Sydney Girls High School
2018

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- All answers should be given in simplest exact form unless otherwise specified.

Total marks – 100

Section I Pages 3 – 9

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 10 – 19

90 Marks

- Attempt Questions 11 – 16
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 2 hours and 45 minutes for this section

Name: ………………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2018 HSC Examination Paper in this subject.

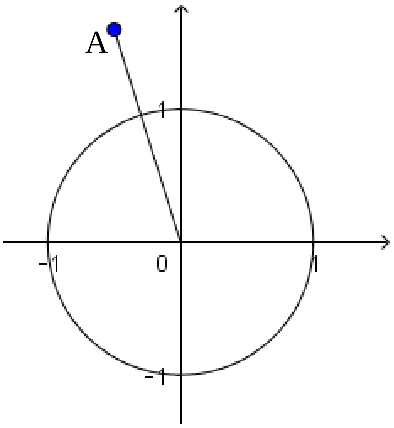
Section I

10 marks

Attempt Questions 1–10

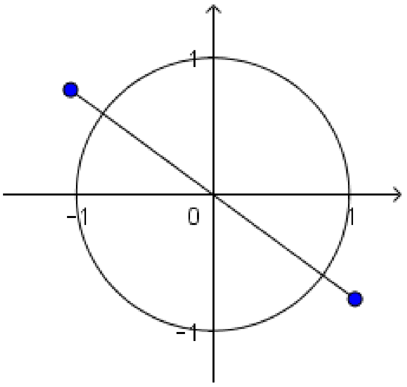
Use the multiple-choice answer sheet for Questions 1–10.

(1) The point A represents the complex number z.

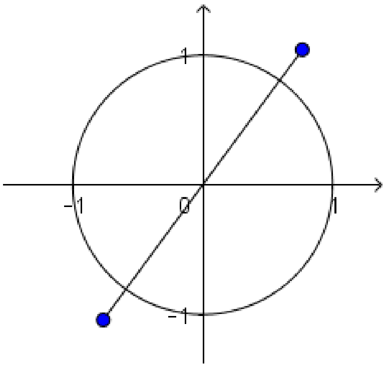


Which diagram shows two points representing the square roots of z?

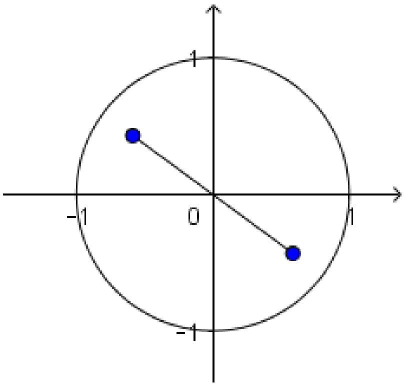
(A)



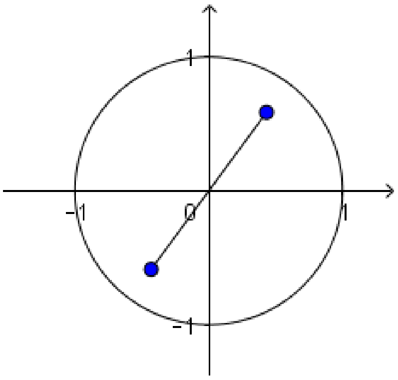
(B)



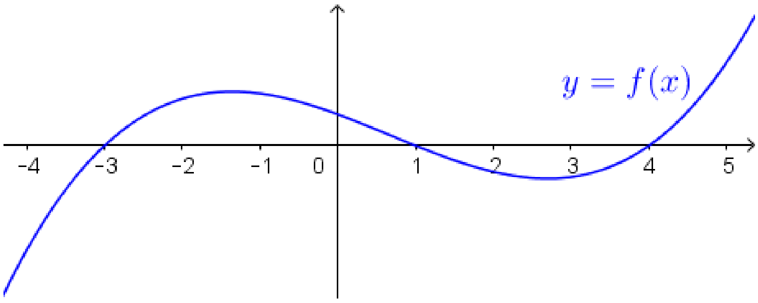
(C)



(D)

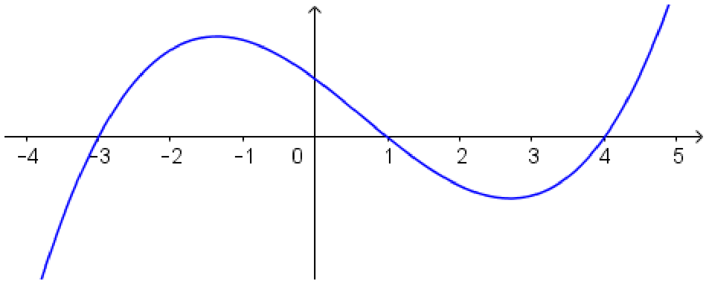


(2)

