SYDNEY GIRLS HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics

Extension 2 2012

General Instructions

- Reading Time- 5 minutes
- Working Time - 3 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A Standard Integrals Sheet is provided at the back of this paper which may be detached and used throughout the paper.

Name:.................................

Teacher:................................

This is a trial paper ONLY. It does not necessarily reflect the format or the content of the 2012 HSC Examination in this subject.

Total Marks 100

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II

90 marks

- Attempt questions 11 – 16
- Answer on the blank paper provided, . Start a new sheet for each question.
- Allow about 2 hours & 45 minutes for this section

Section I – Multiple Choice (10 marks)

1. Realising the denominator of $\dfrac{12-6i}{4+3i}$ gives:

   a) $1.2+2.4i$

   b) $\dfrac{30}{7}+\dfrac{60}{7}i$

   c) $\dfrac{30}{7}-\dfrac{60}{7}i$

   d) $1.2-2.4i$

2. The polynomial $P(x)=x^5-6x^4+13x^3-14x^2+12x-8$ has a root at $x=2$ of multiplicity 3 and $x=-i$ is also a root. Which of the following is a factorised form of $P(x)$ over the complex field?

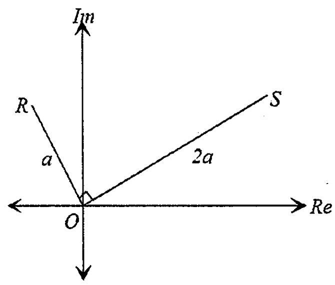
   a) $P(x)=(x-2)^3(x+i)$

   b) $P(x)=(x-2)^3(x+i)(x-i)$

   c) $P(x)=(x+2)^3(x^2+1)$

   d) $P(x)=(x+2)^3(x+i)(x-i)$

3. In the Argand diagram below the points R and S represent the complex numbers w and z, respectively where $\angle SOR=90^\circ$. The distance OS is $2a$ units, and distance OR is a units. Which of the following is correct?



   a) $w=2iz$

   b) $w=i\overline{w}$

   c) $w=-\dfrac{iz}{2}$

   d) $w=-\dfrac{z}{2i}$

4. Find $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}dx$:

a) $y = \sin\left(\sqrt{x}\right) + c$

b) $y = 2\sin\left(\sqrt{x}\right) + c$

c) $y = \frac{1}{\sin\left(\sqrt{x}\right)} + c$

d) $y = \frac{2}{\sin\left(\sqrt{x}\right)} + c$

5. Using the method of integration by parts $\int x^2 \log_e(3x).dx$ is equal to:

a) $\frac{x^3}{9}(3\log_e 3x - 1) + c$

b) $\frac{x^3}{9}(\log_e 3x - 1) + c$

c) $\frac{x^3}{9}(\log_e 3x + 1) + c$

d) $\frac{x^3}{9}(-\log_e 3x + 1) + c$

6. The equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point $\left(cp, \frac{c}{p}\right)$ is:

a) $x + p^2 y = 2c^2$

b) $x + p^2 y = 2cp$

c) $x - p^2 y = 2c^2$

d) $x - p^2 y = 2cp$

7. Given the hyperbola $\frac{x^2}{144}-\frac{y^2}{25}=1$ then:

a) eccentricity $e=\frac{13}{12}$ is and foci are at $\left(\pm\frac{144}{13},0\right)$

b) eccentricity is $e=\frac{13}{5}$ and foci are at $(\pm 13,0)$

c) eccentricity is $e=\frac{13}{12}$ and foci are at $(\pm 13,0)$

d) eccentricity is $e=\frac{13}{5}$ and foci are at $\left(\pm\frac{144}{13},0\right)$

8. The solution to $\frac{x(x-5)}{4-x}<-3$ is:

a) $x<0,\ 4<x<5$

b) $x>5,\ 0<x<4$

c) $x<2,\ 4<x<6$

d) $x>6,\ 2<x<4$

9. The polynomial equation $x^3-2x^2+1=0$ has roots α, β and γ.
Which one of the following equations has roots $2\alpha+\beta+\gamma,\ \alpha+2\beta+\gamma,\ \alpha+\beta+2\gamma$?

a) $x^3-8x^2+20x-15$

b) $x^3+8x^2+20x+15$

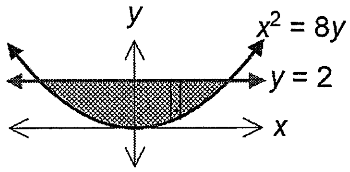
c) x^3-4x^2+4x-1

d) x^3+4x^2+4x+1

10. The volume of the solid generated when the area bounded by $y = 2$ and $x^2 = 8y$ is rotated about the line $y = 2$ using the method of slicing (and taking slices perpendicular to the X-axis) is given by:

a) $V = \pi\int_{-4}^{4}\left(4-\frac{x^4}{64}\right)dx$

b) $V = \pi\int_{-2}^{2}\left(4-\frac{x^4}{64}\right)dx$



c) $V = \pi\int_{-4}^{4}\left(2-\frac{x^2}{8}\right)^2 dx$

d) $V = \pi\int_{-2}^{2}\left(2-\frac{x^2}{8}\right)^2 dx$

Question 11 (15 marks) **Marks**

a) Find $\int \frac{\cos\theta}{\sin^4\theta}d\theta$ [2]

b) Find real numbers A and B such that:

i) $\frac{3x^2+3x-2}{(x-1)(x+1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{1}{(x+1)^2}$ [2]

ii) Hence find $\int \frac{3x^2+3x-2}{(x-1)(x+1)^2}dx$ [2]

c) Find $\int \frac{dx}{\sqrt{2-4x-2x^2}}$ [3]

d) i) Simplify i^{2013} [1]

ii) Sketch the locus of $\arg(z-1)=\frac{\pi}{4}$ [1]

e) Sketch the region in the complex number plane where the inequalities $|z-\overline{z}|\le 1$ and $|z-1|\le 2$ hold simultaneously. [2]

