SYDNEY GIRLS HIGH SCHOOL

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Diagrams are NOT drawn to scale.
- All necessary working should be shown in every question.
- Start each question on a new page.

Total marks – 100

Section I Pages 3-7

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 8-17

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

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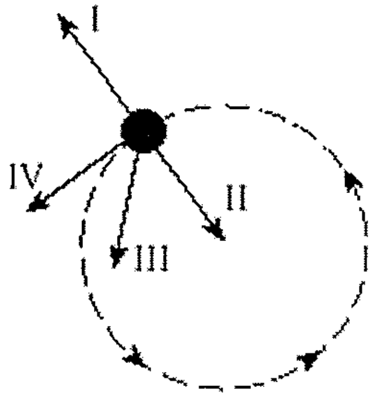
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 An object moves in a circular path at a constant speed. Which vector in the diagram below best represents the object's acceleration?



(A) I

(B) II

(C) III

(D) IV

2. Which of the following cannot be the argument of a complex number z such that $z^9 = -1 + i$?

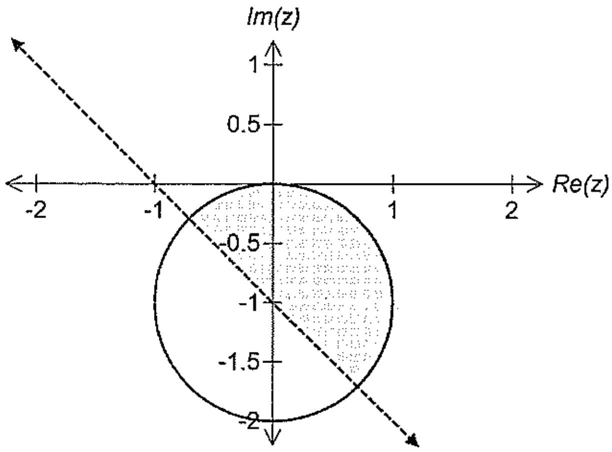
(A) $\frac{11\pi}{36}$

(B) $\frac{\pi}{12}$

(C) $\frac{29\pi}{36}$

(D) $\frac{19\pi}{36}$

3. Consider the Argand diagram below:



Which pair of inequalities correctly define the shaded area?

(A) $|z+i| \leq 1$ and $0 \leq \arg(z+1) < -\frac{\pi}{4}$

(B) $|z-i| \leq 1$ and $0 \leq \arg(z-1) < -\frac{\pi}{4}$

(C) $|z-i| \leq 1$ and $0 \leq \arg(z-1) < \frac{\pi}{4}$

(D) $|z+i| \leq 1$ and $0 \leq \arg(z+1) < \frac{\pi}{4}$

4. A point $P(x,y)$ moves so that the ratio of its distance from the point $(1,0)$ and the line $x=2$ is a constant e, where $0<e<1$. The locus described by this point would be a:

(A) Circle

(B) Parabola

(C) Hyperbola

(D) Ellipse.

5. Given that $(x-1)P(x)=16x^5-20x^3+5x-1$, then if $P(x)=\left(4x^2+ax-1\right)^2$, the value of a is:

(A) 1

(B) 2

(C) $\frac{1}{2}$

(D) 0.

6. $\int_0^{\pi} 5\sin x\cos^4 x\,dx =$

(A) 0

(B) 2

(C) -2

(D) 20

7. Given that $\int \sec^n x.dx = \frac{1}{n-1}\tan x\sec^{n-2}x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$ then $\int_0^{\frac{\pi}{4}} \sec^4 x.dx =$

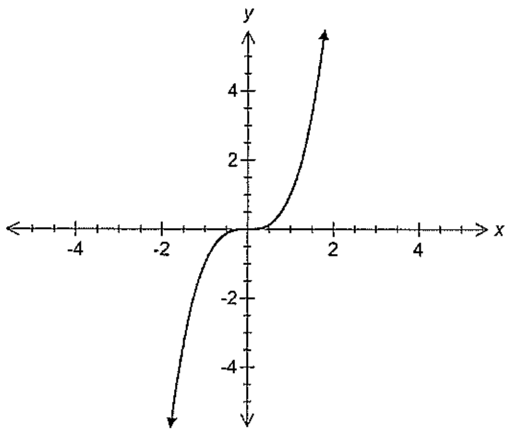
(A) $\frac{4}{3}$

(B) 1

(C) $\frac{5}{6}$

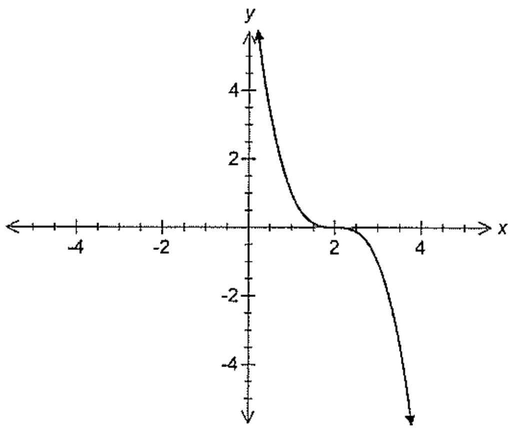
(D) $\frac{6+4\sqrt{2}}{9}$

8. The graph of $y = f(x)$ is shown below.

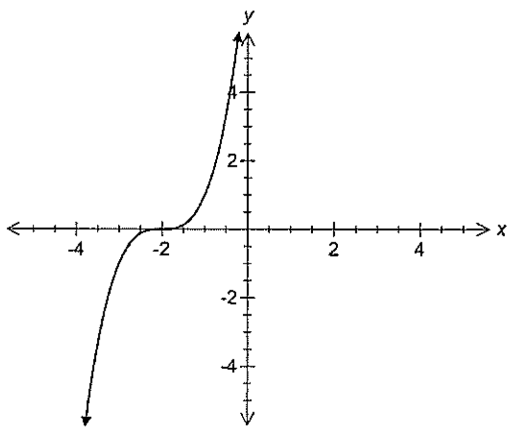


Which graph best represents $y = f(2-x)$?

