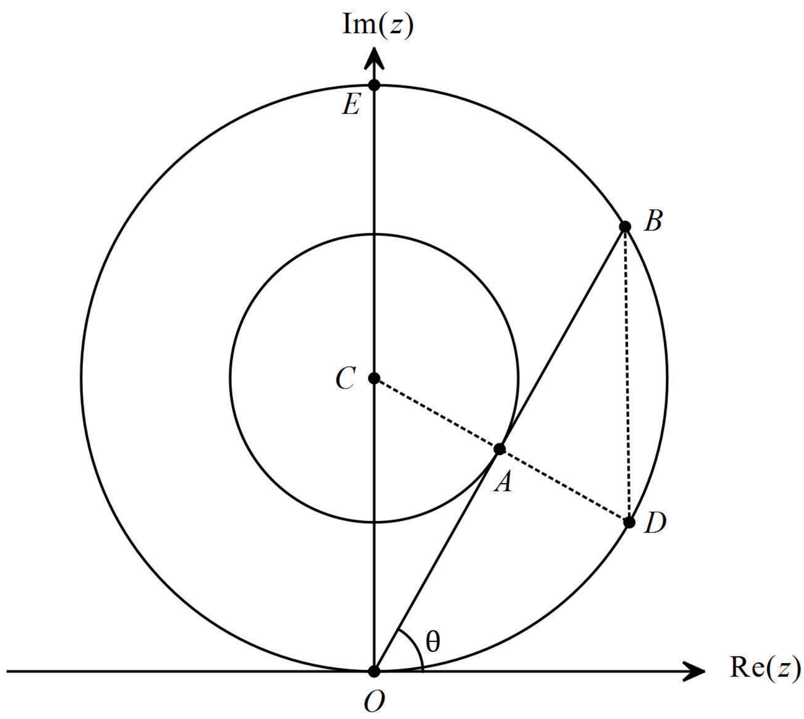
The larger circle touches the real axis only once and that is at the origin.

The chord OB is tangential to the smaller circle.

OB makes an angle θ with the real axis.

Other points are marked for convenience if necessary.

In general, BD is NOT parallel to the imaginary axis.



(i) Given that the equation of the region bounded by the two circles, inclusive of the circles, is $r < \left| z - c \right| < R$, write an expression for the complex number at point B, in terms of r, R and θ, in exponential form. **1**

(ii) Show that the complex number represented by D on the Argand diagram is **2**

$$\sqrt{2R^2 - 2rR} \times e^{\frac{i\theta}{2}},$$

in terms of r, R and θ.

End of Question 14

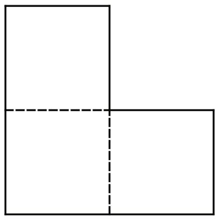
Question 15 (14 marks) Use a SEPARATE Writing Booklet.

(a) A triomino is an L-shaped tile made up of 3 one by one square tiles as shown below. **3**

Use induction to prove that triominoes may be placed on a 2^n by 2^n square board in such a way that there is only ever a single one by one square left uncovered.

The triominoes may not overlap.

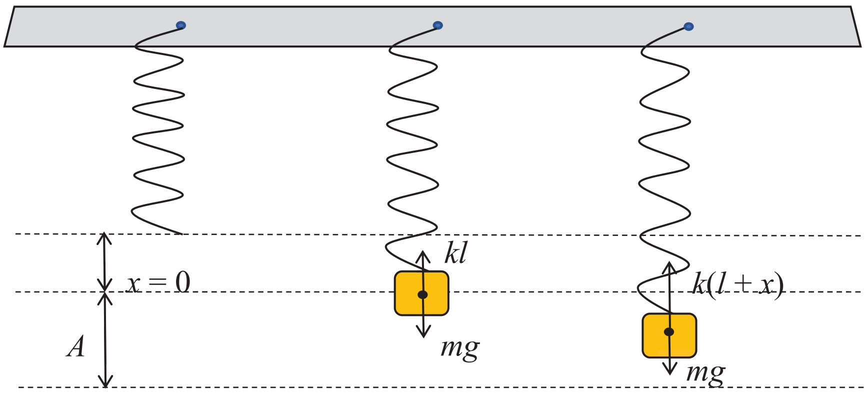
Hint: consider the position of the uncovered square in each inductive step.



(b) A vertical spring is attached to a ceiling in a room. A mass m kg is attached to the spring's other end and as a result the spring stretches by a distance l metres and reaches an equilibrium position. The forces acting on the mass, which is not moving, are the gravity force, mg, and the force of spring, kl, where k is a constant.

The mass is then pulled down by a distance A metres from the equilibrium position and then released. The spring starts to move in simple harmonic motion.

Let x metres be the displacement of the mass at time t seconds and the positive direction measured in a downwards direction. The mass m at a distance x metres ($x \leq A$) experiences, while moving, a spring force $k(l+x)$ and the gravity force mg as shown.



(i) Show that the equation of the motion of the particle is $\ddot{x} = -\dfrac{k}{m}x$. **1**

(ii) Given that $x = A\cos(\omega_0 t + \alpha)$ is a solution of the differential equation in part (i), show that **2**

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

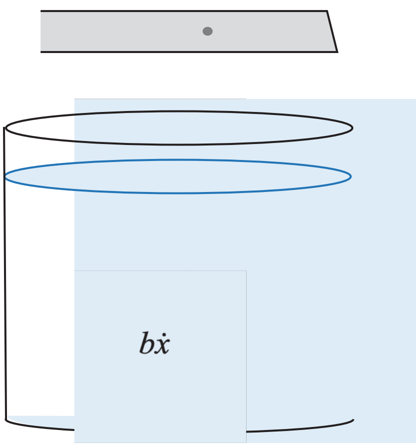
Question 15b continues on page 13

Question 15 (continued)

(b) (continued)

The spring is placed, with a mass m, in a tube containing liquid and again, the mass is then pulled down by a distance A metres from the equilibrium position and released.
The spring will move in damped simple harmonic motion.

The mass m at a distance experiences, while moving, a new resistance force ,
in addition to a spring force and the gravity force mg as shown below.



(iii) Show that the equation of the motion of the mass is $\dot{x}+\frac{k}{m}x=0$. **1**

(iv) Given that $x = Ae^{\frac{-bt}{2m}}\cos(\omega_1 t +$ is a solution of the differential equation in part (iii) and that **3**

$$Ae^{\frac{-bt}{2m}}\cos(\omega_1 t+\phi)\left(\frac{b^2}{4m^2}-\omega_1^2\right)+\frac{Ab\omega_1}{m}e^{\frac{-bt}{2m}}\sin(\omega_1 t+\phi \;,$$

(Do NOT prove this)

show that .

Question 15 continues on page 14

Question 15 (continued)

(c) Bernadette is about to skydive for the first time. The forces acting on her will be gravity (mg), acting downwards which she assigns in the positive direction, and the air resistance is ($R = kv^2$), in the opposite direction.

Let Bernadette's mass be m kg. Hence, $m\dfrac{dv}{dt} = mg - kv^2$.

(i) Show that Bernadette's terminal velocity, $v_T = \sqrt{\dfrac{mg}{k}}$. **1**

(ii) Given that the initial velocity is v_0, show that **3**

$$t = \frac{m}{2kv_T}\ln\left[\left(\frac{v_T + v}{v_T - v}\right)\left(\frac{v_T - v_0}{v_T + v_0}\right)\right],$$

where v is velocity.

End of Question 15

Question 16 (15 marks) Use a SEPARATE Writing Booklet.

(a) Show that there exist complex numbers z and w, such $\frac{1}{z}+\frac{1}{w}=\frac{1}{z+w}$, where $z,\, w \neq 0$. **2**

(b) Let $I_k = \int \cos^k x \, dx$ and $J_n = \int \frac{1}{(x^2+a^2)^n} \, dx$.

