Sydney Girls High School

2006

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 2

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2006 HSC Examination Paper in this subject.

General Instructions

- Reading Time - 5 mins
- Working time - 3 hours
- Attempt **ALL** questions
- **ALL** questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Candidate Number

Question 1

a) If z is the complex number $-1+i$, indicate on the Argand Diagram, the points

$i) z,\ ii) \overline{z},\ iii) iz,\ iv) \frac{1}{z},\ v) z^2$ [5]

b) If $z=\cos\theta+i\sin\theta$ and $w=\frac{dz}{d\theta}$, show that $z+iw=0$. [2]

c) Describe Geometrically the locus in the Argand Diagram represented by z if $z\overline{z}-6(z+\overline{z})=45$. [2]

d) Consider the equation $z^4+1=0$

i) Find the four complex roots expressing them in the form $a+ib$

ii) If the roots are plotted on an Argand Diagram, find the area of the figure those roots form. [3]

e) The interval AB where A is (2,3) and B(4,5) is rotated 60^0 about A in an anticlockwise direction to AB′. Find the co-ordinates of the point B′ [3]

Question 2.

a) Find the following definite integrals:

i) $\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$

ii) $\int_0^{\frac{1}{2}} \sin^{-1} x.dx$ [9]

iii) $\int_0^1 \frac{dx}{(x+1(x+2)^2}$

b) Find $\int \frac{\sqrt{x^2-4}}{x^2}.dx$ [3]

c) Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x.dx$, where n is an integer.

i) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ [3]

ii) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^7 x.dx$

Question 3:

a) If α, β, γ are the roots of the polynomial $3x^3 - 6x^2 - x + 1 = 0$

i) Find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

ii) Hence or otherwise find the value of $(\alpha + \beta - \gamma)(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)$

[6]

b) The polynomial $ax^n + bx^{n-1} - 1$ where n is an even positive integer is divisible by $(x+1)^2$.
Show that $a = 1 - n$ and $b = -n$

[5]

c) Show that the product of 4 consecutive numbers is always one less than a perfect square.

[4]

Question 4:

a) Sketch the following curves showing all the important features of those curves

i) $y = (3+x)(1+x)^3(1-x)(3-x)^2$

ii) $y = \dfrac{x^2(x-3)}{(x-2)^2}$

iii) $4y^2 = x^2 - 4x$ [9]

b) Consider the function $f(x) = 4 - x^2,\ -2 \le x \le 2$ and sketch the following curves

i) $y = \sqrt{f(x)}$

ii) $y = \log_e \{f(x)\}$

iii) $y = 2^{f(x)}$ [6]

Question 5:

a) Consider the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

i) Determine the eccentricity [1]

ii) Find the foci and equation of the directrices [1]

iii) Sketch the ellipse [1]

iv) Find the equation of the tangent at the point $P(3\cos\theta, 5\sin\theta)$ [3]

v) Determine the equation of the tangent at $P(3\cos\theta, 5\sin\theta)$ in terms of the gradient m.

[2]

vi) Write down the equations of the tangents to $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ with gradient 2.

[1]

b) i) Find the equation of the tangent to $\dfrac{x^2}{8} - \dfrac{y^2}{4} = 1$ at the point (x_1, y_1)

[2]

ii) Hence write down the equation of the chord of contact to $\dfrac{x^2}{8} - \dfrac{y^2}{4} = 1$ from the point (x_0, y_0)

[1]

iii) If the equation of the chord of contact to $\dfrac{x^2}{8} - \dfrac{y^2}{4} = 1$ from the point T (x_0, y_0) is
$2x - 3y - 5 = 0$, find the co-ordinates of point T.