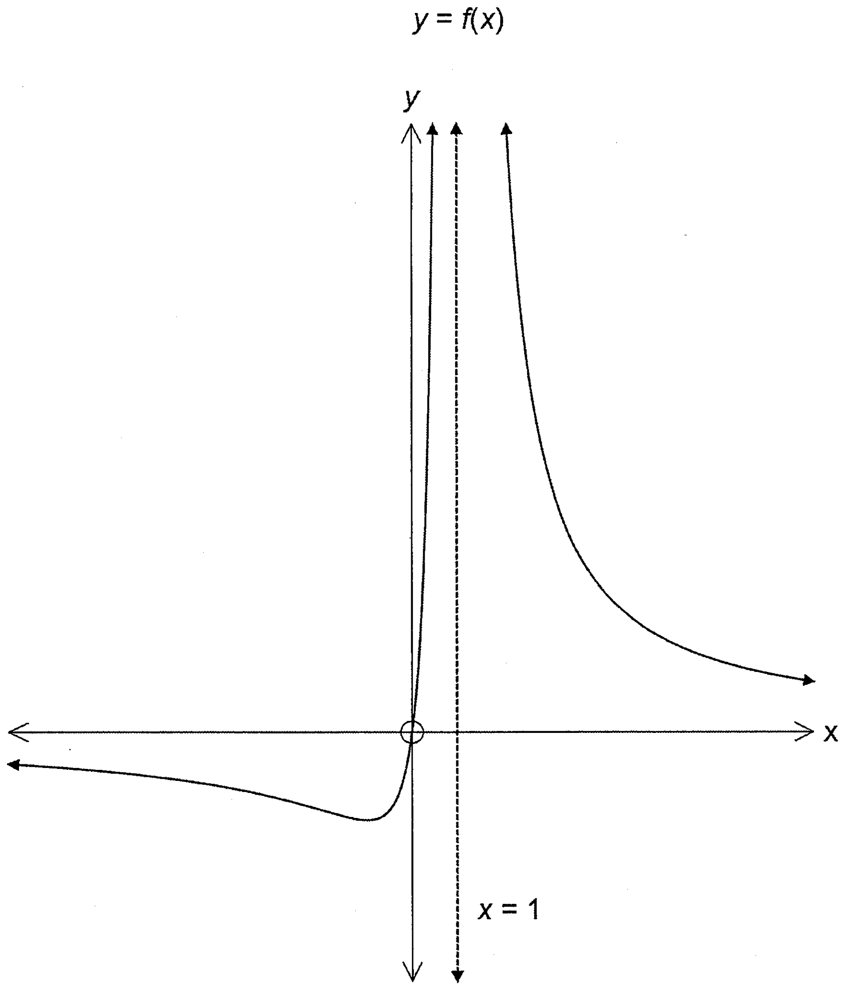
f) Factorise x^4-3x^2-10 over:

i) The rational field [1]

ii) The real field [1]

Question 12 (15 marks) **Marks**

a) The graph of $y = f(x)$ is shown below. The graph has two branches and is asymptotic to the line $x = 1$ and the X- axis.



Sketch the graphs of:

i) $y = f(-x)$ [1]

ii) $f(x+1)$ [1]

ii) $y = f|x|$ [1]

iii) $y = \dfrac{1}{f(x)}$ [2]

Question 12 continues on the next page

Question 12 continued

b) Find the square roots of $1+i\sqrt{3}$ [2]

c) i) Express $z=\frac{\sqrt{3}}{2}+\frac{i}{2}$ in the form $r(\cos\theta+i\sin\theta)$ [1]

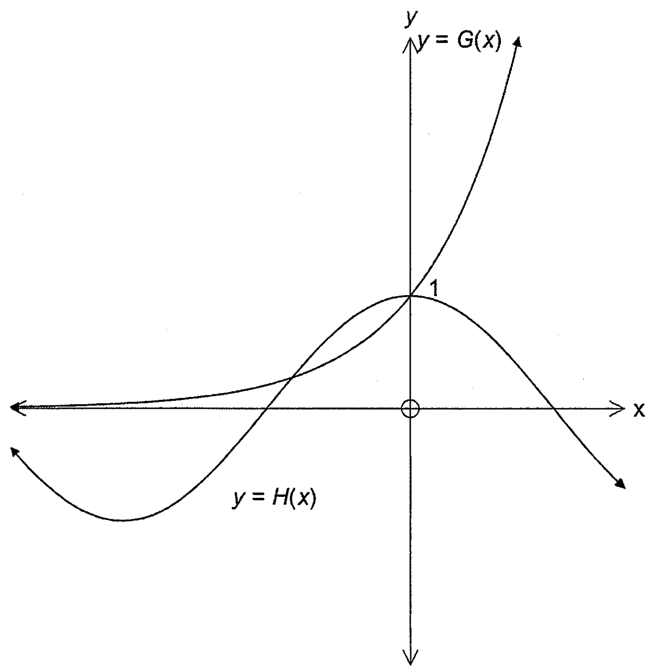
ii) Hence or otherwise find z^{15} in the form $x+iy$ [2]

d) Given that the polynomial $ax^3+bx^2+cx+d=0$ has roots α,β and γ, find the polynomial equation with roots α^2,β^2 and γ^2. [2]

e) Prove by Mathematical Induction that $a^{2n}-b^{2n}$ is divisible by $(a-b)$ for $n\geq 1$ [3]

Question 13 (15 marks) **Marks**

a) The graphs of $y = G(x)$ and $y = H(x)$ are shown below. Note that the graphs intersect at two points one of which is (0, 1).



i) Sketch the graph of $y = G(x) \times H(x)$ [1]

ii) Sketch the graph of $y = \dfrac{G(x)}{H(x)}$ [2]

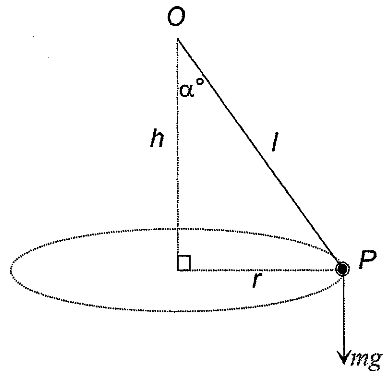
b) The base of a solid is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. Find the volume of the solid if every cross section perpendicular to the X axis is a right isosceles triangle with the hypotenuse on the base of the solid. [4]

c) Find the range of values of k such that $x^3 - 3x^2 - 9x + k = 0$ has:
i) one real solution [2]
ii) three distinct solutions [1]

Question Thirteen continues on the next page

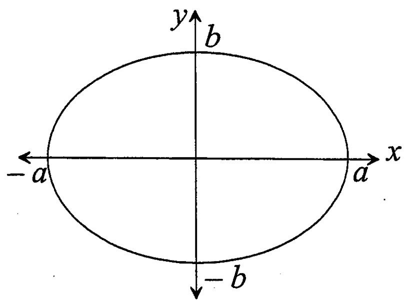
Question Thirteen continued

d) A particle P of mass m is attached by a light, inextensible rod of length l metres to a fixed point O. The particle is made to revolve in a horizontal circle of radius r metres, h metres below O. The angle between the rod and the vertical is $\alpha°$. The forces acting on the particle are its weight mg and the tension in the string T. The particle is moving with constant angular velocity.