(i) Show that $I_k = \frac{1}{k}\sin x \cos^{k-1} x + \frac{k-1}{k}\int \cos^{k-2} x \, dx$. **2**

(ii) Using a suitable substitution, show that $J_n = \frac{1}{a^{2n-1}}\int \cos^{2(n-1)}\theta \, d\theta$ **2**

(iii) Hence or otherwise, give the exact value of $\int_0^{\frac{1}{2}} \frac{1}{\left(x^2+\frac{1}{4}\right)^3} \, dx$. **2**

(c) The distinct complex numbers α, β and γ represent the vertices of an equilateral triangle in an Argand diagram, in order.

(i) Given the three complex numbers above, show that **3**

$$\alpha^2+\beta^2+\gamma^2-\left(\alpha\beta+\alpha\gamma+\beta\gamma\right)=0$$

(ii) Show that the roots of the equation **2**

$$z^3+az^2+bz+c=0 \qquad (1)$$

represent the vertices of an equilateral triangle if $a^2=3b$.

(iii) Let $z=pw+q$, where p and q are complex numbers ($p \neq 0$). **2**
Using $z=pw+q$ transforms equation (1) to

$$w^3+Aw^2+Bw+C=0. \qquad (2) \qquad \text{(Do NOT prove this)}$$

Show that if the roots of the equation (1) represent the vertices of an equilateral triangle, then the roots of equation (2) also represent the vertices of an equilateral triangle.

End of paper

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SYDNEY BOYS HIGH SCHOOL

2020

YEAR 12
TERM 3
TRIAL HSC - ASSESSMENT TASK 3

Mathematics Extension 2

Sample Solutions

Quick MC Answers

1	A
2	D
3	B
4	C
5	D
6	C
7	D
8	A
9	C
10	C

NOTE: Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

2020 Y12 X2 THSC Multiple choice solutions

Mean (out of 10): 7.99

1. A When $x=2$

$v = 3 + \ln 1$

$= 3$

$\ddot{x} = v\frac{dv}{dx}$

$= (3+\ln(x-1)) \cdot \frac{1}{x-1}$

When $x=2$; $\ddot{x} = 3 \cdot 1$

$= 3$ ✓

B When $x=2$, $v^2 = -6$ ✗

C When $x=2$, $v = 3 + 2\cos\frac{3\pi}{2}$

$= 3 + 2 \times -1$

$= 1$ ✗

D When $x=2$, $v = e^0 + 2$

$= 3$

$\ddot{x} = v\frac{dv}{dx}$

$= (e^{-x+2}+2) \cdot -e^{-x+2}$

When $x=2$, $\ddot{x} = 3 \cdot -1$

$= -3$ ✗

A

A	79
B	13
C	5
D	21

2. $\int \tan^4 x \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\, dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

D

A	1
B	3
C	3
D	111

3. $v^2 = -x^2 + 2x + 8$

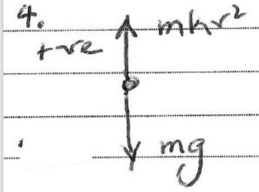
$= 9 - (x^2 - 2x + 1)$

$= 9 - (x-1)^2$

$\therefore$ Amplitude $= 3$ B

A	1
B	101
C	13
D	3

4.



$\therefore \ddot{x} = -g + kv^2$

C

A	26
B	0
C	84
D	8

$$5.\quad \int \frac{\sin x \cos x \, dx}{5+\sin x} \qquad \text{Let } u=\sin x,\ du=\cos x\, dx$$

$$= \int \frac{u\,du}{5+u}$$

$$= \int \frac{5+u}{5+u}\,du - \int \frac{5}{5+u}\,du$$

$$= \int du - 5\ln|5+u| + c$$

$$= u - 5\ln|5+u| + c$$

$$= \sin x - 5\ln|5+\sin x| + c$$

D

A	0
B	2
C	9
D	107

$$6.\quad 4\sqrt{3} - 4i$$

$$= 8\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= 8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 8e^{-i\frac{\pi}{6}}$$

C

A	3
B	0
C	113
D	2

$$7.\quad \int_0^1 \frac{e^x}{(1+e^x)^2}\,dx \qquad \text{Let } u=e^x,\ du=e^x\,dx$$

$$\text{If } x=0,\ u=1; \quad \text{If } x=1,\ u=e$$

$$= \int_1^e \frac{du}{(1+u)^2}$$

D

A	10
B	0
C	1
D	107

$$8.\quad I = \int_0^\pi x\sin x\,dx$$

$$= \int_0^\pi (\pi - x)\sin(\pi - x)\,dx$$

$$= \pi\int_0^\pi \sin x\,dx - \int_0^\pi x\sin x\,dx$$

$$\therefore 2I = \pi\left[-\cos x\right]_0^\pi$$

$$\therefore I = \frac{\pi}{2}\{-\cos\pi - -\cos 0\}$$

$$= \frac{\pi}{2}\{1+1\}$$

$$= \pi$$

A

A	95
B	16
C	5
D	2

9. A $\dfrac{1+\cos(-x)}{(-x)^3} = -\dfrac{1+\cos x}{x^3}$

$\therefore$ ODD FUNCTION

$\therefore$ INTEGRAL $= 0$ ✗

B $\sin^{-1}(-x)^5 = \sin^{-1}(-x^5)$

$= -\sin^{-1} x^5$

$\therefore$ ODD FUNCTION

$\therefore$ INTEGRAL $= 0$ ✗

C $\dfrac{\tan^3(-x)}{(-x)^5} = \dfrac{-\tan^3 x}{-x^5}$

$= \dfrac{\tan^3 x}{x^5}$

$\therefore$ EVEN FUNCTION

$\therefore$ INTEGRAL > 0 ✓

D $((-x)^2-4)\,e^{-(-x)^2} = (x^2-4)e^{-x^2}$

$\therefore$ EVEN FUNCTION

HOWEVER, for $-1 \le x \le 1$,

$(x^2-4)\,e^{-x^2} < 0$

$\therefore$ INTEGRAL < 0 ✗

C

A	19
B	3
C	64
D	32

10. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

$P(\bar{z}) = a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \dots + a_1\bar{z} + a_0$

$= \overline{a_n z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0}$

$= \overline{P(z)}$

$\therefore 2 - i = \overline{P(z)}$

$\therefore P(z) = 2 + i$ C

A	23
B	7
C	82
D	6

Question 11

(a) $z_1 = 2+3i$, $z_2 = 3+2i$ and $z_3 = a+bi$

(i) $|z_1 + z_2|$

$$z_1 + z_2 = 2+3i+3+2i$$
$$= 5+5i$$
$$\therefore |z_1+z_2| = |5+5i|$$
$$= \sqrt{5^2+5^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

(ii) $Re(z_1\overline{z_2})$

$$z_1\overline{z_2} = (2+3i)(3-2i)$$
$$= 6-4i+9i-6i^2$$
$$= 12+5i$$
$$\therefore Re(z_1\overline{z_2}) = 12$$