[3]

Question 6:

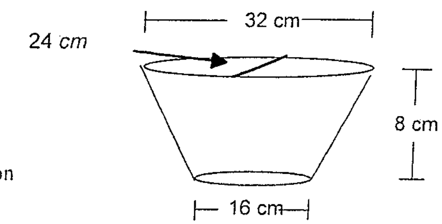
a) i) Find the exact value of $\tan 75^0$ [1]

ii) Last month in the USA a trainee pilot stalled the engine of his plane and it nosedived, hitting the ground at an angle of 75^0 (being lucky enough to have that fall broken by a tree). If he was traveling in a horizontal direction at 108 km/hr when he stalled and the future motion of the plane was that of a projectile, how high was the plane when it stalled?
[3]

b) The area enclosed by a triangle with co-ordinates A(1,1), B(2,2) and C(3,1) is rotated about the Y axis. Use the method of cylindrical shells to find the volume that is formed.

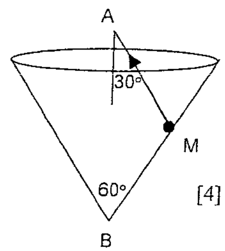
[5]

c) Maisie's birthday cake is in the shape of an ellipse with axes of 32 cm and 24 cm at one end and a circle of radius 8 cm at the other end. If the cake is 8 cm high, find the volume of the cake if each cross section taken perpendicular to the height is an ellipse



Question 7:

a) A mass M of 1 kg is attached to a string of length 4 metres at a point A directly above the vertex of a cone, with semi vertical angle of 30^0. The string also makes an angle of 30^0 with the vertical.



i) If the mass M rotates at 2 rad/sec, find the tension in the string and the normal force exerted by the side of the cone on the mass [4]

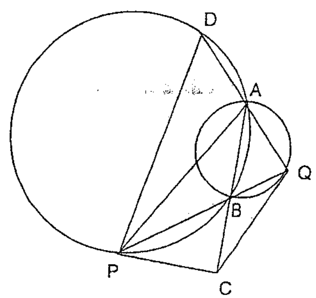
ii) How fast (in rad/sec) should the mass be rotated in the tension (T) and the normal force (F) are to be equal in magnitude. [3]

b) Two circles intersect at A and B AB is produced to a point C, such that when tangents CP and CQ are drawn, PBQ is a straight line.

i) Show that CP = CQ

ii) Show that APCQ is a cyclic quadrilateral

iii) If QA is produced to meet the larger circle, at D, show that PB bisects $\angle CPD$.



Marks: i) [2], ii) [3], iii) [3]

Question 8:

a) If $xe^{-x} = k$ has two solutions, find the range of values of k

{4}

b) If the polynomial $x^4 + qx^2 + rx + s = 0$ has roots $\alpha, \beta, \gamma, \delta$ show that the value of the constant term in the polynomial with roots $1-\alpha^2, 1-\beta^2, 1-\gamma^2, 1-\delta^2$ is $(q+s+1)^2 - r^2$

[6]

c) Evaluate $\int_0^4 \sqrt{2-\sqrt{x}}\,dx$

[5]

....end of paper..