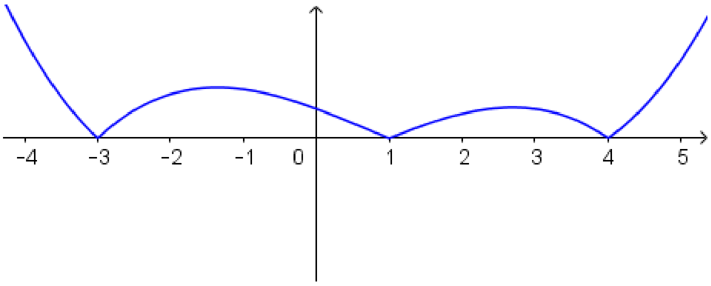


Given the graph above represents $y = f(x)$, which one of the following graphs represents $y = [f(x)]^2$?

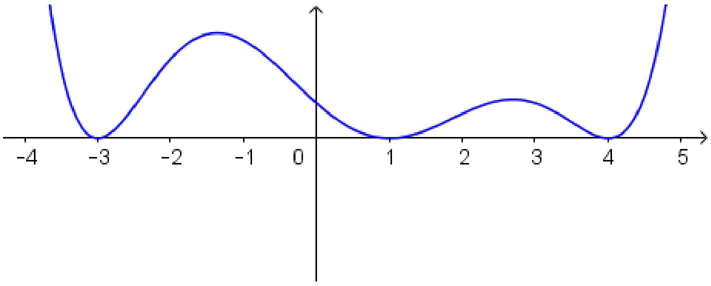
(A)



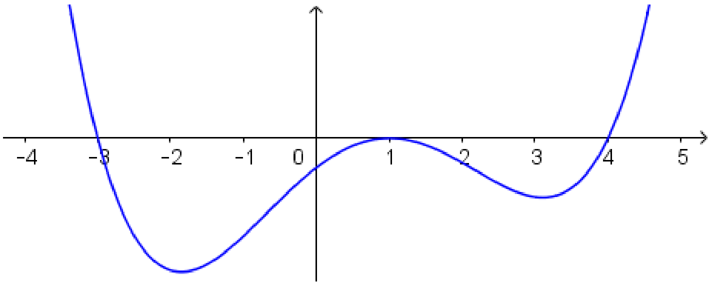
(B)



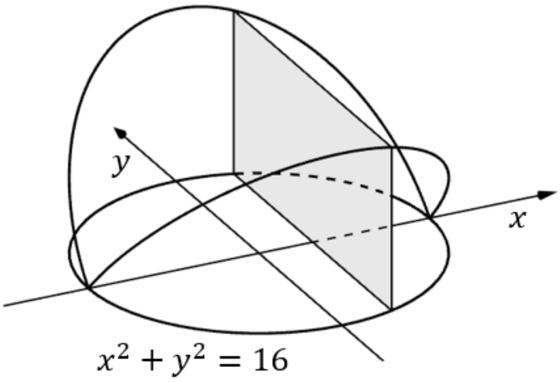
(C)



(D)



(3) A solid has a base that is the circle $x^2 + y^2 = 16$. Cross sections perpendicular to the x-axis are rectangles with heights that are half the widths. Which definite integral will give the volume of the solid?



(A) $\int_0^4 (16 - x^2)dx$

(B) $2\int_0^4 (16 - x^2)dx$

(C) $4\int_0^4 (16 - x^2)dx$

(D) $8\int_0^4 (16 - x^2)dx$

(4) Which one of the following relations does **not** have a graph that is a straight line passing through the origin on the Argand plane?

(A) $z + \bar{z} = 0$

(B) $z = i\bar{z}$

(C) $3Re(z) = Im(z)$

(D) $Re(z) + Im(z) = 1$

(5) Which of the following is an expression for $\int \cos^2 x \sin^7 x \, dx$?