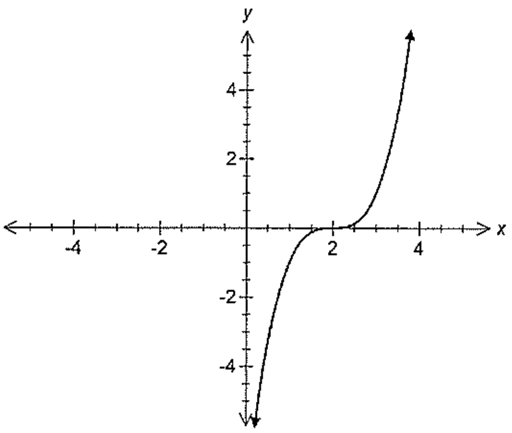
(A)



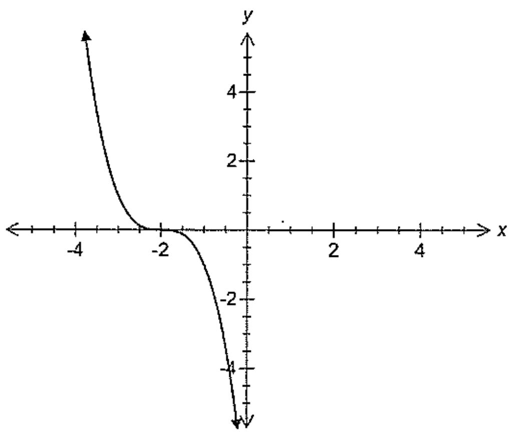
(B)



(C)



(D)



9. The curve $4y^2 - 9x^2 = 36$ is defined parametrically by the equations:

(A) $x = 4\sec\theta,\ y = 9\tan\theta$

(B) $x = 3\sec\theta,\ y = 2\tan\theta$

(C) $x = 2\tan\theta,\ y = 3\sec\theta$

(D) $x = 9\tan\theta,\ y = 4\sec\theta$

10. $\dfrac{3}{(x^2+2)(x-1)} = \dfrac{Px+Q}{x^2+2} + \dfrac{R}{x-1}$ where $x \neq 1$, and P, Q, R are constants. Which one of the following statements is **false**?

(A) $3 = (Px+Q)(x-1) + R(x^2+2)$

(B) $R = 1$

(C) $3 = 2R - Q$

(D) $3 = P + R$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) The complex number w is given by $w = -1 + \sqrt{3}i$.

(i) Show that $w^2 = 2\overline{w}$. **2**

(ii) Evaluate $|w|$ and $\arg w$. **2**

(iii) Show that w is a root of the equation $w^3 - 8 = 0$. **1**

(b) Sketch the region of the Argand diagram whose points satisfy the inequalities

$|z - \overline{z}| \le 4$ and $\frac{-\pi}{3} \le \arg z \le \frac{\pi}{3}$. **2**

Question 11 continues on page 9

(c) $ABCD$ is a quadrilateral whose diagonals AC and BD are equal and bisect each other at the origin. A is the complex number z and $\angle AOB = 30°$.

(i) Find the coordinates of B, C, and D in terms of z. 2

(ii) What type of quadrilateral is $ABCD$? (justify your answer) 1

(d) The complex number z is a function of the real number r given by the rule 3

$$z = \frac{r-i}{r+i}, \qquad 0 \le r \le 1.$$

Evaluate $|z|$ and hence describe the locus of z as r varies from 0 to 1.

(e) By completing the square, find $\displaystyle\int \frac{dx}{x^2+4x-1}$. 2

End of Question 11

Question 12 (15 marks)

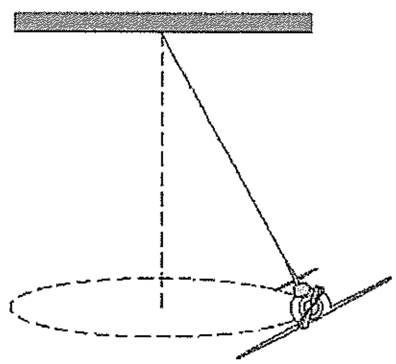
(a) (i) Express $\dfrac{x+7}{x^2(x+2)}$ in the form $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+2}$. 2

(ii) Hence or otherwise find $\displaystyle\int \frac{x+7}{x^2(x+2)}\,dx$. 2

(b) Find $\displaystyle\int \frac{dx}{\sqrt{1+4x^2}}$. 2

(c) Evaluate $\displaystyle\int_0^1 xe^{-x}\,dx$. 4

(d) A toy airplane attached to a string 1.5m long moves in a horizontal circle as shown in the diagram below. The string makes an angle of $30°$ with the vertical. (Use $g = 10\,\mathrm{ms}^{-2}$)



(i) What is the mass of the plane if the tension in the string is 5N? 3

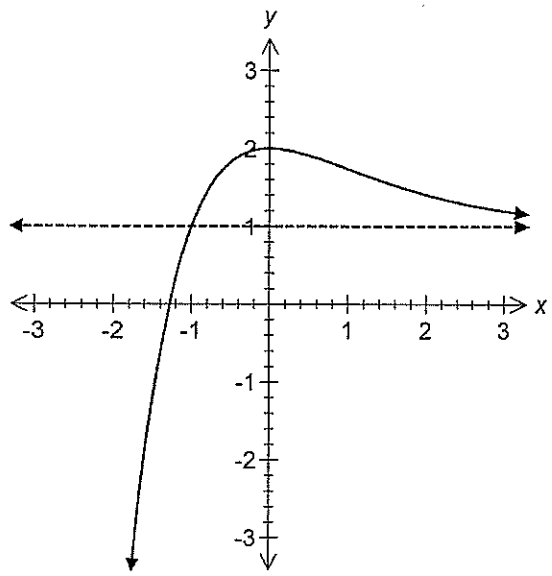
(ii) Calculate the period of the plane's motion. 2

End of Question 12

Question 13 (15 marks)

(a) Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, show that $\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}\, d\theta = \sqrt{3} - 1$. **4**

(b) The diagram shows the graph $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(|x|)$ **1**