1 a) $z = -1+i$

$\bar{z} = -1-i$

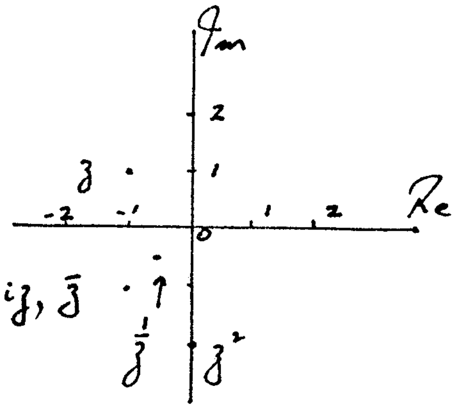
$iz = -i-1 = -1-i$

$\frac{1}{z} = \frac{1}{-1+i} \cdot \frac{-1-i}{-1-i}$

$= \frac{-1-i}{2}$

$z^2 = 1-1-2i$

$= -2i$



b) $z = \cos\theta + i\sin\theta$

$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta = w$

$iw = -i\sin\theta - \cos\theta$

$\therefore z + iw = 0.$

c) $z\bar{z} - 6(z+\bar{z}) = 45$

Let $z = x+iy$

$\therefore x^2+y^2-12x = 45$

$(x-6)^2+y^2 = 81$

circle, centre $(6,0)$, radius 9

d) $z^4+1=0$

$z^4 = -1$

Let $z = \cos\theta + i\sin\theta$

$\therefore (\cos\theta + i\sin\theta)^4 = -1$

$\therefore \cos 4\theta + i\sin 4\theta = -1$ (de Moivre's thm)

$\cos 4\theta = -1$

$4\theta = \pi, 3\pi, 5\pi, 7\pi$

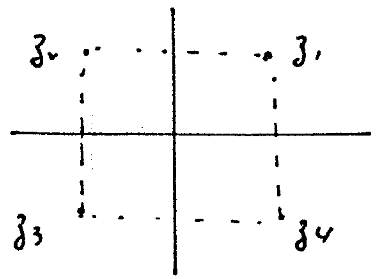
$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

$\therefore z_1 = \cos \pi/4 + i\sin \pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$z_2 = \cos 3\pi/4 + i\sin 3\pi/4 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$z_3 = \cos 5\pi/4 + i\sin 5\pi/4 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

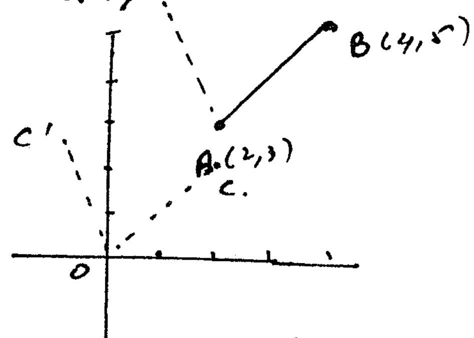
$z_4 = \cos 7\pi/4 + i\sin 7\pi/4 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$



$A = r^2$

$= \left(\frac{2}{\sqrt{2}}\right)^2$

Area $= 2$ sq units



Translate AB to OC

- moving $(-2,-3)$ places

Consider the interval OC where O is $(0,0)$ & C is $(2,2)$

Rotate C by 60° to C'

$C' = (2+2i)(\cos 60 + i\sin 60)$

$= (2+2i)(\frac{1}{2} + i\sqrt{3}/2)$

$= 1 + i\sqrt{3} + i - \sqrt{3}$

$= (1-\sqrt{3}) + i(1+\sqrt{3})$

$\therefore B'$ is $(1-\sqrt{3}+2), i(1+\sqrt{3}+3)$

$= \{3-\sqrt{3}, i(4+\sqrt{3})\}$

Q2. a)

i) $\int_0^{1/3} \frac{dx}{1+9x^2} = \frac{1}{3}\tan^{-1}(3x) \Big]_0^{1/3}$

$= \frac{1}{3}\tan^{-1}(1) - 0$

$= \pi/12$

ii) $\int_0^{1/2} \sin^{-1}x \, dx.$

Let $u = \sin^{-1}x$ $\quad dv = dx$

$du = \frac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$\therefore I = [x\sin^{-1}x]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx$

$= \frac{1}{2}\sin^{-1}(\frac{1}{2}) + [\sqrt{1-x^2}]_0^{1/2}$

$= \frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{3/4} - 1$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

3(i) Let the consecutive numbers be

$x-1,\ x,\ x+1,\ x+2$

& show $(x-1)x(x+1)(x+2)+1 = K^2$

$$LHS = (x^3-x)(x+2)+1$$
$$= x^4+2x^3-x^2-2x+1$$

Now $(x^2+x-1)^2 = x^4+x^2+1$
$$+2x^3-2x^2-2x$$
$$= x^4+2x^3-x^2-2x+1$$

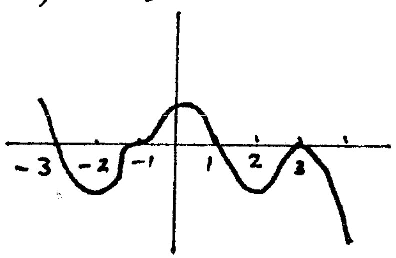
$\therefore$ $(x-1)x(x+1)(x+2)+1$
$$= (x^2+x-1)^2$$