By resolving forces horizontally and vertically show that the period of motion is given by $2\pi\sqrt{\frac{h}{g}}$ [3]

e) The ellipse below has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



i) Show that the area of the ellipse is given by $A=\frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$ [1]

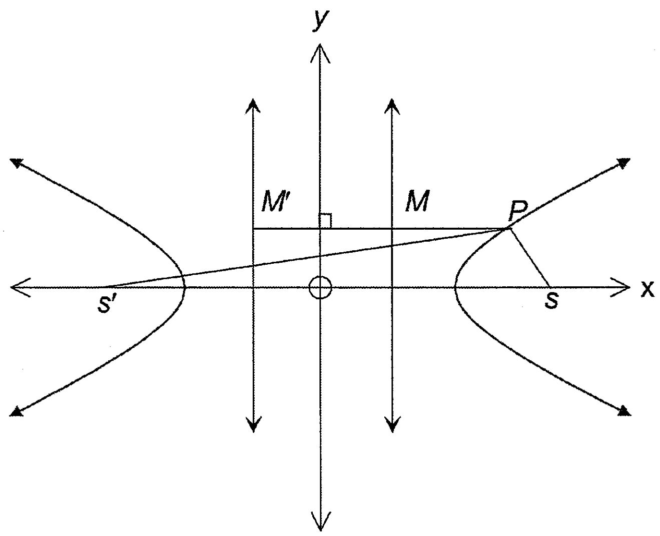
ii) Hence show that $A=\pi ab$ [1]

Question Fourteen (15 marks)

a) The parametric equation of a curve is given by $x = \sin\theta,\ y = \cos 2\theta$. [1]
Find the Cartesian equation of the curve.

b) Use a binomial expansion and De Moivre's Theorem to <u>show</u> that [3]
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

c) The hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ is shown below along with its foci and directrixes.
P is any point on the hyperbola.



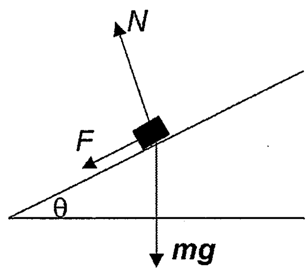
i)	Find the eccentricity e of the hyperbola	[1]
ii)	Find the equations of the directrices	[1]
iii)	Show that $\lvert PS - PS' \rvert = c$ where c is a constant	[1]
iv)	Find the value of c	[1]

d) Find the equation of the tangent to the curve $x^3 + 2y^2 = 1$ at the point [2]
with coordinates $(-1, 1)$

Question Fourteen continues on the next page

Question 14 continued

e) The corner of a speedway track for motorcycles is an arc of a circle radius r metres. The corner is banked at angle θ to the horizontal. The motorcycles travel around the corner at a constant speed v. The motorcycle has mass m, the forces acting on the motorcycle are the gravitational force mg, a sideways frictional force F and a normal reaction N from the track.



i) By resolving the horizontal and vertical components of force, find expressions for $F\cos\theta$ and $F\sin\theta$ [2]

ii) Show that $F = \dfrac{mv^2}{r}\cos\theta - mg\sin\theta$ [2]

iii) The radius of the track is 80 metres and the track is banked at an angle such that there is no tendency for the motorcycles to slip sideways when cornering at 100km/h. Find θ to the nearest degree. Use $g = -10ms^{-2}$ [1]

Question Fifteen (15 marks)

a) Find the equation of the chord of contact to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ [1]

from the point with coordinates $(4,-3)$

b) i) The polynomial $P(x)$ has a double root at $x=\alpha$. Show that $P'(x)$ [1]
also has a root at $x=\alpha$

ii) The polynomial $Q(x)=x^4-ax^2+bx+12$ has a double root at $x=2$ [2]
Find the values of a and b

c) A particle moving along the X-axis starts at rest from the origin and has acceleration given by $\ddot{x}=8-kx$ (where k is a constant).
When the particle passes through $x=12$ its acceleration is $4ms^{-2}$.

i) Find its speed when $x = 12$ [2]

ii) Find the maximum speed and where it occurs [2]

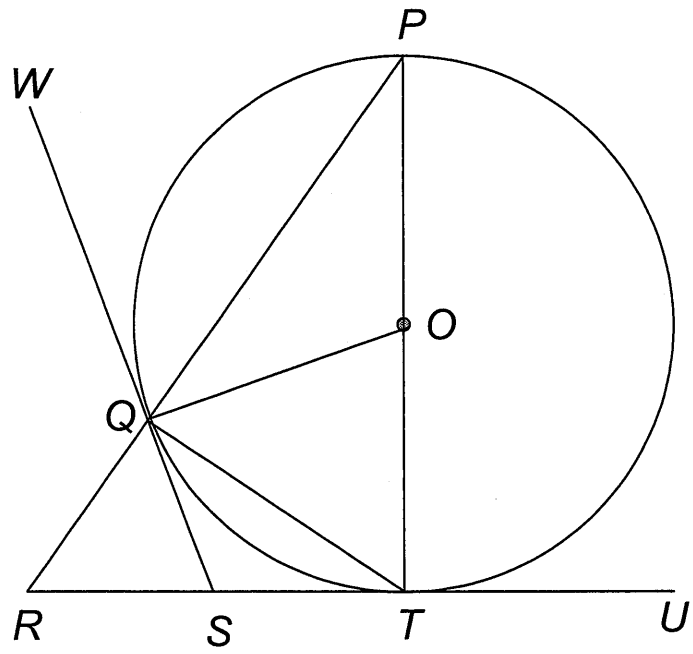
d) Given $I_n=\int_0^1 \frac{dx}{(1+x^2)^n}$ for $n\geq 1$ show that [3]

$$I_n=\frac{1}{2^n}+2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}dx$$

Question Fifteen continues on the next page

Question 15 continued

e) In the diagram below PT is a diameter , RU is a tangent at T, WS is a tangent at Q



i) Prove $\angle QSR = \angle QOT$ [2]

ii) Prove that S is the centre of a circle that passes through R, Q and T [2]