(A) $-\frac{\cos^3 x}{3} + \frac{3\cos^5 x}{5} - \frac{3\cos^7 x}{7} + \frac{\cos^9 x}{9} + c$

(B) $-\cos^3 x + 3\cos^5 x - 3\cos^7 x + \cos^9 x + c$

(C) $\frac{\cos^3 x}{3} - \frac{3\cos^5 x}{5} + \frac{3\cos^7 x}{7} - \frac{\cos^9 x}{9} + c$

(D) $\cos^3 x - 3\cos^5 x + 3\cos^7 x - \cos^9 x + c$

(6) The derivative of the curve

$x^3 + 9x^2 - y^2 + 27x - 4y + 23 = 0$ is:

(A) $\frac{dy}{dx} = \frac{x^2+6x+9}{2y}$

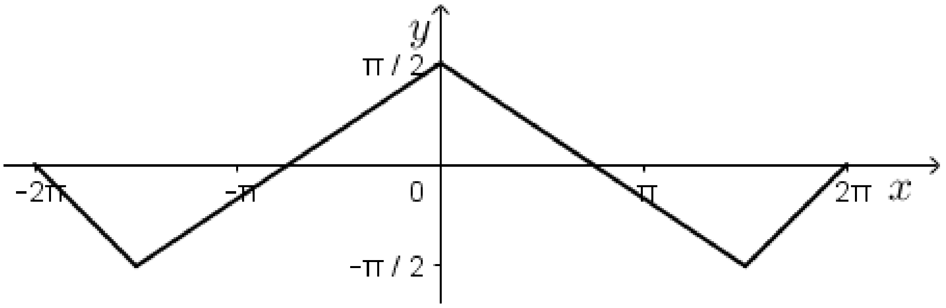
(B) $\frac{dy}{dx} = \frac{x^2+6x+9}{-2y}$

(C) $\frac{dy}{dx} = \frac{3x^2+18x+27}{-2y-4}$

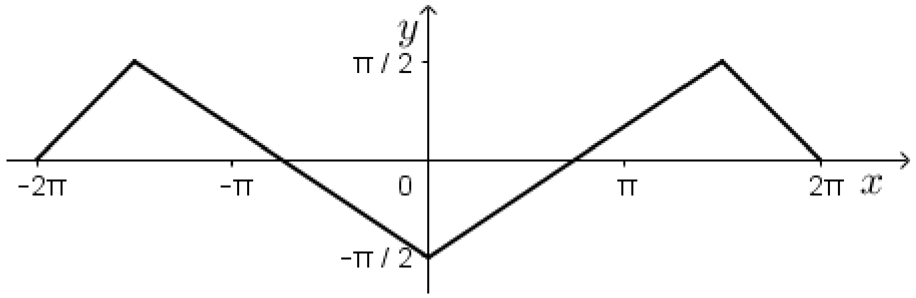
(D) $\frac{dy}{dx} = \frac{3x^2+18x+27}{2y+4}$

(7) Which of the graphs below represents $y = \sin^{-1}(\cos x)$ for $-2\pi \leq x \leq 2\pi$?

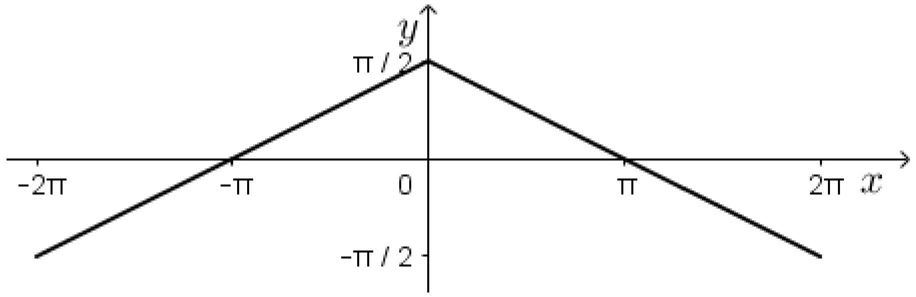
(A)



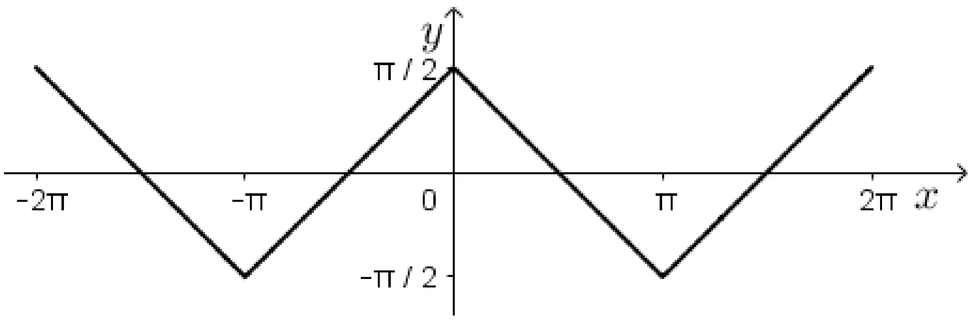
(B)



(C)



(D)



(8) A particle moves in a straight line so that its velocity at any particular time is given by $v = k(b - x)$, where x is its displacement from a given point O. The particle is initially at O.