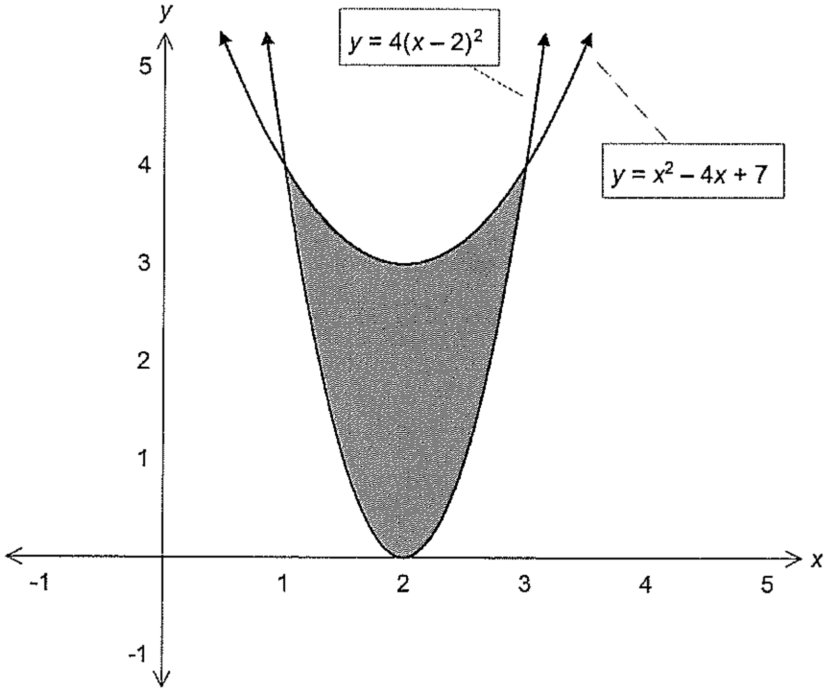
(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y = [f(x)]^2$ **2**

(iv) $y = \ln[f(x)]$. **2**

Question 13 continues on page 12

(c) Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region between the curves $y = 4(x-2)^2$ and $y = x^2 - 4x + 7$ about the y axis. 4



End of Question 13

Question 14 (15 marks)

(a) The Earth's orbit around the sun is elliptical with the Sun as one of the foci. The semi major axis of the Earth's orbit is 1.486×10^8 kilometres and its eccentricity is 0.017.

(i) How close to the sun does the earth come? 2

(ii) What is the greatest possible distance between the sun and the earth? 2

(b) Determine all the roots of the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ given that there is a root of multiplicity 3. 3

(c) The polynomial $P(x) = x^3 + ax^2 + bx + 6$ where a, b are real numbers has $1 - i$ as one zero.

(i) Find a and b. 2

(ii) Factorise $P(x)$ over the complex field. 1

(iii) Factorise $P(x)$ over the real field. 1

(d) Show that the equation $3x^5 + 20x^3 + 45x = c$ can have only one real root, and find the value of the constant c, if the sum of the other (complex) roots is -7. 4

End of Question 14

Question 15 (15 Marks)

(a) For the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, find:

(i) its eccentricity 1

(ii) the coordinates of its foci 1

(iii) the equations of its directrices 1

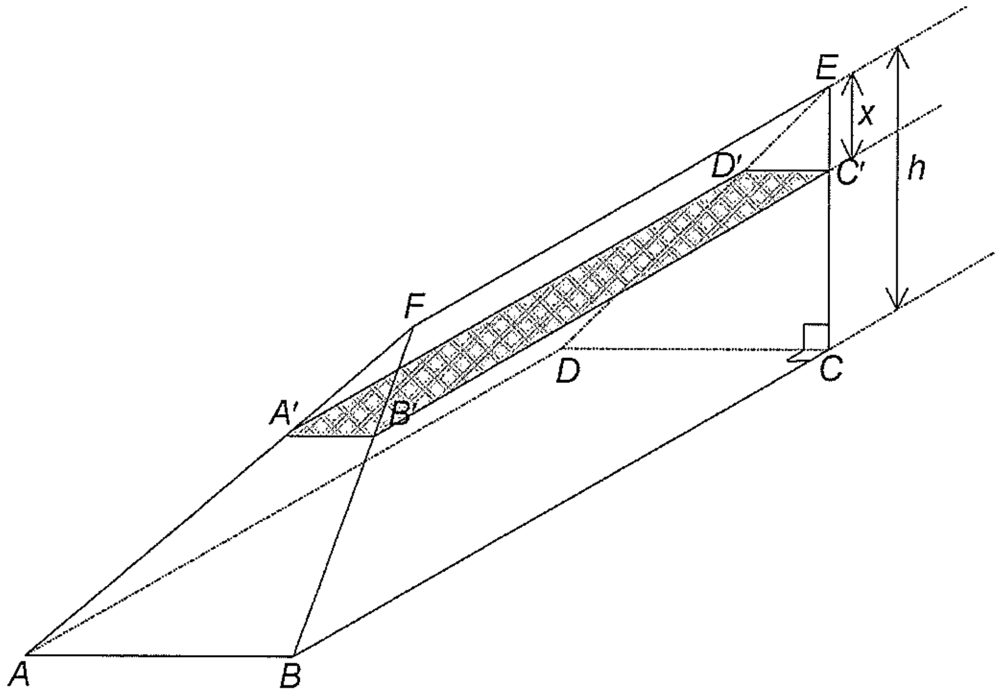
(iv) the equations of its asymptotes 1

(v) the equation of the chord of contact of tangents drawn from the point $(1,2)$ 1

(b) Consider the region enclosed by the curves $x = y^2$ and $x = 2 - y^2$. Find the volume of the solid formed when this region is rotated about the line $x = 3$ by taking slices perpendicular to the axis of rotation. 4

Question 15 continues on page 15

(c) Consider a solid $ABCDEF$ whose height is h, and whose base is a rectangle $ABCD$, where $AB = a$, $BC = b$, and the top edge $EF = c$



Consider a rectangular slice $A'B'C'D'$ (parallel to the base $ABCD$) x units from the top edge, with width Δx .

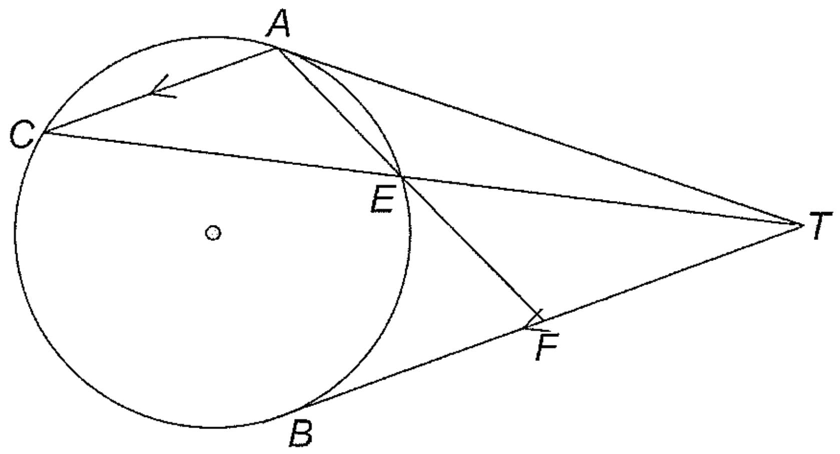
(i) Show that the volume of the slice is $\Delta V = \left(\frac{x}{h}a\right)\left(c + \frac{b-c}{h}x\right)\Delta x$. 3

(ii) Hence show that the volume of the solid is $\frac{ha}{6}(2b+c)$. 3

End of question 15

Question 16 (15 marks)

(a) Two tangents TA, TB are drawn from a point T to a given circle. Through A, a chord AC is drawn parallel to the other tangent TB and TC meets the circle at E.