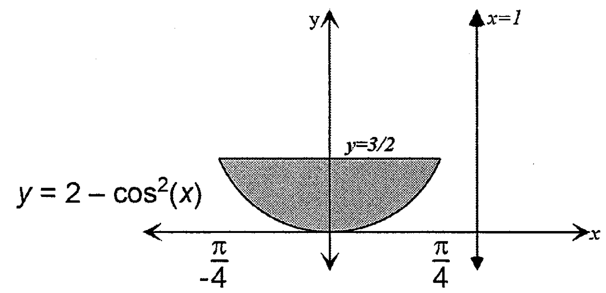
Question Sixteen (15 marks)

a) i) Show that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$ [1]

ii) Hence or otherwise solve: [3]

$\cos 5\theta+\cos\theta=\cos 3\theta$ for $0\le\theta\le 2\pi$

b) A solid is formed by rotating the area bounded by the curve $y=2-\cos^2(x)$, and the line $y=\frac{3}{2}$ around the line $x=1$. The coordinates of the points of intersection of $y=2-\cos^2 x$ and $y=\frac{3}{2}$ are $\left(\pm\frac{\pi}{4},\frac{3}{2}\right)$.



i) Use the method of cylindrical shells to show the volume of the resulting solid is given by $V=\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2x)(1-x)\,dx$ [2]

ii) Hence find the exact volume of the solid [2]

Question Sixteen continues on the next page

Question 16 continued

c) Given $z = r(\cos\theta + i\sin\theta)$ prove $\dfrac{z}{z^2 + r^2}$ is real. [3]

d) i) A projectile is fired from a point O with initial velocity $16ms^{-1}$ and angle of inclination θ. Taking $g = -10ms^{-2}$ show that after t seconds the displacement equations are given by: [1]

$$x = 16t\cos\theta$$

$$y = 16t\sin\theta - 5t^2$$

ii) T seconds later a second particle is fired from the same point with the same velocity but with a different angle of inclination. The two particles collide at a point 6 metres horizontally from O and 10 metres vertically above O. Find the value of T. [3]

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Soln's SGHS Ext 2 Trial 2012

1. $\frac{12-6i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{48+18i^2-60i}{25}$

$= \frac{30-60i}{25}$

$= 1.2 - 2.4i$ d)

2. $(x-2)^3(x+i)(x-i)$ b)

3. $w = \frac{-z}{2i} \times \frac{i}{i} = \frac{-iz}{2i^2}$

$= \frac{iz}{2}$ d)

4. let $u = x^{\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}} \cdot dx$

$I = 2\int \cos u \, du$

$= 2\sin u$

$= 2\sin\sqrt{x} + C$ b)

5. let $u = \log_e 3x)$ $\quad v' = x^2$

$u' = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\log_e 3x - \int\left(\frac{x^3}{3} \times \frac{1}{x}\right)dx$

$= \frac{x^3}{3}\log_e 3x - \frac{1}{3}\int x^2 \, dx$

$= \frac{x^3}{3}\log_e 3x - \frac{x^3}{9} + C$

$= \frac{x^3}{9}(3\log_e 3x - 1) + C$ a)

6. $y = c^2 x^{-1}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

at $x = cp$

$m = \frac{-c^2}{c^2 p^2}$

$= -\frac{1}{p^2}$

Eqn of tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp$

$p^2 y - cp = -x + cp$

$x + p^2 y = 2cp$ b)

7. $\frac{x^2}{144} + \frac{y^2}{25} = 1$ $a^2 = 144,\ b^2 = 25$

$b^2 = a^2(e^2 - 1)$

$25 = 144(e^2 - 1)$

$e^2 = \frac{25}{144} + 1$

$e = \frac{13}{12}$

foci $(\pm ae, 0) \Rightarrow (\pm 13, 0)$ c)

8. $\frac{x(x-5)}{(4-x)} < -3$

$x(x-5)(4-x) < -3(4-x)^2$

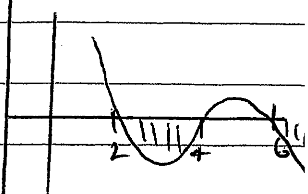
$x(x-5)(4-x) + 3(4-x)^2 < 0$

$(4-x)[x(x-5) + 3(4-x)] < 0$

$(4-x)(x^2 - 8x + 12) < 0$

$(4-x)(x-6)(x-2) < 0$

$2 < x < 4,\ x > 6$



d)

9. $\alpha + \beta + \gamma = 2$

$\therefore$ required roots are $\alpha+2, \beta+2, \gamma+2$

sub $x-2$

$(x-2)^3 - 2(x-2) + 1 = 0$

$x^3 - 8x^2 + 20x - 15 = 0$ a)

10 $x^2 = 8y$, $y = 2$

$\therefore$ $x^2 = 16$ $x = \pm 4$

$r = 2 - \frac{x^2}{8}$

$V = \pi \int_{-4}^{4} \left(2 - \frac{x^2}{8}\right) dx$ c)

A	B	C	D
2	3	2	3

Question 11.

a) let $u = \sin\theta$

$du = \cos\theta \, d\theta$ ✓

$I = \int \frac{du}{u^4}$

$= -\frac{1}{3} u^{-3}$ ✓ (2)

$= \frac{-1}{3\sin^3\theta} + C$

b) i) $3x^2 + 3x - 2 = A(x-1)(x+1) + B(x+1)^2 + 1(x-1)$

put $x = 1$

$4 = 4B \Rightarrow B = 1$ ✓

coeff't x^2 $3 = A + B$ (2)

$A = 2$ ✓

ii) $I = \int \frac{2}{x+1} + \frac{1}{x-1} + (x-1)^{-2} \, dx$

$= 2\ln|x+1| + \ln|x-1| - (x-1)^{-1}$ ✓ ✓ (2)