Which of the following gives an expression for x:

(A) $x = b(1 - e^{kt})$

(B) $x = b(1 + e^{kt})$

(C) $x = b(1 - e^{-kt})$

(D) $x = b(1 + e^{-kt})$

(9) The equation of the locus of the complex number z such that $|z - 3| + |z + 3| = 12$ is

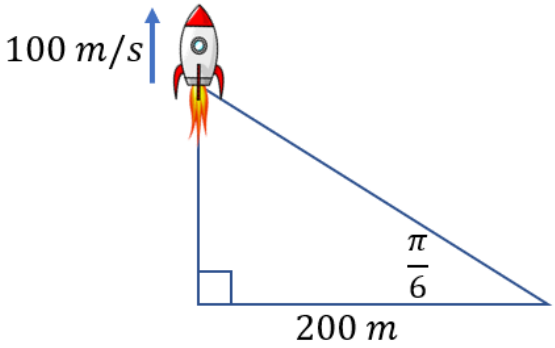
(A) $\frac{x^2}{36} + \frac{y^2}{9} = 1$

(B) $\frac{x^2}{36} + \frac{y^2}{27} = 1$

(C) $\frac{x^2}{144} + \frac{y^2}{9} = 1$

(D) $\frac{x^2}{144} + \frac{y^2}{27} = 1$

(10) A rocket is launched vertically with a tracking station located on the ground 200 metres away from the launch pad. The angle of elevation of the automatic tracking system changes so that it always lined up with the rocket. When the angle of elevation is $\pi/6$, the rocket is travelling at 100 metres/second, the rate of change of the angle of the tracking system is closest to:



(A) 0.13 radians/seconds

(B) 0.38 radians/seconds

(C) 0.51 radians/seconds

(D) 0.76 radians/seconds

Section II

90 marks

Attempt Questions 11–16

Start each question on a NEW piece of paper.

Question 11 (15 marks)

Use a NEW sheet of paper.

(a) If $z_1 = \frac{1}{\sqrt{2}}\left(1 + \sqrt{3}i\right)$

(i) Express z_1 in modulus-argument form. [2]

(ii) Show that z_1 is a solution to $z^8 - 8z^2 = 0$. [2]

(b) Find

(i) [1]

$$\int x^2 \cos(x^3 + 1)\, dx$$

(ii) [2]

$$\int \frac{dx}{\sqrt{3 + 2x - x^2}}$$

(c)

(i) Express $(1 - i)(5 + i)^2$ in the form $a + bi$ where a and b are real. [2]

(ii) Hence, prove that [2]

$$\tan^{-1}\left(\frac{7}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \frac{\pi}{4}$$

(d) Consider the ellipse with a focus at $S(-1, 0)$, a corresponding directrix $x = 5$ and eccentricity of a $\frac{1}{2}$.

(i) Find the centre of the ellipse. [2]

(ii) Sketch the graph of the ellipse, showing its domain and range. [2]

End of Question 11

Question 12 (15 marks)

Use a NEW sheet of paper.

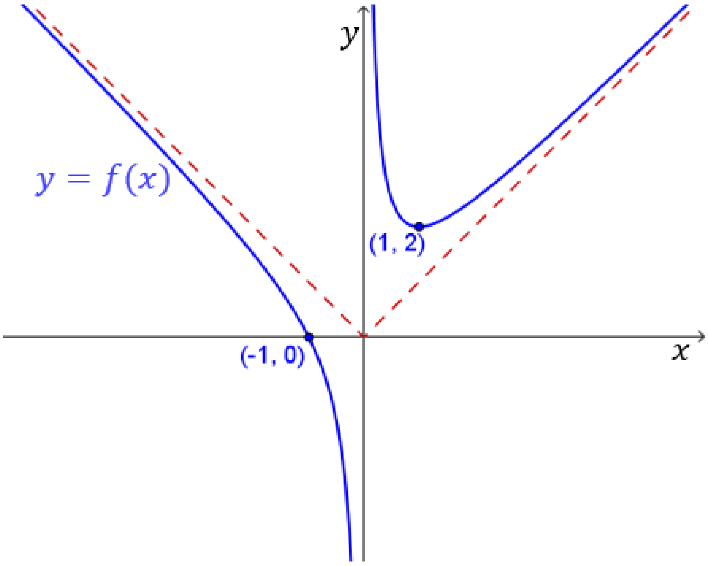
(a)

(i) Express $\frac{5x^2-11}{(x+2)(x-1)^2}$ in the form $\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$, where A, B and C are constants. [2]

(ii) Hence find [2]

$$\int \frac{5x^2-11}{(x+2)(x-1)^2}\,dx$$

(b)



Using four separate graphs sketch, showing important features:

(i) $y = f(-|x|)$ [2]

(ii) $y = \sqrt{f(x)}$ [2]

(iii) $y = \frac{1}{f(x)}$ [2]

(iv) $y = 2^{f(x)}$ [2]

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate [3]

$$\int_0^{\pi/2} \frac{dx}{5+4\cos x+3\sin x}$$

End of Question 12

Question 13 (15 marks)

Use a NEW sheet of paper.