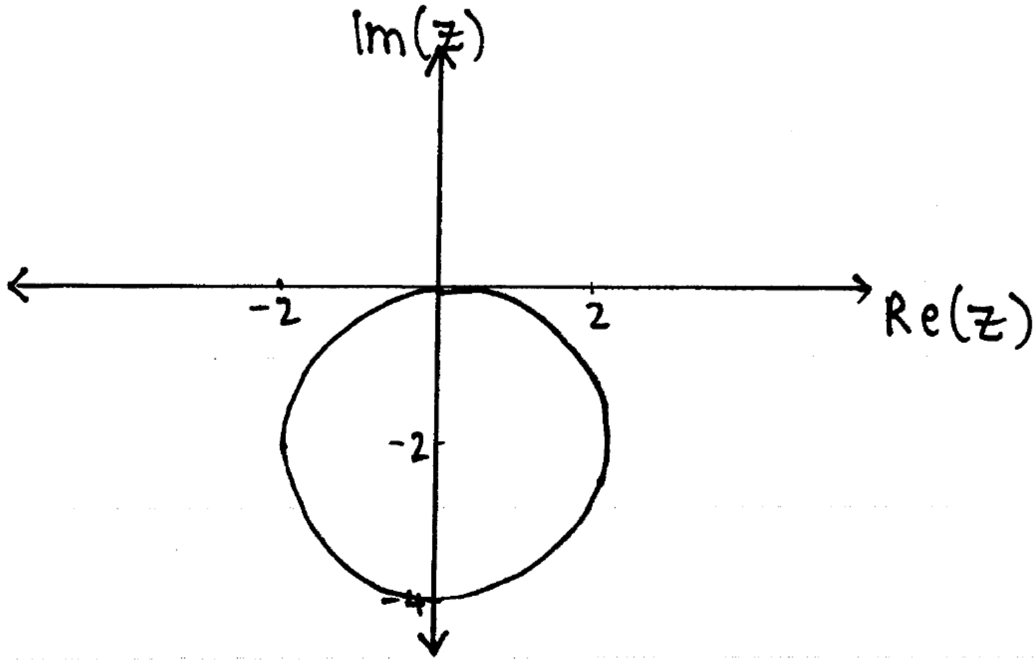
(iii) $w = \dfrac{z_1 z_3}{z_2}$

$$= \frac{(2+3i)(a+bi)}{(3+2i)}$$
$$= \frac{(2a+2bi+3ai+3bi^2)(3-2i)}{(3+2i)(3-2i)}$$
$$= \frac{6a+6bi+9ai-9b-4ai+4b+6a-6bi}{13}$$
$$= \frac{12a-5b}{13} + \frac{(12b+5a)i}{13}$$

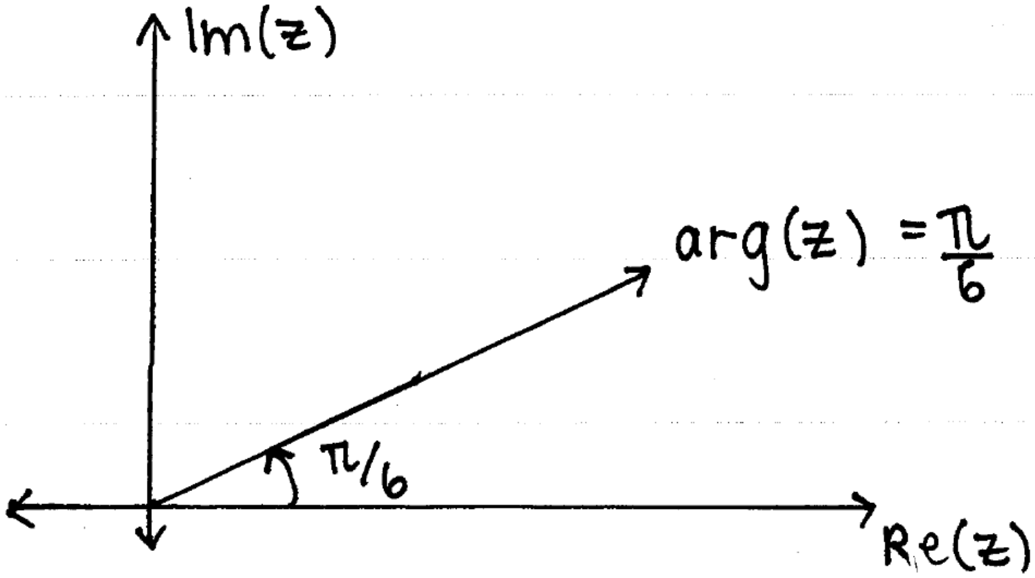
Comments:
Generally, very well done.

(b) (i) $|z + 2i| = 2$

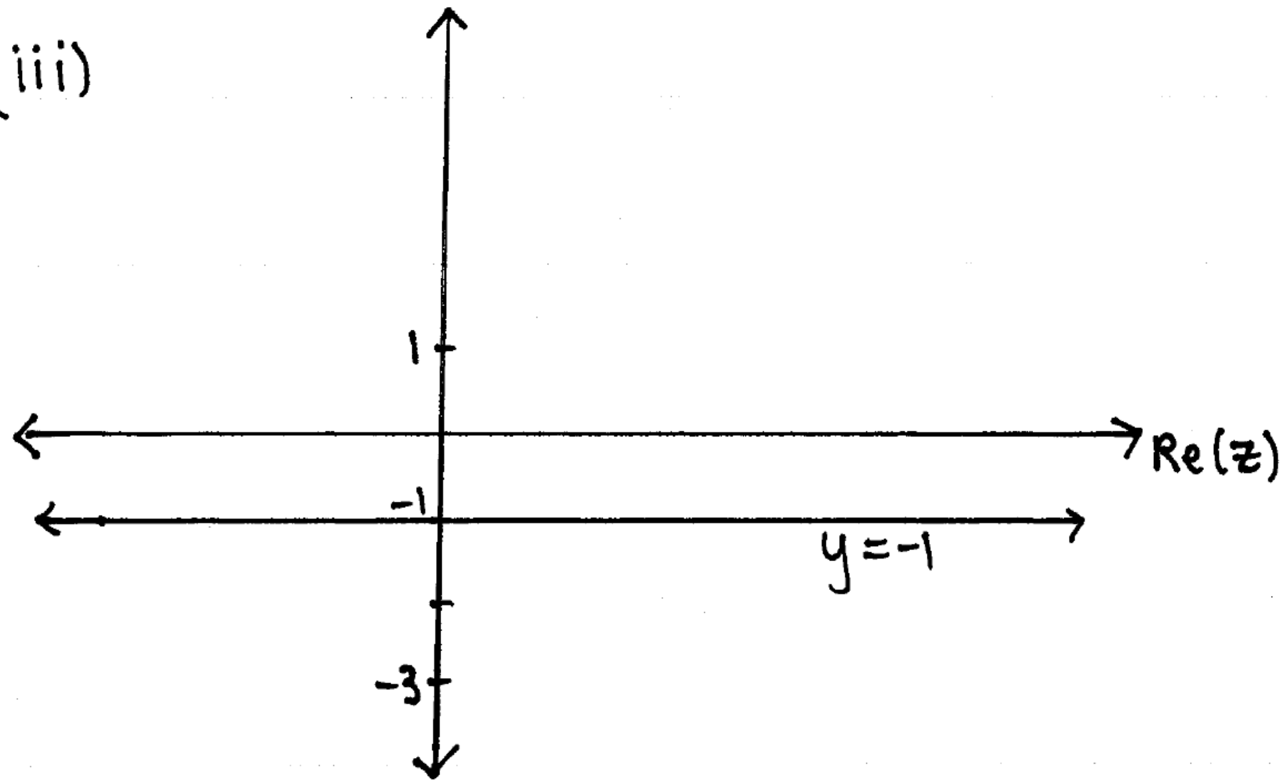
$|z - (-2i)| = 2$



(ii)



(iii)



Comments:
Parts (i) and (ii) generally, very well done.
In part (iii), candidates incorrectly evaluated the modulus.