$= \ln\left|\frac{(x+1)^2}{x-1}\right| - \frac{1}{x-1} + C$

c) $I = \int \frac{dx}{\sqrt{2 - 4x - 2x^2}}$

Note
$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$

$2 - 4x - 2x^2 = 2 - (4x + 2x^2)$

$= 2 - 2(x^2 + 2x + 1) + 2$

$= 4 - 2(x+1)^2$

let $u = x + 1$, $du = dx$

$I = \int \frac{du}{\sqrt{4 - 2u^2}}$

$= \int \frac{du}{\sqrt{2(2 - u^2)}}$

$= \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{(\sqrt{2})^2 - u^2}}$

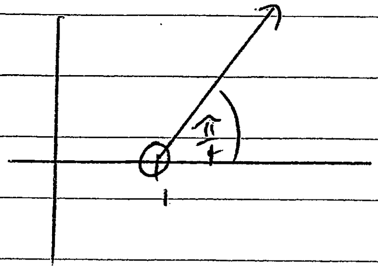
$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{u}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{x+1}{\sqrt{2}} + C$

−1 for each error. (3)

d) i) i^{2013} Now $\frac{2013}{4} = 503$ remainder 1.
$\therefore i^{2013} = i$ ✓ (1)

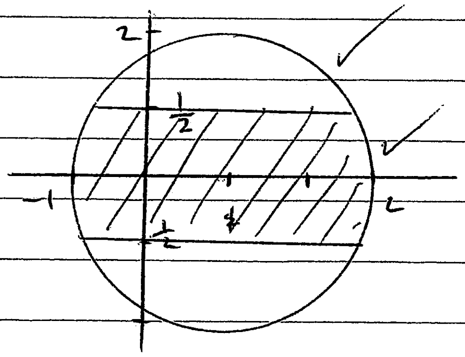
ii)



✓ (1)

e) $|z - \bar{z}| \le 1$, $|z - 1| \le 2$

$|x + iy - (x + iy)| \le 1$
$|2iy| \le 1$ ie $|y| \le \frac{1}{2}$



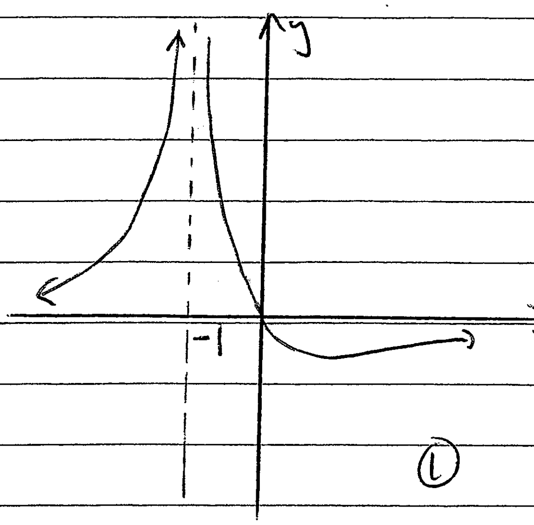
(2)
2

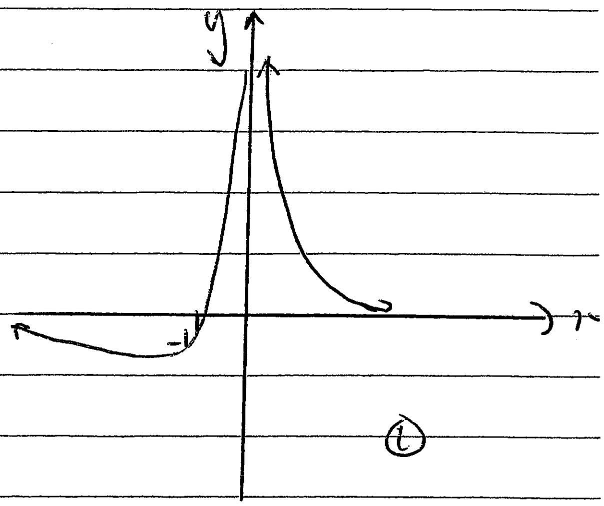
f) i) $x^4 - 3x^2 - 10 = (x^2 - 5)(x^2 + 2)$ C.F.P.A.
ii) $= (x + \sqrt{5})(x - \sqrt{5})(x^2 + 2)$

(1 mark if answers wrong way around
if attempted both parts may get 1

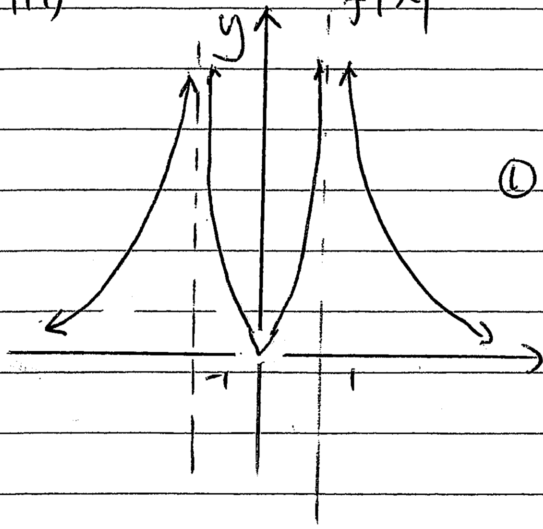
Question 12.

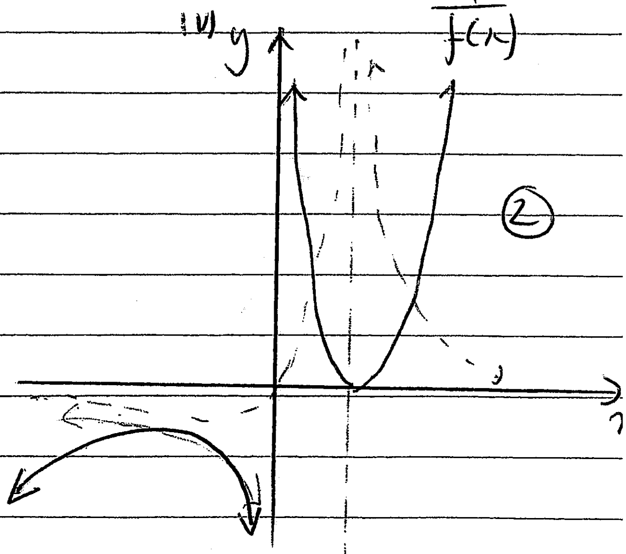
a) i) $f(-x)$ ii) $f(x+1)$





iii) $f|x|$ iv) $\frac{1}{f(x)}$





b) let $a+ib = \sqrt{1+i\sqrt{3}}$

$(a+ib)^2 = 1 + i\sqrt{3}$

$a^2-b^2+2aib = 1+i\sqrt{3}$

$a^2-b^2 = 1$ ①

$2ab = \sqrt{3}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 1+3$

$a^2+b^2 = 2$ ② $(a^2+b^2)>0$

$2a^2 = 3$

$a = \pm\sqrt{\frac{3}{2}}$ ✓

$ab = \frac{\sqrt{3}}{2}$

$b = \frac{\sqrt{3}}{2} \times \left(\pm\frac{\sqrt{2}}{\sqrt{3}}\right)$

$= \pm\frac{1}{\sqrt{2}}$ ✓

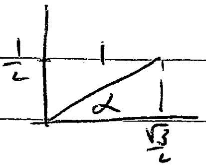
ie $\pm\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