(i) Prove ΔAFT is similar to ΔEFT. 2

(ii) Hence show that $TF^2 = AF \times EF$. 1

(iii) Hence or otherwise prove that AE extended bisects TB. 2

(b) (i) Prove that the equation of the tangent to the curve $x = \dfrac{1}{1+t^4}, y = \dfrac{t^5}{1+t^4}$ 3

at the point with parameter t is $4y + \left(5t + t^5\right)x = 5t$.

(ii) This tangent meets the coordinate axes OX and OY in the points P and Q. 3

Show that the area of the triangle OPQ never exceeds $\left(\dfrac{15}{32}\right)\left(\dfrac{5}{3}\right)^{\frac{1}{4}}$.

Question 16 continues on page 17

(c) Prove by mathematical induction for all positive integers n, 4

$$\tan^{-1}\left(\frac{1}{2\times 1^2}\right)+\tan^{-1}\left(\frac{1}{2\times 2^2}\right)+\ldots+\tan^{-1}\left(\frac{1}{2\times n^2}\right)=\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{2n+1}\right).$$

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension 2 Trial HSC 2013

Solutions.

Multiple Choice.

1) B

2) C

3) All responses marked correct

4) D

5) B

6) B

7) ~~C~~ A

8) A

9) C

10) D

11)

a) i) $w^2 = (-1+\sqrt{3}i)^2$

$= 1 - 2\sqrt{3}i - 3$

$= -2 - 2\sqrt{3}i$

$2\bar{w} = 2(-1-\sqrt{3}i)$

$= -2 - 2\sqrt{3}i$

$= w^2$.

ii) $|w| = \sqrt{(-1)^2 + \sqrt{3}^2}$

$= 2$

$\arg w = 2\pi/3$

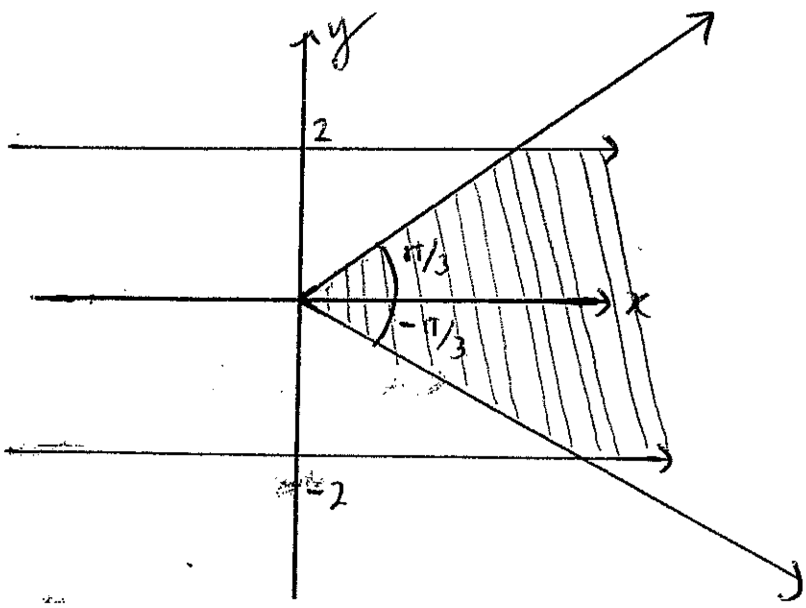
iii) $w^3 - 8 = \left[2 \operatorname{cis} 2\pi/3\right]^3 - 8$

$= 8 \operatorname{cis} 2\pi - 8$

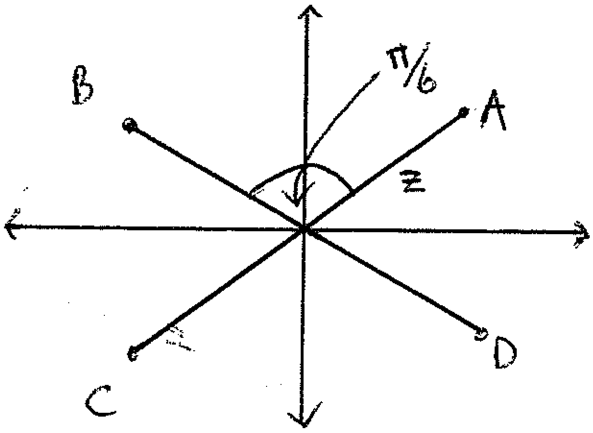
$= 8 - 8$

$= 0$

b)



c)



i) $B = z\,\text{cis}\left(\frac{\pi}{6}\right)$

$C = -z$

$D = -z\,\text{cis}\left(\pi/6\right)$

ii) ABCD is a rectangle. (equal diagonals that bisect each other).

d) $$|z| = \frac{|r-i|}{|r+i|}$$

$$= \frac{\sqrt{r^2+1}}{\sqrt{r^2+1}}$$

$$= 1$$

When $r=0$ $\quad z = \frac{-i}{i} = -1$

$r=1$ $\quad z = \frac{1-i}{1+i} \times \frac{(1-i)}{(1-i)}$

$= -i$

$\therefore$ locus is a quarter of the circle $x^2+y^2=1$ in the 3rd quadrant.

e) $\displaystyle\int \frac{dx}{x^2+4x-1} = \int \frac{dx}{(x+2)^2-5}$

$$= \frac{1}{2\sqrt{5}}\left[\ln\left(\frac{x+2-\sqrt{5}}{x+2+\sqrt{5}}\right)\right] + C$$

12 (a)(i) $x+7 = Ax(x+2) + B(x+2) + Cx^2$

when $x=0$ when $x=-2$

$7 = 2B$ $5 = 4C$

$B = \frac{7}{2}$ $C = \frac{5}{4}$

when $x=1$

$8 = 3A + 3B + C$

$= 3A + \frac{21}{2} + \frac{5}{4}$

$A = -\frac{5}{4}$

(ii) $\int\left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)}\right)dx$

$= -\frac{5}{4}\log x - \frac{7}{2x} + \frac{5}{4}\log(x+2) + c$

(b) $\frac{1}{2}\int\frac{dx}{\sqrt{\frac{1}{4}+x^2}} = \frac{1}{2}\log\left(x+\sqrt{\frac{1}{4}+x^2}\right)+c$