(a) If ω is a complex root of the equation $x^3 - 1 = 0$.

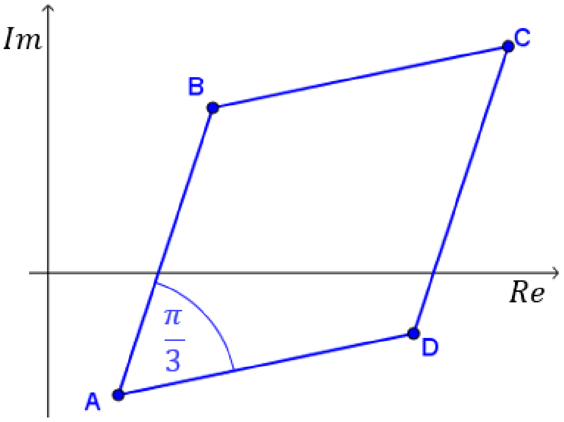
(i) Show that the other complex root is ω^2. [1]

(ii) Show that $1 + \omega + \omega^2 = 0$. [1]

(iii) Find the value of $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega)$. [2]

(iv) If the equations $x^3 - 1 = 0$ and $px^5 + qx + r = 0$ have a common root, evaluate $(p + q + r)(p\omega^2 + q\omega + r)(p\omega + q\omega^2 + r)$. [2]

(b) $ABCD$ is a rhombus with $\angle BAD = {}^{\pi}/_{3}$, where A and B are represented by the complex numbers $(3 - 2i)$ and $(5 + 4i)$ respectively.



(i) Find the complex representation of D in $a + bi$ form. [2]

(ii) Find the complex representation of C in $a + bi$ form. [2]

(c) α, β and γ are the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A and B are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$.

(i) Explain why β is real and both α and γ are not real. [2]

(ii) Show that α and γ are pure imaginary. [1]

(iii) Find the values of A and B. [2]

End of Question 13

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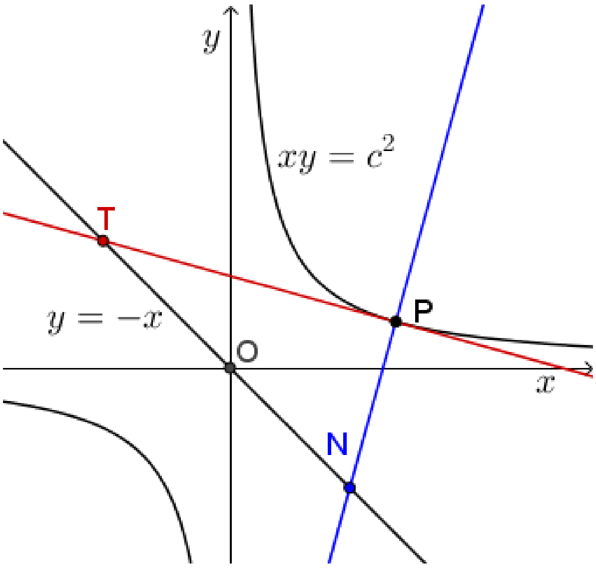
Question 14 (15 marks)

Use a NEW sheet of paper.

(a)

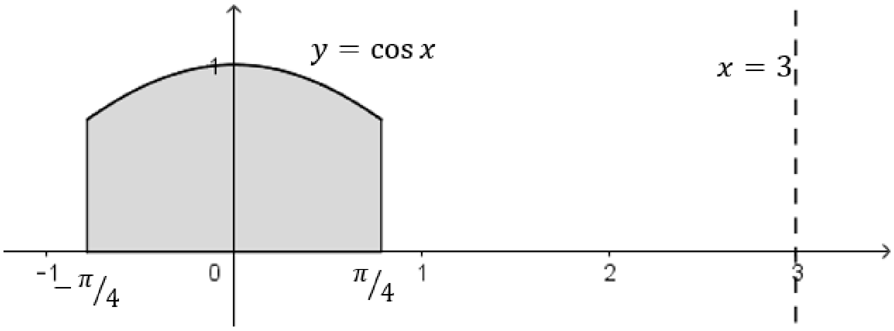
(i) Show that the normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $t^3x - ty = ct^4 - c$. [2]