or $\pm \frac{\sqrt{6}}{2} + \frac{i}{\sqrt{2}}$

②

Q 12

c) $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$



$\alpha = \frac{\pi}{6}$

i) $z = cis \frac{\pi}{6}$ ✓ ①

ii) $z^{15} = cis \frac{15\pi}{6}$ ✓

$= cis \frac{5\pi}{2}$

$= i$ ✓ ②

d) subst $x^{\frac{1}{2}}$

$ax^{\frac{3}{2}} + bx + cx^{\frac{1}{2}} + d = 0$ ✓

$x^{\frac{1}{2}}(ax+c) = -bx - d$

square both sides

$x(a^2x^2 + 2acx + c^2) = b^2x^2 - 2bdx + d^2$ ✓

$a^2x^3 + 2acx^2 + c^2x - b^2x^2 + 2bdx - d^2 = 0$

$a^2x^3 + (2ac - b^2)x^2 + (c^2 + 2bd)x - d^2 = 0$ ②

e) ① when $n = 1$

$a^2 - b^2 = (a-b)(a+b)$

which is divisible by $(a-b)$ ✓

② Assume $a^{2k} - b^{2k}$ divisibly by $(a-b)$

③ Now prove $a^{2k+2} - b^{2k+2}$ divisible by $(a-b)$

$a^{2k+2} - b^{2k+2} = a^{2k+2} - a^2b^{2k} + a^2b^{2k} - b^{2k+2}$

$= a^2\underbrace{(a^{2k} - b^{2k})}_{M} + b^{2k}\underbrace{(a^2 - b^2)}$

Divisible by $(a-b)$ ~~from~~ assumpt ✓

Divisible by $(a-b)$ from step 1 ✓

OR

Next page.

Q 12

② Assume true for $n = k$

$a^{2k} - b^{2k} = (a-b)m$

ie $a^{2k} = (a-b)m + b^{2k}$ [M a factor]

③ Prove true for $n = k+1$

Now $a^{2k+2} - b^{2k+2}$

$= a^2 a^{2k} - b^2 b^{2k}$

$= a^2[(a-b)m^* + b^{2k}] - b^{2k} b^2$ * from assumpt

$= (a-b)Ma^2 + b^{2k}a^2 - b^{2k}b^2$

$= (a-b)Ma^2 + b^{2k}(a^2-b^2)$

$= (a-b)Ma^2 + b^{2k}(a+b)(a-b)$

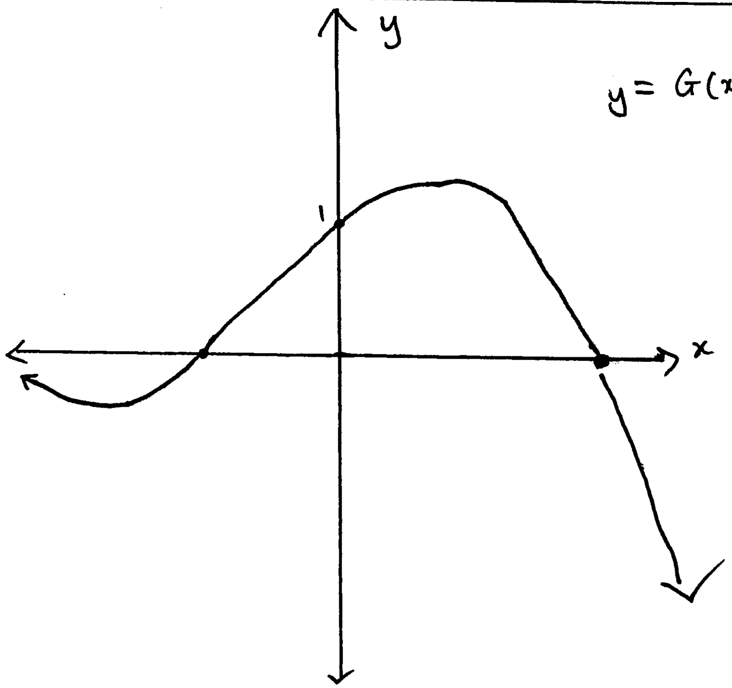
$= (a-b)[Ma^2 + b^{2k}(a+b)]$

Hence,

Question 13

(a) (i)

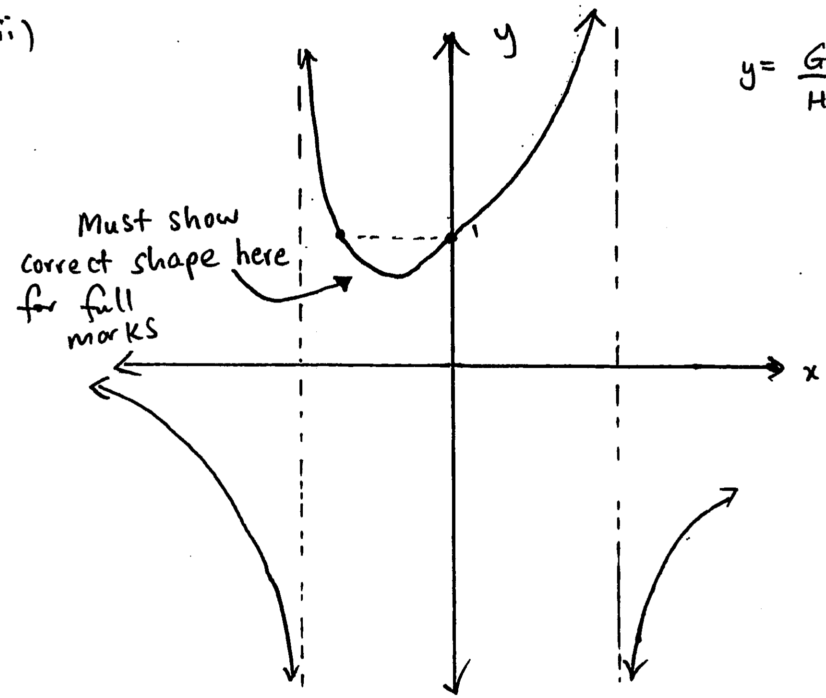
$y = G(x)H(x)$



(1)