(c) $z^2 = 4 + 4i$

(i) Let $z = a + bi$, where $a, b \in \mathbb{R}$

$$z^2 = a^2 + 2abi + b^2i^2$$
$$= a^2 - b^2 + (2ab)i$$

$$\therefore 4 + 4i = (a^2 - b^2) + (2ab)i$$

Equating real and imaginary parts

$a^2 - b^2 = 4$ ① $2ab = 4$ ②

also $(a^2+b^2)^2 = (a^2 - b^2) + (2ab)^2$

$$= 4^2 + 4^2$$
$$= 32$$

$$\therefore a^2+b^2 = \sqrt{32}$$
$$= \pm 4\sqrt{2}$$

$a, b \in \mathbb{R} \quad \therefore a^2 \geq 0, \; b^2 \geq 0$ and $a^2+b^2 \geq 0$

$$\therefore a^2+b^2 = 4\sqrt{2} \quad ③$$

③ + ①: $$2a^2 = 4\sqrt{2}+4$$
$$a^2 = 2\sqrt{2}+2$$
$$a = \pm\sqrt{2\sqrt{2}+2}$$

③ − ①: $$2b^2 = 4\sqrt{2}-4$$
$$b^2 = 2\sqrt{2}-2$$
$$b = \pm\sqrt{2\sqrt{2}-2}$$

From ② $\quad 2ab = 4, \quad ab > 0$

$\therefore$ a and b have the same sign

$\therefore a = \sqrt{2\sqrt{2}+2}, \; b = \sqrt{2\sqrt{2}-2}$

or

$a = -\sqrt{2\sqrt{2}+2}, \; b = -\sqrt{2\sqrt{2}-2}$

(ii) $z^2 w = z^2 - (\overline{z}^2)\sqrt{3}$

$$(4+4i)w = (4+4i) - (4-4i)\sqrt{3}$$
$$(1+i)w = (1+i) - (1-i)\sqrt{3}$$
$$(1+i)w = (1-\sqrt{3}) + (1+\sqrt{3})i$$
$$w = \frac{(1-\sqrt{3}) + (1+\sqrt{3})i}{1+i}$$
$$= \frac{1-\sqrt{3} - (1-\sqrt{3})i + (1+\sqrt{3})i - (1+\sqrt{3})i^2}{1^2+1^2}$$
$$= \frac{2+2\sqrt{3}i}{2}$$

$$= 1 + \sqrt{3}\, i$$

$$|w| = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= 2$$

$$\arg(w) = \tan^{-1}(\sqrt{3})$$
$$= \frac{\pi}{3}$$

$$\therefore\ w = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

(iii) $z^2 = 4 + 4i$

$$|z^2| = \sqrt{4^2 + 4^2}$$
$$= 4\sqrt{2}$$

$$\arg(z^2) = \arg(4 + 4i)$$
$$= \frac{\pi}{4} \quad \text{(first quadrant)}$$

$$\therefore\ z^2 = 4\sqrt{2}\, e^{i(\pi/4)}$$
$$z = \pm\sqrt{4\sqrt{2}}\, e^{i(\pi/8)}$$

when $z = \sqrt{4\sqrt{2}}\, e^{i(\pi/8)}$ (one value of n)

$$\therefore\ iz = \sqrt{4\sqrt{2}}\, e^{i(5\pi/8)}$$

$$(iz)^n = \left[\left(2^{5/2}\right)^{1/2}\right]^n \cdot \left(e^{i(5\pi/8)}\right)^n$$

$(iz)^n + k = 0$, $(iz)^n = -k$, where $k \in \mathbb{Z}$

$$(iz)^n = 2^{5n/4}\left(e^{i(5n\pi/8)}\right)$$

$\frac{5n}{4} = \lambda$, $\frac{5n\pi}{8} = m\pi$, $(m, \lambda \in \mathbb{Z})$

$5n = 4\lambda$, $5n\pi = 8m\pi$

when $n = 8$

$5 \times 8 = 4\lambda$ $\quad \lambda = 10 \in \mathbb{Z}$

and $5 \times 8 = 8m$ $\quad m = 5 \in \mathbb{Z}$

$\therefore \quad n = 8.$

Comments:
Parts (i) and (ii) generally, very well done.
In part (iii), very poorly done.

12) a) i) $\displaystyle\int_0^3 \frac{9}{9+x^2}\,dx$

$$= 9\int_0^3 \frac{dx}{9+x^2}$$

$$= 9\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= 9\left(\frac{1}{3}\tan^{-1}\frac{3}{3} - \cancel{\frac{1}{3}\tan^{-1}\frac{0}{3}}\right)$$

$$= 9\cdot\frac{1}{3}\cdot\frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

COMMENT: Given this is on the reference sheet. It should be an easy 2 marks, which it was for most.

ii) $\displaystyle\int_1^2 \frac{1}{x+\sqrt{x}}\,dx$

$$= \int_1^2 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$$

$$= 2\int_1^2 \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}+1}\,dx$$

$$= 2\left[\ln|\sqrt{x}+1|\right]_1^2$$

$$= 2\left[\ln|\sqrt{2}+1| - \ln|\sqrt{1}+1|\right]$$

$$= 2\ln\left(\frac{\sqrt{2}+1}{2}\right)$$

COMMENT: A substitution could have been used. Partial fractions is not a good idea. Depending on the method used the answer may be written differently.

iii) $\displaystyle\int_2^5 \frac{1+x}{\sqrt{x-1}}\,dx$

$$= \int_2^5 \frac{x-1+2}{\sqrt{x-1}}\,dx$$

$$= \int_2^5 \left((x-1)^{\frac{1}{2}} + 2(x-1)^{-\frac{1}{2}} \right) dx$$

$$= \left[\frac{2}{3}(x-1)^{3/2} + 4(x-1)^{\frac{1}{2}} \right]_2^5$$

$$= \frac{2}{3}(5-1)^{3/2} + 4(5-1)^{\frac{1}{2}} - \left(\frac{2}{3}(2-1)^{3/2} + 4(2-1)^{\frac{1}{2}} \right)$$

$$= \frac{26}{3}$$

COMMENT: A substitution could have been used. Again, since the reference sheet has the reverse chain rule results it can be bypassed. A lot of students didn't get the answer.

b) i) $\displaystyle\int x \ln x \, dx$

$u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

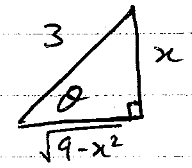
COMMENT: This was mostly done well, as long as students know the method of integration by parts (which is on the reference sheet).

ii) $\int (9-x^2)^{\frac{1}{2}}\,dx$

let $x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$



$$= \int \sqrt{9-(3\sin\theta)^2}\,.\,3\cos\theta\, d\theta$$

$$= \int \sqrt{9(1-\sin^2\theta)}\,.\,3\cos\theta\, d\theta$$

$$= \int 3\cos\theta\,.\,3\cos\theta\, d\theta$$

$$= \int 9\cos^2\theta\, d\theta$$

$$= \int 9\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{9\theta}{2} + \frac{9}{4}\sin 2\theta + C$$

$$= \frac{9\theta}{2} + \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C$$

COMMENT:

Students should recognise when trig. substitutions can be helpful ($\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$)
The primitive needs to be a function of x and should be simplified (not just putting $\sin^{-1}\left(\frac{x}{3}\right)$ wherever θ is).

c) $\dfrac{13-4x}{(2x+1)^2(x+3)} = \dfrac{6}{(2x+1)^2} + \dfrac{A}{2x+1} + \dfrac{B}{x+3}$

$13-4x = 6(x+3) + A(2x+1)(x+3) + B(2x+1)^2$

let $x=-3$

$13-4(-3) = B(2(-3)+1)^2$

$25B = 25$

$\underline{B = 1}$

equate coefficient of x^2

$0 = 2A + 4B$

$0 = 2A + 4(1)$

$2A = -4$

$\underline{A = -2}$

$\displaystyle\int \frac{13-4x}{(2x+1)^2(x+3)}\,dx = \int\left(\frac{6}{(2x+1)^2} - \frac{2}{2x+1} + \frac{1}{x+3}\right)dx$

$\displaystyle = \int\left(3.2(2x+1)^{-2} - \frac{2}{2x+1} + \frac{1}{x+3}\right)dx$

$\displaystyle = \frac{3(2x+1)^{-1}}{-1} - \ln|2x+1| + \ln|x+3| + C$

$\displaystyle = -\frac{3}{2x+1} + \ln\left|\frac{x+3}{2x+1}\right| + C$

COMMENT:

Should have been an easy 4 marks. Given that it was worth 4 marks, a mistake attracted a penalty of 1 mark.

16) a) $\frac{1}{z} + \frac{1}{w} = \frac{1}{z+w}$

$$\frac{z+w}{z} + \frac{z+w}{w} = 1$$

$$1 + \frac{w}{z} + \frac{z}{w} + 1 = 1$$

$$\frac{z}{w} + \frac{w}{z} + 1 = 0$$

let $A = \frac{z}{w}$

$$A + \frac{1}{A} + 1 = 0$$

$$A^2 + 1 + A = 0$$

$$A^2 + A + \frac{1}{4} + \frac{3}{4} = 0$$

$$\left(A + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2 = 0$$

$$\left(A + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(A + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$$

$$A = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

since $A = \frac{z}{w}$ there exist complex numbers z & w (for instance $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ & $w = 1$) such that $\frac{1}{z} + \frac{1}{w} = \frac{1}{z+w}$.