(ii) The normal at P meets the line $y = -x$ at N and the tangent at P meets $y = -x$ at T. Find the coordinates of N and T, in terms of c and t. [3]

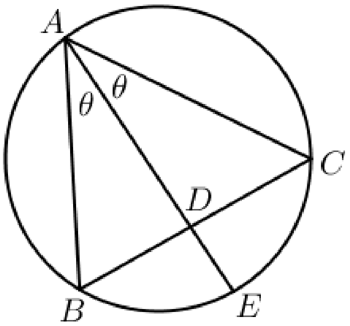
(iii) If O is the origin, show that $OT \times ON = 4c^2$. [2]

(b)



The area bounded by the curve $y = \cos x$ and the x-axis between $x = -\pi/4$ and $x = \pi/4$ is rotated through one revolution about the line $x = 3$. By taking the limiting sum of the volumes of cylindrical shells find the volume of the solid. [3]

(c)



In the diagram, the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E. Copy the diagram on to the Answer Sheet.

(i) Prove that the triangles ABE and ADC are similar. [2]

(ii) Show that $AB \times AC = AD \times AE$. [1]

(iii) Prove that $AD^2 = AB \times AC - BD \times DC$. [2]

End of Question 14

Question 15 (15 marks)

Use a NEW sheet of paper.

(a) Find [3]

$$\int \cos(\log_e x)\,dx$$

(b) The complex numbers represented by z are such that

$$\left|z - 3\sqrt{3} - 3i\right| = 3.$$

(i) Find the maximum value of $|z|$. [1]

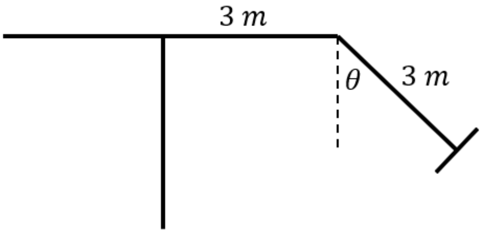
(ii) Find the range of values of $\arg(z)$. [2]

(c)

(i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(a\sec\theta, b\tan\theta)$ is given by $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. [2]

(ii) If the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ cuts the x-axis at A and the y-axis at B, show that $\frac{PA}{PB} = \sin^2\theta$. [3]

(d) An amusement ride (chair-o-plane) is designed with a rigid circular structure, with radius 3 metres, at the top. Chairs are attached to the circumference of the rigid circle with cables, that are 3 metres long, so that as the ride spins the chairs rise up and out from the centre of motion. This motion creates an angle θ between the vertical and the cable. Let the mass of the chair and rider be m kg and the acceleration due to gravity be 10 m/s^2.



(i) Riders find it intensely uncomfortable if angle θ exceeds $45°$. Find the maximum angular speed in radians per second so that the angle will not exceed $45°$. [2]

(ii) Riders tend to be disappointed if the tangential velocity is less than 4 metres per second. Show that the angle created at this velocity is between $21°$ and $22°$. [2]

End of Question 15

Question 16 (15 marks)

Use a NEW sheet of paper.

(a) Given that the polynomial equation $x^4 + 2x^2 + 8x + 5 = 0$ has a real double root, find all roots of the equation. [3]

(b) Let

$$I_n = \int_0^1 \sqrt{x}(1-x)^n dx \quad \text{where } n = 0, 1, 2, \ldots$$

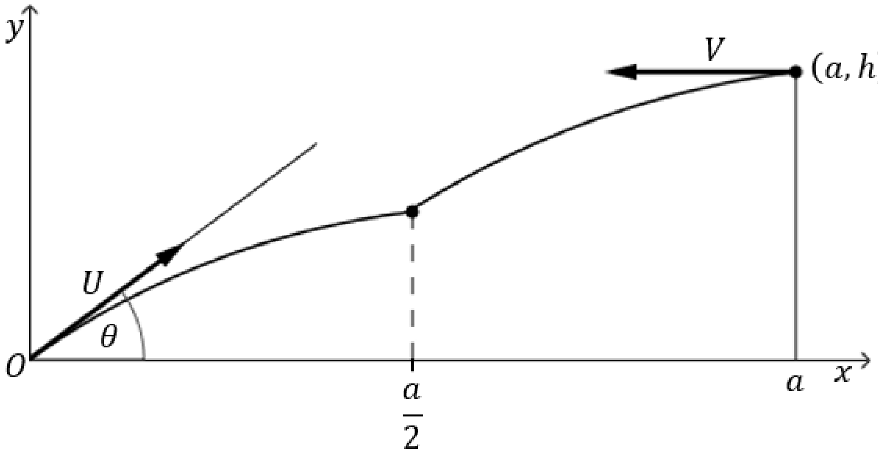
(i) Show that [3]

$$I_n = \frac{2n}{2n+3} I_{n-1}$$

(ii) Hence evaluate [2]

$$\int_0^1 \sqrt{x}(1-x)^3 dx$$

(c)