(c) Let $u = x$ $v' = e^{-x}$

$u' = 1$ $v = -e^{-x}$

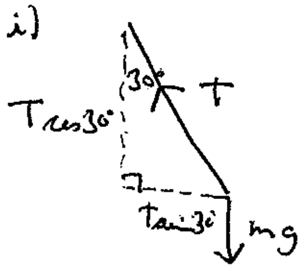
$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx$

$= -e^{-1} + 0 + \left[-e^{-x}\right]_0^1$

$= -e^{-1} - e^{-1} + 1$

$= 1 - 2e^{-1}$

(d) i)



$T\cos\theta = mg$

$5\cos 30° = m \times 10$

$m = \frac{1}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{4}$

ii) $T = ml\omega^2$

$5 = \frac{\sqrt{3}}{4} \times \frac{3}{2} \times \omega^2$

$\frac{40}{3\sqrt{3}} = \omega^2$

$\omega = \sqrt{\frac{40}{3\sqrt{3}}}$

Period $= \frac{2\pi}{\sqrt{\frac{40}{3\sqrt{3}}}}$

$= 2.264596406$

$\doteqdot 2.30$

Question 13:

a)

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}d\theta = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\left(1+\frac{2t}{1+t^2}\right)}.\frac{2dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2+2t}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} 2(t+1)^{-2}dt$$

$$= \left[\frac{2}{t+1}\right]_{\frac{1}{\sqrt{3}}}^{0}$$

$$= 2-\frac{2\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{2+2\sqrt{3}-2\sqrt{3}}{1+\sqrt{3}}$$

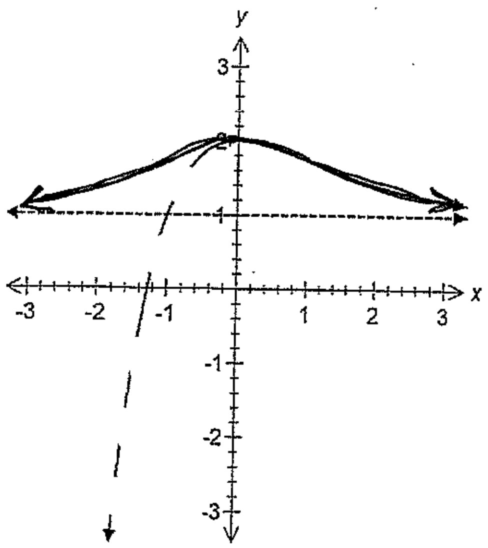
$$= \frac{2}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{2(1-\sqrt{3})}{-2}$$

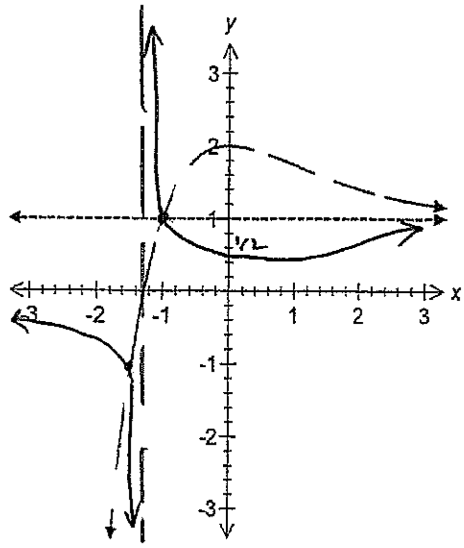
$$= \sqrt{3}-1$$

b)