OR $(x-1)x(x+1)(x+2) = K^2-1$

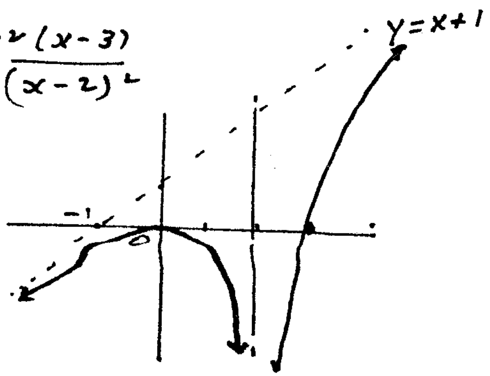
where $K = x^2+x-1$.

Q4. a)

i) $y = (3+x)(1+x)^3(1-x)(3-x)^2$



ii) $y = \dfrac{x^2(x-3)}{(x-2)^2}$



$$\begin{array}{r} x+1 \\ x^2-4x+4\ \overline{)\,x^3-3x^2+0x+0} \\ \underline{x^3-4x^2+4x} \\ x^2-4x+0 \\ \underline{x^2-4x+4} \\ -4 \end{array}$$

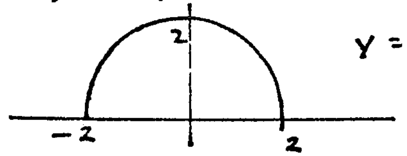
$$y = x+1 - \frac{4}{(x-2)^2}$$

$\therefore$ Curve is always below $y=x+1$.

iii) → see end.

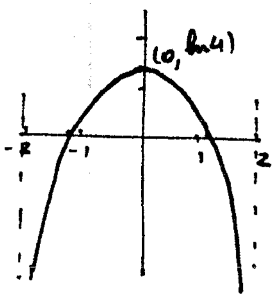
Q4 b) $f(x) = 4-x^2$

i)



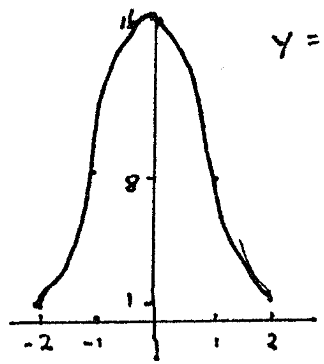
$y = \sqrt{4-x^2}$

ii)



$y = \ln(4-x^2)$

iii)



$y = 2^{4-x^2}$

Q5 a) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

i) $b^2 = a^2(1-e^2)$

$\cancel{25} = \cancel{9}$

$9 = 25(1-e^2)$

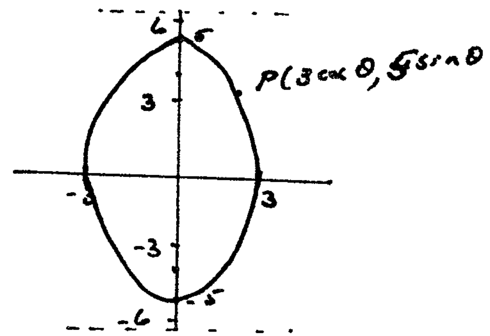
$\dfrac{9}{25} = 1-e^2$, $e^2 = 16/25$

$e = 4/5$

ii) foci $(0, \pm 4)$

directrices. $y = \pm 25/4$

iii)



iv) $\dfrac{2x}{9} + \dfrac{2y}{25} \cdot \dfrac{dy}{dx} = 0$

$\therefore$ $\dfrac{dy}{dx} = -\dfrac{x}{9} \cdot \dfrac{25}{y}$

at $(3\cos\theta, 4\sin\theta)$

$$m = \frac{-\cancel{25}\,5 \times \cancel{3}\cos\theta}{3\,\cancel{9} \times \cancel{5}\sin\theta}$$

$$= \frac{-5\cos\theta}{3\sin\theta}$$

$\therefore$ T is $y - 5\sin\theta = \dfrac{-5\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