A gun is aimed so that the shell it fires strikes a target released simultaneously from an aeroplane flying horizontally towards the gun at a speed of V m/s and at a height 'h' metres. The aeroplane was at a horizontal distance 'a' metres from the gun when the target was released, and the shell strikes the target at half this horizontal distance 'a', as shown on the diagram. The initial velocity of the shell is U m/s and the angle of projection is θ. The equations of motion of the shell are:

$$\dot{x} = U\cos\theta \qquad \dot{y} = U\sin\theta - gt$$

$$x = Ut\cos\theta \qquad y = Ut\sin\theta - \frac{1}{2}gt^2$$ (Do NOT prove these.)

(i) Show that the equations of motion of the target are: [3]

$$\dot{x} = -V \qquad \dot{y} = -gt$$

$$x = a - Vt \qquad y = h - \frac{1}{2}gt^2$$

(ii) Show that the gun was aimed at a point h metres vertically above the aeroplane at the instant the target was released. [2]

(iii) Also, show that [2]

$$U = \frac{V}{a}\sqrt{a^2 + 4h^2}$$

End of Question 16

End of Exam

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics Extension 2

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = ? (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ☒ B ☒ C ○ D ○

Student Number: ANSWERS.

Completely fill the response oval representing the most correct answer.

1. A ○ B ● C ○ D ○
2. A ○ B ○ C ● D ○
3. A ○ B ○ C ● D ○
4. A ○ B ○ C ○ D ●
5. A ● B ○ C ○ D ○
6. A ○ B ○ C ○ D ●
7. A ○ B ○ C ○ D ●
8. A ○ B ○ C ● D ○
9. A ○ B ● C ○ D ○
10. A ○ B ● C ○ D ○

(1) Ⓑ $\frac{1}{2}\arg(z)$ and $\sqrt{z}\,|z|$

$\frac{1}{2}\arg(z)+\pi$ and $\sqrt{z}\,|z|$.

(2) Ⓒ Single roots become double roots.

(3) Ⓒ $\delta V = 2y^2\,\delta x.$

$\delta V = 2(16-x^2)\,\delta x$

$V = 2\int_{-4}^{4}(16-x^2)\,dx$

$V = 4\int_{0}^{4}(16-x^2)\,dx.$

(4) Ⓓ $x+y=1$

(5) $\int \cos^2 x(\sin^6 x)\sin x\,dx.$

$= \int \cos^2 x(1-\cos^2 x)^3 \sin x\,dx$

$= \int \cos^2 x(1-3\cos^2 x+3\cos^4 x-\cos^6 x)\sin x\,d$

$= \int (\cos^2 x - 3\cos^4 x + 3\cos^6 x - \cos^8 x)\sin x\,d$

$u = \cos x$

$\underline{du} = -\sin x\,dx.$

$= \int(-u^2+3u^4-3u^6+u^8)\,du$

$= \int u^8\,du + \ldots$

$= \frac{u^9}{9} + \ldots$

$= \frac{\cos^9 x}{9} + \ldots$ Ⓐ

(6) $3x^2+18x-2yy'+27-4y'=0.$

$$y'(-2y-4)=-3x^2-18x-27.$$

$$y'=\frac{3x^2+18x+27}{2y+4}.$$ Ⓓ

(7) $y=\sin^{-1}(\cos(0))$ $\qquad$ $y=\sin^{-1}(\cos(2\pi))$

$y=\sin^{-1}(1)$ $\qquad$ $y=\sin^{-1}(1)$

$y=\frac{\pi}{2}.$ $\qquad$ $y=\frac{\pi}{2}.$

Ⓓ.

(8)

$$\frac{dx}{dt}=k(b-x)$$

$$\int\frac{1}{b-x}\,dx=\int k\,dt.$$

$$-\ln(b-x)=kt+C.$$

when $t=0$ $\quad x=0.$

$$-\ln(b)=C.$$

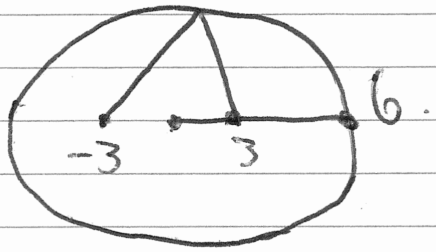
$$-\ln(b-x)=kt-\ln(b).$$

$$-kt=\ln\left(\frac{b-x}{b}\right).$$

$$\frac{b-x}{b}=e^{-kt}$$ $\qquad$ $x=b(1-e^{-kt}).$

Ⓒ

(9)



$ae = 3$

$a = 6$

So $e = \frac{1}{2}$.