COMMENT: Most students struggled with this question. If it is clear that there exist complex numbers z & w (as mentioned by some) then they should be stated. Just having a lot of working doesn't mean marks are gained.

Question 13

(a) $x = A\cos\left(\sqrt{2}t + \frac{\pi}{4}\right)$, where $A^2 = (a-\sqrt{2})^2 + 4$

$\dot{x} = -\sqrt{2}A\sin\left(\sqrt{2}t + \frac{\pi}{4}\right)$

$$\therefore \quad A\sqrt{2} = 2\sqrt{5}$$
$$A\sqrt{2} = \sqrt{20}$$
$$2A^2 = 20$$
$$A^2 = 10$$
$$A = \pm\sqrt{10} \quad \text{but } A > 0$$

$$\therefore A = \sqrt{10}$$

Comments:
Generally, very well done.

(b) Vertically: $F + T\cos 45° = 120\cos 30°$

$$F + \frac{T}{\sqrt{2}} = 60\sqrt{3}$$

Horizontally: $T\sin 45° = 120\sin 30°$

$$\frac{T}{\sqrt{2}} = 60$$
$$\therefore T = 60\sqrt{2}$$

$$\therefore F + 60 = 60\sqrt{3}$$
$$F = 60\sqrt{3} - 60$$
$$\therefore F = 60(\sqrt{3} - 1)$$

Comments:
Generally, very well done. However, many students incorrectly calculated the exact values.

(c) $P(z) = z^4 - 2z^3 + 126z^2 - 242z + 605$

(i) $P(ki) = (ki)^4 - 2(ki)^3 + 126(ki)^2 - 242(ki) + 605$

$k^4 + 2k^3 i - 126k^2 - 242ki + 605 = 0$

$k^4 - 126k^2 + 605 = 0$ or $2k^3 - 242k = 0$

$\therefore 2k^3 - 242k = 0$

$2k(k^2 - 121) = 0$

$k = 0$ or $k^2 = 121$

$k = \pm 11$

$121^2 - 121 \times 126 + 605 = 121 \times -5 + 605$

$= 0$

$\therefore k = \pm 11$

2 zeros are $11i, -11i$.

(ii) Let the other 2 roots be α and β

$\alpha + \beta + 11i + (-11i) = 2$

$\alpha + \beta = 2$ (1)

$\alpha . \beta . 11i . (-11i) = 605$

$121 \alpha\beta = 605$

$\alpha\beta = 5$

$\therefore z^2 - (\alpha + \beta) z + \alpha\beta = 0$

$z^2 - 2z + 5 = 0$

$$\therefore z = \frac{-(-2) \pm \sqrt{4 - 4.1.5}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= \frac{2(1 \pm 2i)}{2}$$

$$= 1 \pm 2i$$

$\therefore$ the zeros are $11i, -11i, 1+2i$ and $1-2i$

Comments:
Generally, very well done. However, some students the second 2 zeroes.

(d) • For particle A:

$$y_A = -\frac{gt^2}{2} + 50t \sin 0^\circ + 100$$

$$= -\frac{gt^2}{2} + 100$$

$$x_A = 50t . \cos 0^\circ + 0$$

$$= 50t$$

• For particle B:

$$y_B = -\frac{gt^2}{2} + 100t \sin 60^\circ + 0$$

$$= -\frac{gt^2}{2} + \frac{100\sqrt{3}}{2}t$$

$$= -\frac{gt^2}{2} + 50\sqrt{3}t$$

$$x_B = 100t \cos 60^\circ + 0$$

$$= 50t$$

$\therefore$ The 2 particles have the same x coordinate.

For the particles to collide $y_A = y_B$

$$-\frac{gt^2}{2} + 100 = -\frac{gt^2}{2} + 50\sqrt{3}t$$

$$100 = 50\sqrt{3}\,t$$

$$\therefore \quad t = \frac{2}{\sqrt{3}}$$

$$\therefore \quad x = 50 \cdot \frac{2}{\sqrt{3}} = \frac{100\sqrt{3}}{3} \approx 58 \text{ m}$$

$$y = \frac{-10 \cdot \left(\frac{2}{\sqrt{3}}\right)^2}{2} + 100 = \frac{280}{3} \approx 93 \text{ m}$$

$\therefore$ they collide at $(58, 93)$.

Comments:
Poorly done. Many students incorrectly calculated the exact values, or used an incorrect value for *t*.

(e) $F = m\ddot{x}$

$$\ddot{x} = -\frac{2}{5} - \frac{v^2}{200}$$

$$= -\frac{80 + v^2}{200}$$

$$v\frac{dv}{dx} = -\frac{80 + v^2}{200}$$

$$\int \frac{v\,dv}{80+v^2} = -\frac{1}{200}\int dx$$

$$\frac{1}{2}\ln\left|80 + v^2\right| = -\frac{1}{200}x + c$$

when $x = 0$, $v = 1000$

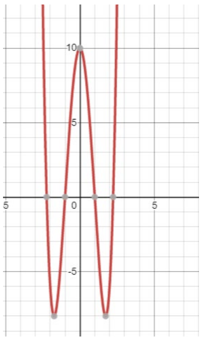
$$\therefore \frac{1}{200}x = \frac{1}{2}\ln\left|80 + 100^2\right| - \frac{1}{2}\ln\left|80+v^2\right|$$

$$\therefore x = 100\ln\left|\frac{80 + 100^2}{80+v^2}\right|$$

Comments:
Generally, very well done.

Question 14

Solution	Marking Scheme	Marker's Comments
a) Let $u = \sqrt{x+1}$ $\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$ $2\sqrt{x+1}\ du = dx$ $2u\ du = dx$ $\int e^{2\sqrt{x+1}} dx = \int e^{2u}(2u)\, du$ $\int e^{2u}(2u)\, du = 2u \times \frac{1}{2}e^{2u} - \int 2 \times \frac{1}{2}e^{2u}\ du$ $= ue^{2u} - \int e^{2u} du$ $= ue^{2u} - \frac{1}{2}e^{2u} + C$ $\therefore \int e^{2\sqrt{x+1}}\ dx = \sqrt{x+1}e^{2\sqrt{x+1}} - \frac{1}{2}e^{2\sqrt{x+1}} + C$	**1 mark** for correct substitution or equivalent. **1 mark** for correct use of integration by parts. **1 mark** for correct answer.	- Candidates who used the right substitution tend to correctly answer the question correctly. - Significant number of candidates made error with integrating by parts section of their working.
b) Prove true for $n = 2$ LHS = $1 + \frac{1}{2} = \frac{3}{2}$ RHS = $\frac{2(2)}{2+1} = \frac{4}{3}$ Since LHS > RHS $\therefore$ true for n = 2 Assume the statement is true for $n = k$ where $k \in Z^+$, $k \geq 2$ i.e. $1 + \frac{1}{2} + ... + \frac{1}{k} > \frac{2k}{k+1}$ To prove true for $n = k + 1$ i.e. $1 + \frac{1}{2} + ... + \frac{1}{k} + \frac{1}{k+1} > \frac{2k+2}{k+2}$ LHS $> \frac{2k}{k+1} + \frac{1}{k+1}$ (From Assumption) **(SEE NEXT PAGE)**	**1 mark** for correctly proving statement is true for $n = 2$. **2 marks** for correctly proving the statement is true for $n = k+1$.	- Majority of the candidates were able to prove the n = 2 case correctly. - Many candidates were not able to gain full marks for proving the case for $n = k+1$. - Candidates need to prove **strictly** one side is greater than the other side with **REASONING**. Marks were not given if candidates fail to do so.