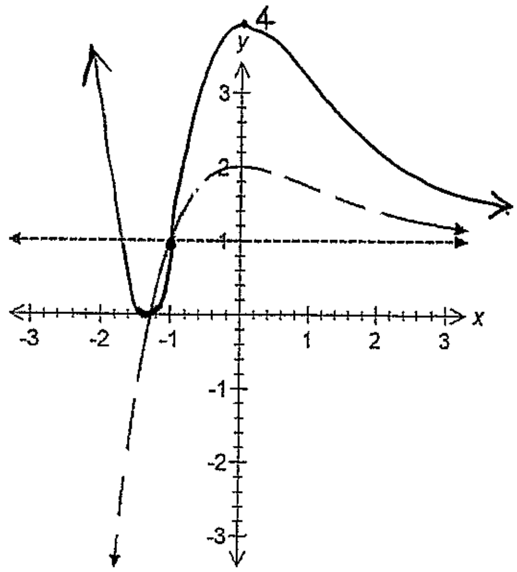
i) $y = f(|x|)$



ii) $y = \frac{1}{f(x)}$

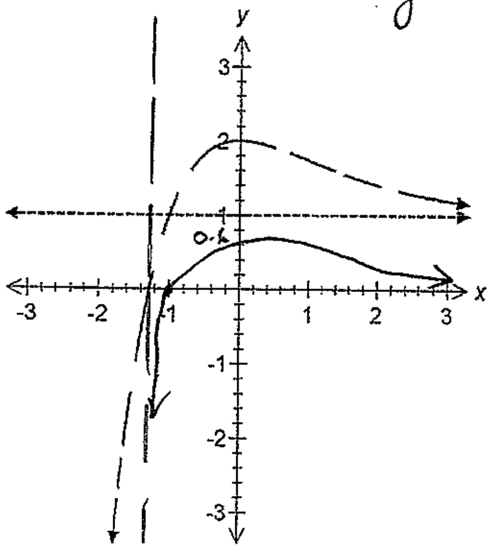


iii) $y = [f(x)]^2$

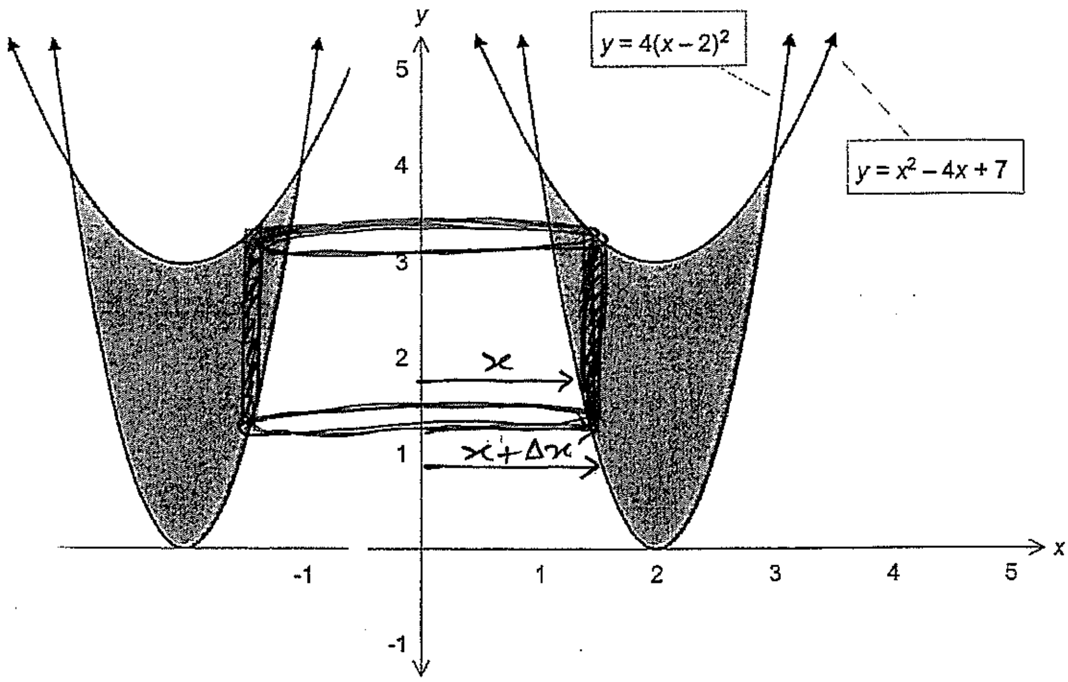


iv) $y = \ln[f(x)]$

$y = \ln 2 \doteqdot 0.6$



c)



Area of base of cylindrical shell: $\Delta A(x) = \pi(R^2 - r^2)$

$$\Delta A(x) = \pi((x+\Delta x)^2 - r^2)$$

$$\Delta A(x) = 2\pi x \Delta x, \text{ (since } \Delta x^2 \text{ is very small).}$$

Volume of shell: $\Delta V(x) = \Delta A(x) \times \text{height}$

$$\Delta V(x) = \Delta A(x) \times \left(x^2 - 4x + 7 - 4(x-2)^2\right)$$

$$\Delta V(x) = \Delta A(x) \times \left(-3x^2 + 12x - 9\right)$$

$$\Delta V(x) = 2\pi x \times 3\left(-x^2 + 4x - 3\right)\Delta x$$

Volume of solid: $V = \lim_{\Delta x \to 0} \sum_{x=1}^{3} 6\pi(-x^3 + 4x^2 - 3x)\Delta x$

$$= 6\pi\left[-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^3$$

$$= 6\pi\left(\left(-\frac{81}{4} + \frac{108}{3} - \frac{27}{2}\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right)$$

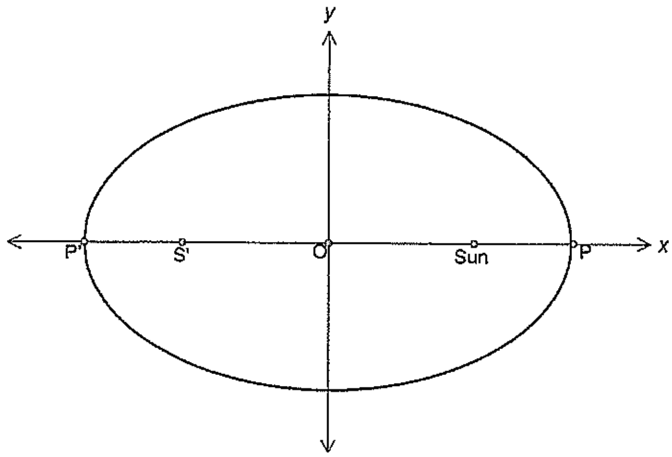
$$= 6\pi\left(\frac{9}{4} - \left(-\frac{5}{12}\right)\right)$$

$$= 6\pi \times \frac{8}{3}$$

$$\therefore V = 16\pi \text{ units}^3$$

Question 14:

a)



i) Earth is closest to the Sun at P.

$$PS = PO - ae$$
$$= 1.486\times10^8 - (1.486\times10^8\times0.017)$$
$$= 1.486\times10^8 - 2526200$$
$$= 146073800\text{km}$$
$$= 1.460738\times10^8\text{km}$$

ii) Earth is furthest from the Sun at P'.

$$P'S = PO + ae$$
$$= 1.486\times10^8 + (1.486\times10^8\times0.017)$$
$$= 1.017\times1.486\times10^8$$
$$= 15112\ 6200\text{km}$$
$$= 1.511262\times10^8\text{km}$$

b)

Let

$$P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$$
$$P'(x) = 4x^3 - 15x^2 - 18x$$
$$P''(x) = 12x^2 - 30x - 18$$

For a root of multiplicity 3, $P''(x) = 0$.

That is:

$$0 = 12x^2 - 30x - 18$$
$$0 = 2x^2 - 5x - 3$$
$$0 = (x-3)(2x+1)$$
$$\therefore x = 3 \text{ or } x = -\frac{1}{2}$$

Testing roots: $P(3) = 0$

$\therefore x = 3$ is a triple root.

Consider the sum of the roots of $P(x)$:

$$3+3+3+\alpha = 5$$
$$\alpha = -4$$

$\therefore P(x) = (x-3)^3(x+4)$ and

$P(x)$ has roots 3, 3, 3 and -4.

c)

i) $P(x)=x^3+ax^2+bx+6$

Since all the coefficients are real, then by the conjugate root theorem both $(1-i)$ and $(1+i)$ are factors of $P(x)$.

Product of roots of $P(x)$:

$$(1-i)(1+i)\alpha=-6$$
$$(1-i^2)\alpha=-6$$
$$2\alpha=-6$$
$$\alpha=-3$$

Now:

$$P(-3)=0$$
$$0=(-3)^3+a(-3)^2+b(-3)+6$$
$$0=-27+9a-3b+6$$
$$0=-21+9a-3b\ldots\ldots\ldots\ldots\ldots(1)$$

Sum of roots:

$$(1-i)+(1-i)-3=-a$$
$$2-3=-a$$
$$a=1\ldots\ldots\text{sub into (1)}$$

$$0=-21+9(1)-3b$$
$$0=-12-3b$$
$$3b=-12$$
$$b=-4$$

ii)

Over the complex field:

$$P(x)=(x-1+i)(x-1-i)(x+3)$$

iii)

Over the real field:

$$P(x)=\left((x-1)^2-i^2\right)(x+3)$$
$$=(x^2-2x+1+1)(x+3)$$
$$=(x^2-2x+2)(x+3)$$

d)

Let

$$P(x)=3x^5+20x^3+45x-c$$
$$P'(x)=15x^4+60x^2+45$$

For stationary points $P'(x)=0$.

$$0=15x^4+60x^2+45$$
$$0=x^4+4x^2+3$$
$$0=(x^2+3)(x^2+1)$$

No real solutions

No turning points.