(ii)

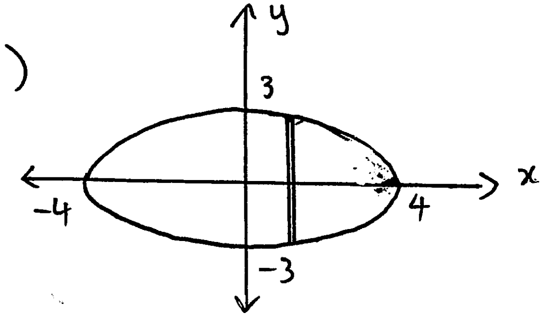
$y = \frac{G(x)}{H(x)}$



(2)

Question 13

(b)



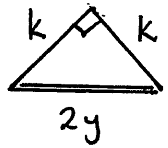
$$V = 2\int_0^4 y^2\,dx$$

$$= 18\int_0^4 \left(1 - \frac{x^2}{16}\right) dx$$

$$= 18\left[x - \frac{x^3}{48}\right]_0^4$$

$$= 18\left(4 - \frac{4^3}{48} - (0)\right)$$

$$\therefore V = 48 \text{ units}^3$$



$$A = \frac{k^2}{2}$$

$$k^2 + k^2 = (2y)^2$$

$$2k^2 = 4y^2$$

$$\frac{k^2}{2} = y^2$$

$$\therefore A = y^2$$

(4)

Question 13

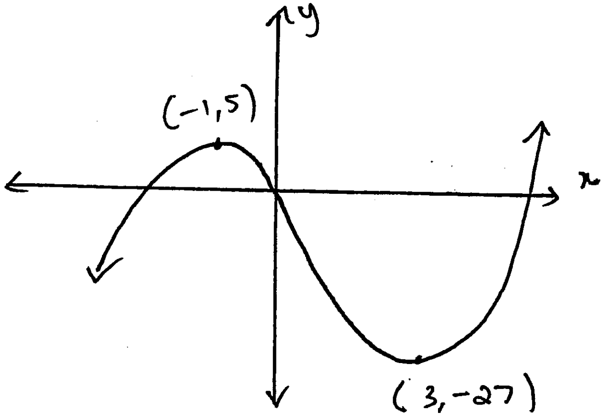
(c)(i) Consider $P(x) = x^3 - 3x^2 - 9x$

stat. pts $P'(x) = 0$ $\quad P'(x) = 3x^2 - 6x - 9$

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$



For one real solution $K > 27$

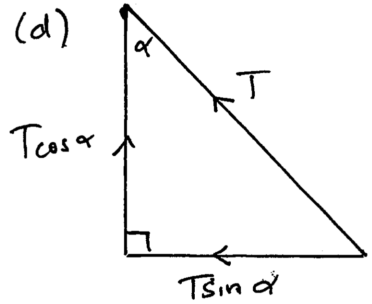
or $K < -5$. (2)

(ii) For three distinct solutions

$-5 < K < 27$. (1)

Question 15

(d)



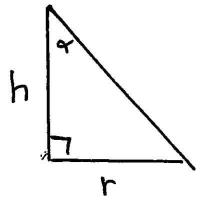
Hor.

$T\sin\alpha = mr\omega^2$

Vert.

$T\cos\alpha = mg$

$\therefore \tan\alpha = \dfrac{r\omega^2}{g}$



$\dfrac{r}{h} = \dfrac{r\omega^2}{g}$

$\therefore \omega = \sqrt{\dfrac{g}{h}}$

$P = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{h}{g}}$ (3)

(e) (i) $A = 2\displaystyle\int_{-a}^{a} y\,dx$ $\qquad y^2 = b^2\left(1 - \dfrac{x^2}{a^2}\right)$

$y = \sqrt{b^2\left(\dfrac{a^2 - x^2}{a^2}\right)}$

$= \dfrac{b}{a}\sqrt{a^2 - x^2}$

$\therefore A = \dfrac{2b}{a}\displaystyle\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx$ (1)

(ii) $A = \dfrac{2b}{a} \times \dfrac{\pi \times a^2}{2}$ (Area of semi-circle, radius a units) (1)

$\therefore A = \pi ab$

Question 14 Solutions:

a.

$x = \sin\theta \quad y = \cos 2\theta$

$y = 1 - 2\sin^2\theta$

$= 1 - 2x^2$

b.

$(cis\theta)^5 = c^5 + 5c^4(is) + 10c^3(is)^2 + 5c(is)^4 + (is)^5$

$cis5\theta = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

equating real parts:

$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$

$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$

$= c^5 - 10c^3 + 10c^2 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

c.

$i) b^2 = a^2(e^2 - 1)$

$9 = 25(e^2 - 1)$

$\frac{25}{9} = e^2 - 1$

$e = \frac{\sqrt{34}}{5}$

$ii) x = \pm\frac{a}{e}$

$= \pm\frac{25}{\sqrt{34}}$

$iii) PS = ePM \ (1)$

$PS' = ePM' (2)$

$(1) - (2)$

$PS - PS' = ePM - ePM'$

$|PS - PS'| = e|PM - PM'|$

$|PS - PS'| = e \times \frac{2a}{e}$

$= 2a$

$= c$

$iv) c = 2 \times 5$

$= 10$

d.