$= \dfrac{2k+1}{k+1}$ $= \dfrac{(2k+1)(k+2)}{(k+1)(k+2)}$ where $k \geq 2$ i.e. $(k+2) \neq 0$ $= \dfrac{2k^2+5k+2}{(k+1)(k+2)}$ $= \dfrac{2k^2+4k+2+k}{(k+1)(k+2)}$ $= \dfrac{(2k+2)(k+1)}{(k+1)(k+2)} + \dfrac{k}{(k+1)(k+2)}$ $= \dfrac{(2k+2)}{k+2} + \dfrac{k}{(k+1)(k+2)}$ $>$ RHS $\left(\text{As } \dfrac{k}{(k+1)(k+2)} > 0 \text{ for } k \geq 2\right)$ $\therefore$ true for $n = k+1$ By the principle of Mathematical Induction, the result is true for all positive integers $n \geq 2$.		- Candidates did not gain full marks by just stating $\dfrac{2k+1}{k+1} > \dfrac{2k+2}{k+2}$ for $k \geq 2$. More details were needed to explain the statement above. - Candidates should avoid proving both sides of the inequality at the same time.
c) i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 4x^3 - 12x$ $\dfrac{1}{2}v^2 = \int 4x^3 - 12x \, dx$ $\dfrac{1}{2}v^2 = x^4 - 6x^2 + C$ At $x = 0$, $v = \sqrt{10}$ $5 = C$ $\therefore v^2 = 2x^4 - 12x^2 + 10$	**1 mark** for correct working. **1 mark** for correct answer	- Majority of the candidates did well in this question.
ii)  $v^2 = 2(x^4 - 6x^2 + 5)$ $= 2(x^2-5)(x^2-1)$ $= 2\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)\left(x-1\right)\left(x+1\right)$ For $v^2 \geq 0 : x \leq -\sqrt{5}$ or $-1 \leq x \leq 1$ or $x \geq \sqrt{5}$ As $v^2 \geq 0$ and the particle **starts at the origin**, it must remain within the interval $-1 \leq x \leq 1$ or it will reach negative for v^2.	**1 mark** for correctly working out the correct intervals for x for which $v^2 \geq 0$ or equivalent solution **1 mark** for stating the particle starts at the origin and is bounded by the region.	- Significant number of candidates made a silly mistake $x^2 = 5$ $x = \pm 5$ And use that to explain their result. Marks were penalised. - This is NOT SHM, check equation. Candidates were penalised for saying it was.

		- The information that the particle starts at origin is important and must be stated clearly in the solution to support their reasoning.
d) i) let $x = \tan u$ $\frac{dx}{du} = \sec^2 u$ $dx = \sec^2 u \, du$ $x = 1 \sim \sqrt{3}$ $u = \frac{\pi}{4} \sim \frac{\pi}{3}$ $I = \int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)}$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 u}{\tan u \,(1+\tan^2 u)} du$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 u}{\tan u \,(\sec^2 u)} du$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan u} du$ where $a = \frac{\pi}{4}$ *and* $b = \frac{\pi}{3}$	1 mark for correct substitution and working. 1 mark for finding correct values for *a* and *b*.	- Majority of candidates did well in this question.
ii) $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot u \, du$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos u}{\sin u} du$ $= [\ln(\sin u)]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \ln\left(\sin\left(\frac{\pi}{3}\right)\right) - \ln\left(\sin\left(\frac{\pi}{4}\right)\right)$ $= \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)$ $= \ln\left(\frac{\sqrt{3}}{2} \div \frac{1}{\sqrt{2}}\right)$ $= \ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$ $= \ln(\sqrt{3}) - \ln(\sqrt{2})$ $= \frac{1}{2}(\ln 3 - \ln 2)$	**1 mark** for correct integration. **1 mark** for showing ALL necessary working to achieve the result.	- Candidates were penalised for not showing all necessary steps. Since it is a SHOW question, candidates need to provide every step, not assume the marker's will follow through with shortcuts. - Some candidates did not how to integrate cot *u* which is a concern.

<table>
<tr><td>e)
i) Let α be the complex number at point B
$\therefore\ \alpha = ke^{i\theta}$ where $k = \lvert OB \rvert$
OB is a chord in a circle of radius R (given)
CA is perpendicular to OB (tangent perpendicular to radius)
CD bisects OB (radius to chord bisects the chord)
$\therefore OB = 2 \times OA$
$\therefore OC^2 = CA^2 + OA^2$ (Pythagoras' Theorem)
$OA = \sqrt{OC^2 - CA^2}$
$= \sqrt{R^2 - r^2}$
$\therefore OB = 2\sqrt{R^2 - r^2}$
$\therefore \alpha = 2\sqrt{R^2 - r^2}e^{i\theta}$</td>
<td>1 mark for the correct answer</td>
<td>- Many candidates did not attempt this question or did not get the modulus at point B correct.</td></tr>
<tr><td>ii)
$\lvert DC \rvert = R$ and DC has an angle of depression of
$-\left(\frac{\pi}{2} - \theta\right)$ from point C.

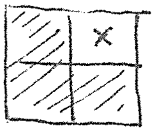
Then let D be represented by the complex number γ
$\gamma = iR + \mathrm{Re}^{-i\left(\frac{\pi}{2}-\theta\right)}$
$e^{i\left(\theta - \frac{\pi}{2}\right)} = \sin\theta - i\cos\theta$
$\gamma = R\left(\left(\sin\theta + i(1-\cos\theta)\right)\right)$
$\lvert\gamma\rvert = R\sqrt{(\sin\theta + i(1-\cos\theta)(\sin\theta + i(\cos\theta - 1)}$
$= R\sqrt{\sin^2\theta - 2\cos\theta + \cos^2\theta + 1}$
$= R\sqrt{2(1-\cos\theta)}$
$= \sqrt{2R^2(1-\cos\theta)}$
But from the diagram $\cos\theta = \frac{r}{R}$
$\therefore \sqrt{2\left(R^2 - Rr\right)}$
Let $Arg\left(\gamma\right) = \tan\phi$ (1)
$Arg\left(\gamma\right) = \frac{1-\cos\theta}{\sin\theta}$ (From diagram)
$= \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$ (Using half angle)
$= \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$
$= \tan\frac{\theta}{2}$ (2)</td>
<td>1 mark for showing how the modulus is derived.

1 mark for how the argument is derived.</td>
<td>- Majority of the candidates did not do well in this question.
- Whilst many candidates did not attempt this question, many candidates “fudged” their result by starting from the end or showed minimal working to verify the result. Marks were penalised for lack of logical working to verify the result.</td></tr>
</table>

Equating (1) and (2) $\phi = \frac{\theta}{2}$ $\therefore \gamma = \sqrt{2R^2 - 2Rr}\ e^{\frac{i\theta}{2}}$		

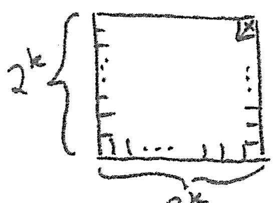
Q15

a) <u>Step 1</u> - show for $n=1$



As per the diagram, a single triomino can be placed on a $2^1 \times 2^1$ square board such that the top-right square is left uncovered.

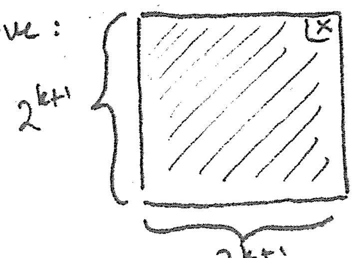
<u>Step 2</u> - assume for $n=k$



The assumption is made that on a $2^k \times 2^k$ square board, triominoes can be placed so that only the top-right square is left uncovered.