$3y\sin\theta - 15\sin^2\theta = -5x\cos\theta + 15\cos^2\theta$

$\therefore$ $5x\cos\theta + 3y\sin\theta = 15$

OR $\dfrac{x\cos\theta}{3} + \dfrac{y\sin\theta}{5} = 1$

v) $5x\cos\theta + 3y\sin\theta = 15$

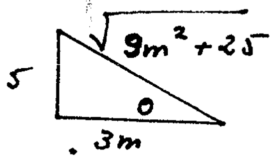
$3y\sin\theta = -5x\cos\theta + 15$

$y = -\frac{5\cos\theta}{3\sin\theta}\cdot x + \frac{15}{3\sin\theta}$

$= -\frac{5x}{3}\cot\theta + 5\csc\theta.$

Let $m = \frac{-5\cot\theta}{3}$

$\therefore \frac{3m}{5} = \cot\theta$



$\therefore y = mx \pm \frac{5\cdot\sqrt{9m^2+25}}{5}$

$= mx \pm \sqrt{9m^2+25}$

vi) If $m = 2$

$y = 2x \pm \sqrt{61}$

b i) $\frac{x^2}{8} - \frac{y^2}{4} = 1$ $[x^2 - 2y^2 = 8]$

$\frac{2x}{8} - \frac{2y}{4}\cdot\frac{dy}{dx} = 0$

$\therefore -\frac{y}{4}\frac{dy}{dx} = -\frac{x}{8}$

$\therefore \frac{dy}{dx} = \frac{x}{2y}$

at (x_1, y_1) $\frac{dy}{dx} = \frac{x_1}{2y_1}$

$\therefore T$ is

$y - y_1 = \frac{x_1}{2y_1}(x - x_1)$

$\therefore 2y_1y - 2y_1^2 = x_1x - x_1^2$

$\therefore x_1^2 - 2y_1^2 = x_1x - 2y_1y$

$\therefore 8 = x_1x - 2y_1y$

ii) C of C from (x_0, y_0) is

$x_0x - 2y_0y = 8$

Now $2x - 3y = 5$

$\therefore \frac{8}{5}\cdot 2x - \frac{8}{5}\cdot 3y = \frac{8}{5}\cdot 5$

$\therefore \frac{16x}{5} - \frac{24y}{5} = 8$

$\therefore x_0 = \frac{16}{5}$, $y_0 = -\frac{24}{5}$

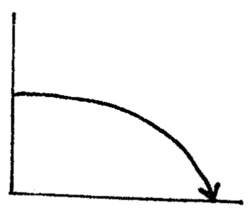
6 a) i) $\tan 75 = \tan(45+30)$

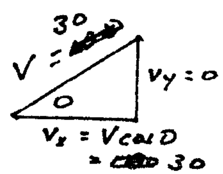
$= \frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{3+2\sqrt{3}+1}{2}$