For points of inflexion $P''(x)=0$.

$$P''(x)=60x^3+120x$$
$$0=x(x^2+2)$$

$\therefore$ at $x=0$ $(y=c)$ there is a point of inflexion.

Hence there must be only one real root and two pairs of conjugate complex roots of $P(x)$.

Sum of 'other' roots is given as -7.

Consider the sum of the roots of $P(x)$:

$$-7+\alpha=0$$
$$\alpha=7 \text{ (is the real root)}$$

Now $P(7)=0$ and by substitution $c=57596$.

15 (a) (i) $16 = 25(e^2 - 1)$

$\frac{16}{25} = e^2 - 1$

$\frac{41}{25} = e^2$

$e = \frac{\sqrt{41}}{5}$

(ii) $\left(\pm 5 \times \frac{\sqrt{41}}{5}, 0\right)$

$= (\pm\sqrt{41}, 0)$

(iii) $y = \pm \frac{5}{\frac{\sqrt{41}}{5}}$

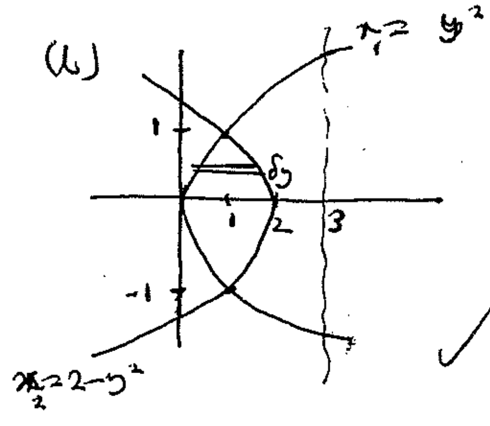
$= \pm \frac{25}{\sqrt{41}}$

(iv) $y = \pm \frac{4}{5}x$

(v) $\frac{1 \times x}{25} - \frac{2 \times y}{16} = 1$

$\frac{x}{25} - \frac{y}{8} = 1$

(b)



$V_{slice} = \pi(R^2 - r^2)$

$= \pi\{(3 - x_1)^2 - (3 - x_2)^2\}\delta y$

$= \pi(9 - 6x_1 + x_1^2 - 9 + 6x_2 - x_2^2)\delta y$

$= \pi(6x_2 - 6x_1 + x_1^2 - x_2^2)\delta y$

$V_{solid} = \lim_{\delta y \to 0} \sum_{y=-1}^{1} \pi(6x_2 - 6x_1 + x_1^2 - x_2^2)\delta y$

$= \pi\int_{-1}^{1}(6x_2 - 6x_1 + x_1^2 - x_2^2)\,dy$

$= \pi\int_{-1}^{1}(12 - 6y^2 - 6y^2 + y^4 - 4 + 4y^2 - y^4)\,dy$

$= \pi\int_{-1}^{1}(8 + 8y^2)\,dy$

$= 16\pi\int_0^1(1 + y^2)\,dy$

$= 16\pi\left[y + \frac{y^3}{3}\right]_0^1$

$= 16\pi\left(1 + \frac{1}{3}\right)$

$= \frac{32\pi}{3}\ u^3$

(c) (i) $V = Ah$

$= A'B' . B'C' . \delta x$

$A'B' = mx + k$

$0 = m \times 0 + k$

$k = 0$

$\therefore A'B' = mx$

$a = mh$

$m = \frac{a}{h}$

$\therefore A'B' = \frac{a}{h}x$

$B'C' = mx + b$

$c = m \times 0 + b$

$b = c$

$B'C' = mx + c$

$b = mh + c$

$m = \frac{b-c}{h}$

$\therefore B'C' = \frac{b-c}{h}x + c$

$\therefore V = \left(\frac{a}{h}x\right)\left(\frac{b-c}{h}x + c\right)\delta x$

(ii) $V = \lim_{\delta x \to 0} \sum_{x=0}^{h}\left(\frac{x}{h}a\right)\left(c + \left(\frac{b-c}{h}\right)x\right)dx$