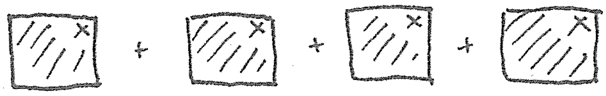
<u>Step 3</u> - show for $n=k+1$

Required to prove:

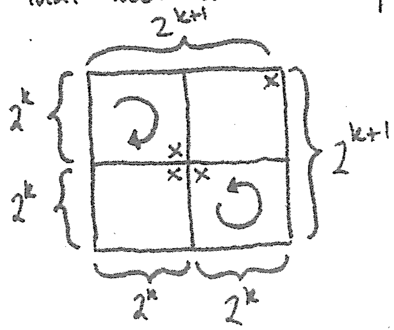


that a $2^{k+1} \times 2^{k+1}$ square board can be filled with triominoes such that only the top-right square is left uncovered.

As the inductive step, take 4 copies of a $2^k \times 2^k$ square board, each configured, as per step 2, so that only the top-right square is left uncovered.



These are placed in a 2×2 configuration, with the top-left and bottom-right boards rotated so that their uncovered square is nearest the centre of the new $2^{k+1} \times 2^{k+1}$ board



The uncovered L-shape in the centre of the $2^{k+1} \times 2^{k+1}$ square board is then covered by a single triomino, leaving only the top-right square uncovered.

$\therefore$ true for $n=k+1$

<u>Step 4</u> - Hence true by Mathematical Induction

Comments

* A difficult question, very poorly done.
① was awarded if students show for $n=1$ and that the empty square in the $n=k$ case is in the corner. A half-mark was not awarded for $n=1$ or stating the $n=k$ case without specifying the empty square correctly.
* Many students used induction to prove that $2^n \times 2^n$ leaves a remainder of 1 when divided by 3. This is a much weaker statement, and no marks were awarded for this.
* Better responses included clear diagrams.
* Poor responses did not make clear that $2^{k+1} = 2 \times 2^k$, with dimensions in the $n=k+1$ case often being 2^k+1 or 2^k+2.

b) i) $F_{NET} = m\ddot{x} = mg - kl = 0$ when in equilibrium

$$m\ddot{x} = mg - k(l+x)$$
$$= mg - kl - kx$$
$$= -kx$$
$$\therefore \ddot{x} = \frac{-k}{m}x$$

* Students must demonstrate using forces, and from the condition that $F=0$ when $x=0$.
* Poorer responses began at $m\ddot{x} = -kx$, or 'disappeared' $mg-kl$ without explanation

ii) $$x = A\cos(\omega_0 t + \alpha)$$
$$\dot{x} = -A\omega_0 \sin(\omega_0 t + \alpha)$$
$$\ddot{x} = -A\omega_0^2 \cos(\omega_0 t + \alpha)$$
$$= -\omega_0^2 x$$
Since $\ddot{x} = \frac{-k}{m}x$,
$$\omega_0^2 = \frac{k}{m}$$
$$\omega_0 = \sqrt{\frac{k}{m}}$$

* Question was well completed.
* Many students assumed the result $\ddot{x} = -n^2 x$ without penalty, but should ensure they can derive the result if necessary.

iii) Method A

$$m\ddot{x} = mg - kl - kx - b\dot{x}$$
$$= -kx - b\dot{x}$$
$$m\ddot{x} + b\dot{x} + kx = 0$$
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Method B

$$m\ddot{x} = -kx - b\dot{x}$$
$$m\ddot{x} + b\dot{x} + kx = 0$$
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

* Question was well completed.
* Students needed to derive result using forces.
* Note that $-b\dot{x}$ is downward because $\dot{x}$ is upward.

b) iv) $x = Ae^{\frac{-bt}{2m}} \cos(\omega_1 t + \phi)$

$$\dot{x} = \frac{-b}{2m} \times Ae^{\frac{-bt}{2m}} \cos(\omega_1 t + \phi) - A\omega_1 e^{\frac{-bt}{2m}} \sin(\omega_1 t + \phi)$$

$$= -Ae^{\frac{-bt}{2m}} \left[\frac{b}{2m} \cos(\omega_1 t + \phi) + \omega_1 \sin(\omega_1 t + \phi) \right]$$

$$\ddot{x} = Ae^{\frac{-bt}{2m}} \left[\cos(\omega_1 t + \phi)\left(\frac{b^2}{4m^2} - \omega_1^2\right) + \frac{b\omega_1}{m} \sin(\omega_1 t + \phi) \right] \quad \text{(Given)}$$

Substituting into $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$, then dividing by common factor $Ae^{\frac{-bt}{2m}}$,

$$\cos(\omega_1 t + \phi)\left(\frac{b^2}{4m^2} - \omega_1^2\right) + \frac{b\omega_1}{m} \sin(\omega_1 t + \phi) - \frac{b}{m}\left[\frac{b}{2m} \cos(\omega_1 t + \phi) + \omega_1 \sin(\omega_1 t + \phi)\right] + \frac{k}{m}\cos(\omega_1 t + \phi) = 0$$

$$\cos(\omega_1 t + \phi)\left[\frac{b^2}{4m^2} - \omega_1^2 - \frac{b^2}{2m^2} + \frac{k}{m}\right] + \sin(\omega_1 t + \phi)\left[\frac{b\omega_1}{m} - \frac{b\omega_1}{m}\right] = 0$$

$$\frac{-b^2}{4m^2} + \frac{k}{m} = \omega_1^2$$

$$\omega_1 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{as required.}$$

* Students who correctly found $\dot{x}$ received 1 mark.
* Students who correctly and clearly substituted their values for $x, \dot{x}$ and $\ddot{x}$ into the equation from (iii) received 1 mark.
* Students needed to collect terms and simplify to receive the last mark.
* Many students did not apply the product rule to find $\dot{x}$, and so made minimal headway answering the question
* Poorly attempted, overall.

c) i) At terminal velocity, $\frac{dv}{dt} = 0$

$$mg - kv_T^2 = 0$$

$$kv_T^2 = mg$$

$$v_T^2 = \frac{mg}{k}$$

$$v_T = \sqrt{\frac{mg}{k}}$$

* Completed well, although students should state $a=0$ as a condition of terminal velocity.

c) ii) $m\frac{dv}{dt} = mg - kv^2$

$$\frac{dv}{mg-kv^2} = \frac{dt}{m}$$

$$\frac{dv}{\left(\frac{mg}{k}\right)-v^2} = \frac{k\,dt}{m}$$

$$\frac{dv}{v_T^2 - v^2} = \frac{k}{m}dt$$

$$\frac{1}{2v_T}\left(\frac{1}{v_T-v} + \frac{1}{v_T+v}\right)dv = \frac{k}{m}dt$$

$$\frac{m}{2kv_T}\int_{v_0}^{v}\left(\frac{1}{v_T-v} + \frac{1}{v_T+v}\right)dv = \int_0^t dt$$

$$t = \frac{m}{2kv_T}\Big[\ln|v_T+v| - \ln|v_T-v|\Big]_{v_0}^{v}$$

$$= \frac{m}{2kv_T}\left[\ln\left|\frac{v_T+v}{v_T-v}\right|\right]_{v_0}^{v}$$

$$= \frac{m}{2kv_T}\left[\ln\left|\frac{v_T+v}{v_T-v}\right| - \ln\left|\frac{v_T+v_0}{v_T-v_0}\right|\right]$$

$$= \frac{m}{2kv_T}\ln\left[\frac{(v_T+v)(v_T-v_0)}{(v_T-v)(v_T+v_0)}\right]$$ as required.