$x^3 + 2y^2 = 1$

$3x^2 + 4y\frac{dy}{dx} = 0$

$4y\frac{dy}{dx} = -3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{4y}$

When $x = -1, y = 1$

$\frac{dy}{dx} = \frac{-3}{4}$

$y - 1 = -\frac{3}{4}(x+1)$

$y = -\frac{3}{4}x + \frac{1}{4}$

e.

i. Vertically: $F\sin\theta + mg = N\cos\theta$

$F\sin\theta = N\cos\theta - mg \quad (1)$

Horizontally: $N\sin\theta + F\cos\theta = \frac{mv^2}{r}$

$F\cos\theta = \frac{mv^2}{r} - N\sin\theta \quad (2)$

ii. $(1) \times \sin\theta$

$F\sin^2\theta = N\sin\theta\cos\theta - mg\sin\theta \quad (3)$

$(2) \times \cos\theta$

$F\cos^2\theta = \frac{mv^2}{r}\cos\theta - N\sin\theta\cos\theta \quad (4)$

$(3) + (4)$

$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

iii. $r = 80, v = \frac{250}{9}, F = 0$

$\tan\theta = \frac{v^2}{rg}$

$= \frac{\left(\frac{250}{9}\right)^2}{80 \times 10}$

$= \frac{625}{645}$

$\theta = 44°$

15 a) $\frac{4x}{16} - \frac{3y}{9} = 1$ ✓

b) i) $P(x) = (x-a)^2 Q(x)$

$P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x)$ ✓

$P'(a) = 0$

$\therefore P'(x)$ has a root at $x = a$

ii) $Q(2) = 2^4 - a\times 2^2 + b\times 2 + 12$

$0 = 28 - 4a + 2b$

$Q'(x) = 4x^3 - 2ax + b$ ✓

$Q'(2) = 32 - 4a + b$ ✓

$0 = 32 - 4a + b$

$\therefore 0 = -4 + b$

$b = 4$

$32 - 4a + 4 = 0$

$-4a + 36 = 0$ ✓

$a = 9$

c) i) $4 = 8 - 12k$

$12k = 4$

$k = \frac{1}{3}$ ✓

$\frac{d(\frac{1}{2}v^2)}{dx} = 8 - \frac{x}{3}$

$\frac{1}{2}v^2 = 8x - \frac{x^2}{6} + c$

$0 = 0 - 0 + c$

$c = 0$ ✓

$\frac{1}{2}v^2 = 8\times 12 - \frac{12^2}{6} + 0$

$v^2 = 144$

speed $= \sqrt{144}$

$= 12$

ii) $\ddot{x} = -\frac{1}{3}(x - \quad)$

max speed at $x = 24$ ✓

$v = \sqrt{16\times 24 - \frac{24^2}{6}\times 2}$ ✓

$= \sqrt{192}$ ms^{-1}

d) let $u = (1+x^2)^{-n}$ $\quad v' = 1$

$u' = -n(1+x^2)^{-n-1} \times 2x$ ✓ $\quad v = x$

$= -\frac{2nx}{(1+x^2)^{n+1}}$ ✓

$I_n = \left[\frac{x}{(1+x^2)^n}\right]_0^1 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}dx$ ✓

$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}dx$

(e)(i) $O\hat{Q}T = 90°$ (radius ⊥ tangent)

$O\hat{T}G = 90°$ (" " ")

OGST is a cyclic quad (opposite ∠s are supplementary)

$O\hat{G}T = G\hat{S}R$ (exterior ∠ of a cyclic quad) ✓✓

(ii) $R\hat{G}T + O\hat{G}T = 180°$ (straight ∠)

$\therefore R\hat{G}T = 90°$

∴ RT is a diameter of the circle through R, G and T (∠ in semi circle)

GS = ST (tangents from an external p[oint])

∴ S is the centre of the circle through R, G and T. ✓✓

16 a) i) $\cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B$

$= 2\cos A\cos B$ ✓

ii) $2\cos 3\theta\cos 2\theta = \cos 3\theta$ ✓

$2\cos 3\theta\cos 2\theta - \cos 3\theta = 0$

$\cos 3\theta(2\cos 2\theta - 1) = 0$

$\cos 3\theta = 0$, $\cos 2\theta = \frac{1}{2}$ ✓

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ ✓

b) i) $V_{shell} = \pi(R^2 - r^2)h$

$= \pi\{(1-x)^2 - (1-x-\delta x)^2\}(\frac{3}{2}-y)$ ✓

$= 2\pi(1-x)(\frac{3}{2}-y)\delta x$

$V_{solid} = 2\pi \lim_{\delta x \to 0} \sum_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)(\frac{3}{2}-y)\delta x$

$= 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)(\frac{3}{2}-y)\,dx$

$= 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)(\frac{3}{2} - 2 + \cos^2 x)\,dx$ ✓

$= 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)(-\frac{1}{2} + \frac{1+\cos 2x}{2})\,dx$

$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)(\cos 2x)\,dx$

ii) $V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x\,dx - \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\cos 2x\,dx$

$= 2\pi\int_{0}^{\frac{\pi}{4}} \cos 2x\,dx$ ✓

$= \frac{\pi}{2}\Big[\sin 2x\Big]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\sin\frac{\pi}{2}$ ✓

$= \pi\,u^3$

c) $\frac{r\,\text{cis}\,\theta}{r^2\text{cis}\,2\theta + r^2}$ ✓

$= \frac{r\,\text{cis}\,\theta}{r^2(\text{cis}\,2\theta + 1)}$

$= \frac{r\,\text{cis}\,\theta}{r^2(\cos^2\theta - \sin^2\theta + 1 + i\sin 2\theta)}$ ✓

$= \frac{r\,\text{cis}\,\theta}{r^2(2\cos^2\theta + 2i\sin\theta\cos\theta)}$

$= \frac{\text{cis}\,\theta}{2\cos\theta(\cos\theta + i\sin\theta)}$ ✓

$= \frac{1}{2\cos\theta}$ Q.E.D.

d) i) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = 16\cos\theta$ $\dot{y} = 16\sin\theta - 10t$

$x = 16t\cos\theta$ $y = 16t\sin\theta - 5t^2$ ✓

ii) $6 = 16t\cos\theta$

$3 = 8t\cos\theta$

$t = \frac{3}{8\cos\theta}$ ✓

$10 = 16t\sin\theta - 5t^2$

$= 16\times\frac{3}{8\cos\theta}\sin\theta - 5\times\frac{3^2}{8^2\cos^2\theta}$

$640 = 384\tan\theta - 45 - 45\tan^2\theta$

$45\tan^2\theta - 384\tan\theta + 685 = 0$ ✓

$\tan\theta = \frac{384 \pm\sqrt{384^2 - 4\times45\times685}}{90}$

$= \frac{384 \pm\sqrt{24156}}{90}$

$= 5.9\ldots$ or $2.5\ldots$

$\theta = 80.5\ldots°$ or $68.5\ldots$

$t = \frac{3}{8\cos 80.5\ldots°}$ or $\frac{3}{8\cos 68.5\ldots°}$

$= 2.278\ldots$ or $1.023\ldots$

$T = 2.2\ldots - 1.0\ldots$

$= 1.255\ldots$ ✓