$= 2+\sqrt{3}.$





$\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = c_1$ $\quad \dot{y} = c_2 - 10t$

at $t = 0$, $\dot{x} = 30 = c_1$, $\dot{y} = 0 = c_2$

$\therefore \dot{x} = 30$ $\quad \dot{y} = -10t$

$x = 30t + c_3$, $\quad y = c_4 - 5t^2$

at $t = 0$, $x = 0 = c_3$, $y = H = c_4$

$\therefore x = 30t$, $\quad y = H - 5t^2$

At any time,

$\tan\theta = \frac{\dot{y}}{\dot{x}}$

$\therefore \tan 75 = \left|\frac{-10t}{30}\right|$

$\therefore 2+\sqrt{3} = t/3$

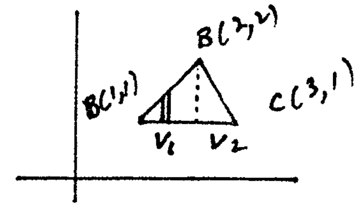
$t = 3(2+\sqrt{3})$ secs

Hits ground at $y = 0$

$\therefore H = 5\times 3(2+\sqrt{3})$ m

$= 15(2+\sqrt{3})$ m.

b)



Line AB is $y = x$

Line BC is $y = 4 - x$

$V_{shell} = \pi R^2H - \pi r^2h.$

V_1: $R = x + dx$, $r = x$, $H = y - 1 = h$

$\therefore V_{shell} = \pi(y-1)\{(x+dx)^2 - x^2\}$

$= \pi(x-1)(x^2 + 2x\,dx + dx^2 - x^2)$

$= \pi(x-1)(2x\,dx)$, $dx^2 \doteqdot 0$

$V_{solid} = \pi\sum_{x=1}^{2}(x-1)2x\cdot dx$

$= 2\pi\int_1^2 (x^2 - x)\,dx$

$= 2\pi\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$

$= 2\pi[\frac{8}{3} - 2] - 2\pi[\frac{1}{3} - \frac{1}{2}]$

8c) $\int_0^4 \sqrt{2-\sqrt{x}}\ dx.$

Let $x = 4\sin^4\theta$

$\therefore dx = 16\sin^3\theta.\cos\theta.\ d\theta.$

at $x = 0,\ \theta = 0$

$x = 4,\ \theta = \pi/2$

$\therefore I = \int_0^{\pi/2} \sqrt{2 - 2\sin^2\theta}.\ 16\sin^3\theta\cos\theta.\ d\theta$

$= \int_0^{\pi/2} \sqrt{2}\sqrt{1-\sin^2\theta}\ .16\sin^3\theta.\cos\theta.\ d\theta$

$= \sqrt{2}\int_0^{\pi/2} \cos\theta.\ 16\sin^3\theta.\cos\theta.d\theta$

$= 16\sqrt{2}\int_0^{\pi/2} \sin^3\theta\cos^2\theta.\ d\theta$

$= 16\sqrt{2}\int_0^{\pi/2} \sin\theta.\cos^2\theta.(1-\cos^2\theta)d\theta$

$= 16\sqrt{2}\int_0^{\pi/2} \cos^2\theta\sin\theta d\theta$

$- 16\sqrt{2}\int_0^{\pi/2} \cos^4\theta\sin\theta.\ d\theta$

$= 16\sqrt{2}\left\{\left[\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right]_0^{\pi/2}\right\}$

$= 16\sqrt{2}\left\{\ 0 - (\frac{1}{3} - \frac{1}{5}\right\}$

$= 16\sqrt{2}\ .\ \frac{2}{15} = \frac{32\sqrt{2}}{15}$

4aiii) $4y^2 = x^2 - 4x.$

$x^2 - 4x - 4y^2 = 0$

$x^2 - 4x + 4 - 4y^2 = \cancel{0}\ 4$

$(x-2)^2 - 4y^2 = 4$

$\therefore \frac{(x-2)^2}{4} - \frac{y}{1} = 1$

Hyperbola, centre at (2,0)

as $y \to \infty$ $\quad 4y^2 \to (x-2)^2 - 4$

$4y^2 \to (x-2)^2$

$2y \to \pm(x-2)$

$y \to \pm\frac{x-2}{2}$

$\therefore$ Asymptotes are

$y = \frac{x}{2} - 1,\ \ -\frac{1}{2}x + 1$