* Many students were successful answering this. However, they didn't use the question as a guide to introduce v_T early, and get rid of $\sqrt{mg}$, $\sqrt{k}$ etc terms, making it more challenging.
* Students had to successfully split their integral into partial fractions to get the first mark, but many had difficulty with this.
* Students had to integrate successfully for the second mark, then re-arrange correctly to earn the third mark.
* Students cannot assume $\int\frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$, especially for a 'show' question.

b) i) $I_k = \int \cos^k x \, dx$

$u = \cos^{k-1} x \qquad v' = \cos x$

$u' = (k-1)\cos^{k-2} x(-\sin x) \qquad v = \sin x$

$$I_k = \sin x \cos^{k-1} x + (k-1)\int \cos^{k-2} x . \sin^2 x \, dx$$

$$I_k = \sin x \cos^{k-1} x + (k-1)\int \cos^{k-2} x \left(1 - \cos^2 x\right) dx$$

$$I_k = \sin x \cos^{k-1} x + (k-1)\int \cos^{k-2} x \, dx - (k-1) I_k$$

$$I_k + (k-1)I_k = \sin x \cos^{k-1} x + (k-1)\int \cos^{k-2} x \, dx$$

$$k I_k = \sin x \cos^{k-1} x + (k-1)\int \cos^{k-2} x \, dx$$

$$I_k = \frac{1}{k} \sin x \cos^{k-1} x + \frac{k-1}{k} \int \cos^{k-2} x \, dx$$

ii) $J_n = \int \frac{1}{(x^2+a^2)^n} dx$

let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta . d\theta$

$$J_n = \int \frac{a\sec^2\theta \, d\theta}{((a\tan\theta)^2 + a^2)^n}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{(a^2\sec^2\theta)^n}$$

$$= \frac{1}{a^{2n-1}} \int \sec^{2-2n}\theta \, d\theta$$

$$= \frac{1}{a^{2n-1}} \int \cos^{2n-2}\theta \, d\theta$$

$$= \frac{1}{a^{2n-1}} \int \cos^{2(n-1)}\theta \, d\theta$$

iii) Note: To use (ii) for the definite integral the limits must change when $x=0$, $\theta=0$; $x=\frac{1}{2}$, $\theta=\frac{\pi}{4}$

$$\int_0^{\frac{1}{2}} \frac{1}{(x^2+\frac{1}{4})^3}\,dx = \frac{1}{(\frac{1}{2})^5}\int_0^{\frac{\pi}{4}} \cos^4\theta\, d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} \cos^4\theta\, d\theta$$

$$= 32\left[\left[\tfrac{1}{4}\sin\theta\cos^3\theta\right]_0^{\frac{\pi}{4}} + \frac{3}{4}\int_0^{\frac{\pi}{4}}\cos^2\theta\, d\theta\right]$$

$$= \left[8\sin\theta\cos^3\theta\right]_0^{\frac{\pi}{4}} + 24\left[\left[\tfrac{1}{2}\sin\theta\cos\theta\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \overset{1}{\cancel{\cos^0\theta}}\, d\theta\right]$$

$$= \left[8\sin\theta\cos^3\theta + 12\sin\theta\cos\theta + 12\theta\right]_0^{\frac{\pi}{4}}$$

$$= 8\sin\tfrac{\pi}{4}\cos^3\tfrac{\pi}{4} + 12\sin\tfrac{\pi}{4}\cos\tfrac{\pi}{4} + 12\left(\tfrac{\pi}{4}\right) - (0)$$

$$= 8\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)^3 + 12\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) + 3\pi$$

$$= 8 + 3\pi$$

COMMENT:

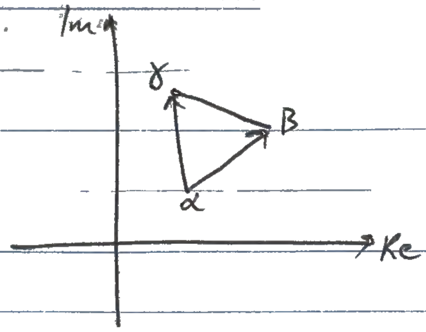
(i) This is the standard reduction formula for $\int \cos^k x\, dx$ which involves integration by parts

(ii) Given that J_n is in terms of θ and involves the cosine function should suggest a trig. substitution.

(iii) Very few students got the answer. Some students did not change the limits. Others made mistakes with the choice of a & n.

Should have been some easy marks late in the exam.

c) i)



$$\gamma - \alpha = (\beta - \alpha)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\frac{\gamma - \alpha}{\beta - \alpha} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \quad \text{—— } ①$$

Similarly, $\frac{\alpha - \beta}{\gamma - \beta} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ —— ②

sub ② into ①

$$\frac{\gamma - \alpha}{\beta - \alpha} = \frac{\alpha - \beta}{\gamma - \beta}$$

$$(\gamma - \alpha)(\gamma - \beta) = (\alpha - \beta)(\beta - \alpha)$$

$$\gamma^2 - \gamma\beta - \alpha\gamma + \cancel{\alpha\beta} = \cancel{\alpha\beta} - \alpha^2 - \beta^2 + \alpha\beta$$

$$\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \alpha\gamma + \beta\gamma) = 0$$

ii) let the roots of $z^3 + az^2 + bz + c = 0$ be z_1, z_2, z_3.

$$z_1 + z_2 + z_3 = -\frac{B}{A}$$

$$= -a$$

$$z_1z_2 + z_1z_3 + z_2z_3 = \frac{C}{A}$$

$$= b$$

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_1z_3 + z_2z_3)$$

$$z_1^2 + z_2^2 + z_3^2 - (z_1z_2 + z_1z_3 + z_2z_3) = (z_1 + z_2 + z_3)^2 - 3(z_1z_2 + z_1z_3 + z_2z_3)$$

$$= (-a)^2 - 3(b)$$

$$= a^2 - 3b$$

If $a^2 = 3b$

$$= (3b) - 3b$$

$$= 0$$

satisfying the condition in (i)
Hence, the roots of $z^3 + az^2 + bz + c = 0$ represent vertices of an equilateral triangle if $a^2 = 3b$.

iii) $z = pw + q$

$pw = z - q$

$w = \dfrac{z-q}{p}$

And so the roots of $w^3 + Aw^2 + Bw + C = 0$ are $\dfrac{z_1 - q}{p}, \dfrac{z_2 - q}{p}, \dfrac{z_3 - q}{p}$.

Consider $\left(\dfrac{z_1-q}{p}\right)^2 + \left(\dfrac{z_2-q}{p}\right)^2 + \left(\dfrac{z_3-q}{p}\right)^2 - \left(\left(\dfrac{z_1-q}{p}\right)\left(\dfrac{z_2-q}{p}\right) + \left(\dfrac{z_1-q}{p}\right)\left(\dfrac{z_3-q}{p}\right) + \left(\dfrac{z_2-q}{p}\right)\left(\dfrac{z_3-q}{p}\right)\right)$

$$= \frac{z_1^2 - \cancel{2z_1q} + \cancel{q^2}}{p^2} + \frac{z_2^2 - \cancel{2z_2q} + \cancel{q^2}}{p^2} + \frac{z_3^2 - \cancel{2z_3q} + \cancel{q^2}}{p^2} - \left(\frac{z_1z_2 - \cancel{z_1q} - \cancel{z_2q} + \cancel{q^2}}{p^2} + \frac{z_1z_3 - \cancel{z_1q} - \cancel{z_3q} + \cancel{q^2}}{p^2} + \frac{z_2z_3 - \cancel{z_2q} - \cancel{z_3q} + \cancel{q^2}}{p^2}\right)$$

$$= \frac{z_1^2 + z_2^2 + z_3^2 - (z_1z_2 + z_1z_3 + z_2z_3)}{p^2}$$

$$= \frac{0}{p^2} \quad \text{from part (ii)}$$

$$= 0$$

satisfying the condition in (i)
And so the roots of $w^3 + Aw^2 + Bw + C = 0$ also represent vertices of an equilateral triangle.

COMMENT:

(i) It is important to remember that α, β, γ are the vertices.
The sides will be represented by the vectors $\beta-\alpha$, $\gamma-\beta$, $\alpha-\gamma$.
For the triangle to be equilateral the modulus of these vectors will be equal with the angle between them $60°$.
Hence, $\gamma-\alpha = (\beta-\alpha)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

This could be shown in other ways but it is important to maintain generality.

(ii) Students needed to link the roots of the equation with the result from part (i).
However, it is important that (i) is satisfied if $a^2=3b$.
Not if (i) is satisfied then $a^2=3b$.

(iii) This could be shown in other ways. However, it is setup to link with the previous parts.

If this question was earlier in the exam more students would likely find success.