$= \frac{a}{h}\int_0^h x\left(c + \frac{b-c}{h}x\right)dx$

$= \frac{a}{h}\int_0^h\left(cx + \frac{b-c}{h}x^2\right)dx$

$= \frac{a}{h}\left[\frac{cx^2}{2} + \frac{b-c}{3h}x^3\right]_0^h$

$= \frac{a}{h}\left(\frac{ch^2}{2} + \frac{b-c}{3h}.h^3\right)$

$= \frac{ach}{2} + \frac{ab - ac}{3}h$

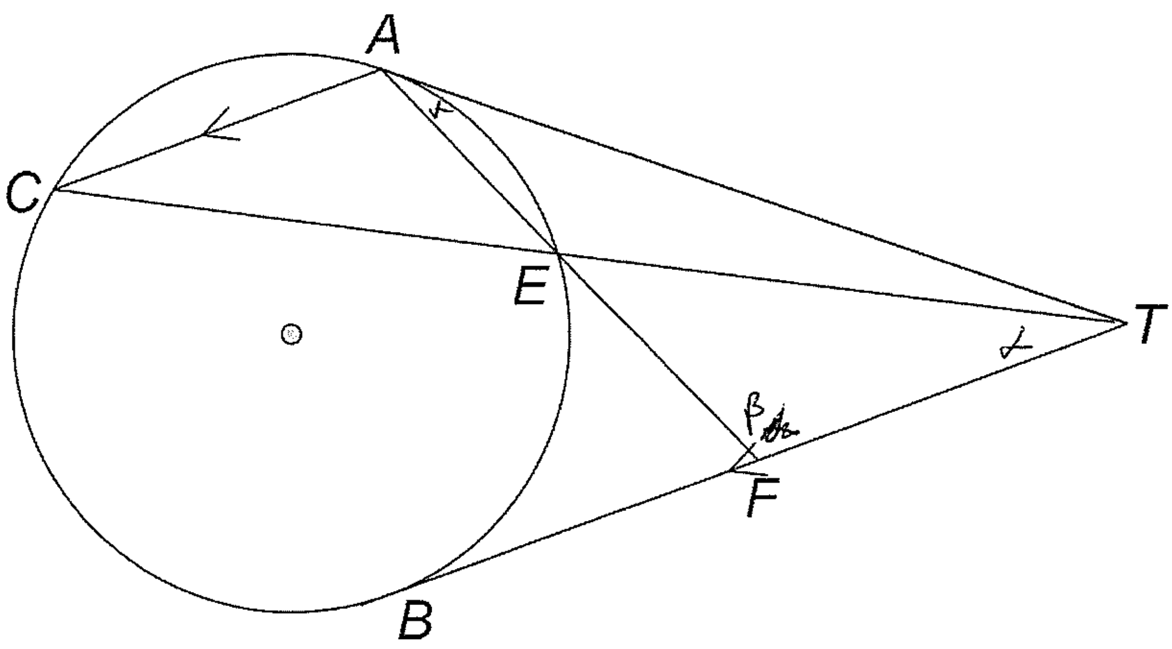
$= \frac{ach}{6} + \frac{abh}{3}$

$= \frac{ah}{6}(c + 2b)$

Q.E.D.

Question 16.

a)



i) In ΔTFA and ΔETF

$\angle TAF = \angle ACE = \alpha$ ($\angle$ in alt segment)

$\angle ACE = \angle ETF = \alpha$ (alt $\angle$s $AC \parallel BT$)

$\therefore \angle TAF = \angle ETF$

$\angle EFT = \angle TFA$ (common)

$\therefore \Delta TFA \sim \Delta ETF$ (equiangular)

ii) $\dfrac{TF}{EF} = \dfrac{AF}{TF}$ (corresp sides of sim Δs in same ratio)

$TF^2 = AF \times EF$

iii) $BF^2 = AF \times EF$ (tangent^2 = prod. of intercepts)

$\therefore TF^2 = BF^2$

$TF = BF$

$\therefore$ AE extended bisects BT.

b) i) $x = \frac{1}{1+t^4}$ $\quad$ $y = \frac{t}{1+t^4}$

$\frac{dx}{dt} = \frac{-4t^3}{(1+t^4)^2}$

$\frac{dy}{dt} = \frac{5t^4(1+t^4) - t^5(4t^3)}{(1+t^4)^2}$

$= \frac{5t^4 + 8t^8}{(1+t^4)^2}$

$= \frac{t^4(5+t^4)}{(1+t^4)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{t^4(5+t^4)}{(1+t^4)^2} \cdot \frac{(1+t^4)^2}{-4t^3}$

$= \frac{t(5+t^4)}{-4}$

$y - \frac{t^5}{1+t^4} = \frac{t(5+t^4)}{-4}\left[x - \frac{1}{1+t^4}\right]$

$4y + x(5t+t^5) = \frac{4t^5}{1+t^4} + \frac{5t+t^5}{1+t^4}$

$= \frac{5t\cancel{(1+t^4)}}{\cancel{(1+t^4)}}$

ii) When $x=0$

$4y = 5t$

$y = \frac{5t}{4}$

$\therefore OY = \frac{5t}{4}$

When $y=0$

$x = \frac{5t}{5t+t^5}$

$= \frac{5}{5+t^4}$

$\therefore OX = \frac{5}{5+t^4}$

$\therefore A = \frac{1}{2} \times \frac{5t}{4} \times \frac{5}{5 + t^4}$

$= \frac{25t}{8(5+t^4)}$

$\frac{dA}{dt} = \frac{25 \times 8(5+t^4) + 32t^3(25t)}{64(5+t^4)^2}$

For a max area $\frac{dA}{dt} = 0$

$200(5+t^4) - 800t^4 = 0$

$1000 + 200t^4 - 800t^4 = 0$

$t^4 = \frac{5}{3}$

$t = \left(\frac{5}{3}\right)^{1/4}$

$\therefore$ Max $A = \frac{1}{2} \times \frac{5}{4}\left(\frac{5}{3}\right)^{1/4} \times \frac{5}{5 + 5/3}$

$= \frac{5}{8}\left(\frac{5}{3}\right)^{1/4} \times \frac{3}{4}$

$= \frac{15}{32}\left(\frac{5}{3}\right)^{1/4}$

c) Show true for $n=1$

$$LHS = \tan^{-1}\left(\tfrac{1}{2}\right)$$

$$\begin{aligned} RHS &= \tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{1}{3}\right) \\ &= \tan^{-1}(1) - \tan^{-1}\left(\tfrac{1}{3}\right) \\ &= \tan^{-1}\left(\frac{1-1/3}{1+1/3}\right) \\ &= \tan^{-1}\left(\tfrac{1}{2}\right) \\ &= LHS \end{aligned}$$

$\therefore$ true for $n=1$

Assume true for $n=k$

$$\tan^{-1}\left(\frac{1}{2\times1^2}\right) + \cdots + \tan^{-1}\left(\frac{1}{2k^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right)$$

Prove true for $n=k+1$

R.T.P

$$\tan^{-1}\left(\frac{1}{2\times1^2}\right) + \cdots + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+3}\right)$$

$$LHS = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) \quad \text{[using assumption]}$$

$$= \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) - \tan^{-1}\left(\frac{1}{2k+1}\right)$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\left\{\frac{1}{2(k+1)^2} - \frac{1}{2k+1}\right\} \div \left\{1 + \frac{1}{2(k+1)^2} \times \frac{1}{2k+1}\right\}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{2k+1-2(k+1)^2}{2(2k+1)(k+1)^2} \div \frac{2(k+1)^2(2k+1)+1}{2(2k+1)(k+1)^2}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{2k+1-2(k+1)^2}{2(k+1)^2(2k+1)+1}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{2k+1-2k^2-4k-2}{(2k^2+4k+2)(2k+1)+1}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{-2k^2-2k-1}{4k^3+8k^2+4k+2k^2+4k+2+1}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{-(2k^2+k+1)}{(2k+3)(2k^2+k+1)}\right]$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{-1}{2k+3}\right]$$

$$= \frac{\pi}{4} - \tan^{-1}\left[\frac{1}{2k+3}\right]$$

$= RHS$

$\therefore$ Proven true for $n = k+1$.

$\therefore$ Proposition is true for all $n \geq 1$ by induction