$b^2 = a^2(1-e^2)$

$b^2 = 36(1-\frac{1}{4})$

$b^2 = 27$ and $a^2 = 36$.

Ⓑ

(10) $\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$

$\tan\theta = \frac{y}{200}$

$y = 200\tan\theta$

$\frac{dy}{d\theta} = 200\sec^2\theta$

$\frac{dy}{d\theta} = 200 \times \left(\frac{4}{3}\right)$

$\frac{dy}{d\theta} = \frac{800}{3}$.

$100 = \frac{800}{3} \times \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = \frac{3}{8} \simeq 0.375$ rad/s.

Ⓑ

Question 11

(a) (i) $z_1 = \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\, i$

$$|z_1| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{\tfrac{3}{2}}\right)^2}$$

$$= \sqrt{2}$$

$$\arg(z_1) = \tan^{-1}\left(\frac{\sqrt{\frac{3}{2}}}{\frac{1}{\sqrt{2}}}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{3} \qquad \therefore z_1 = \sqrt{2}\operatorname{cis}\frac{\pi}{3}$$

Mostly well done but some students need to be more careful when finding the argument.

(ii) If z_1 is a solution, then $(z_1)^8 - 8(z_1)^2 = 0$.

$$\text{LHS} = \left(\sqrt{2}\operatorname{cis}\frac{\pi}{3}\right)^8 - 8\left(\sqrt{2}\operatorname{cis}\frac{\pi}{3}\right)^2$$

$$= (\sqrt{2})^8 \operatorname{cis}\frac{8\pi}{3} - 8(\sqrt{2})^2 \operatorname{cis}\frac{2\pi}{3}$$ using De Moivre's Theorem

$$= 16\operatorname{cis}\frac{2\pi}{3} - 16\operatorname{cis}\frac{2\pi}{3}$$

$$= 0$$

$$= \text{RHI} \qquad \therefore z_1 \text{ is a solution.}$$

The quality of solutions varied. Solutions should state a conclusion at the end. Some solutions involved more calculations than needed.

Question 11 (continued)

(b) (i) $\int x^2 \cos(x^3+1)\,dx$ let $u = x^3+1$

$du = 3x^2\,dx$

$I = \frac{1}{3}\int \cos u\,du$

$= \frac{1}{3}\sin u + c$

$\therefore I = \frac{1}{3}\sin(x^3+1) + c$

Don't forget to express the answer in terms of x

(ii) $\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x+1)+4}}$

$= \int \frac{dx}{\sqrt{4-(x-1)^2}}$

$= \sin^{-1}\left(\frac{x-1}{2}\right) + c$

Mostly well done.

(c) (i) $(1-i)(5+i)^2 = (1-i)(25+10i+i^2)$

$= (1-i)(24+10i)$

$= 24 - 24i + 10i - 10i^2$

$= 34 - 14i$

Well done.

(ii) $\arg(34-14i) = \arg\left((1-i)(5+i)^2\right)$

$= \arg(1-i) + \arg\left((5+i)^2\right)$

$= \arg(1-i) + 2\arg(5+i)$

$\tan^{-1}\left(\frac{-14}{34}\right) = \tan^{-1}(-1) + 2\tan^{-1}\left(\frac{1}{5}\right)$

Question 11 (continued)

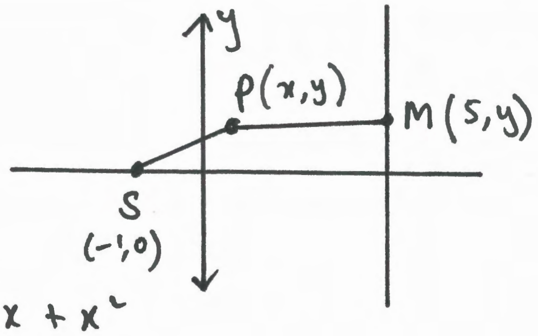
(c) (ii) (continued)

$$-\tan^{-1}\left(\frac{7}{17}\right) = -\pi + 2\tan^{-1}\left(\frac{1}{5}\right)$$

$$\therefore \quad \tan^{-1}\left(\frac{7}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \pi$$

Many students did not use part (i) which is required ("Hence"). Some solutions could be communicated more effectively.

(d) (i) $\frac{PS}{PM} = e \qquad PS^2 = e^2 PM^2$



$$(x+1)^2 + y^2 = \frac{1}{4}(5-x)^2$$

$$4x^2 + 8x + 4 + 4y^2 = 25 - 10x + x^2$$

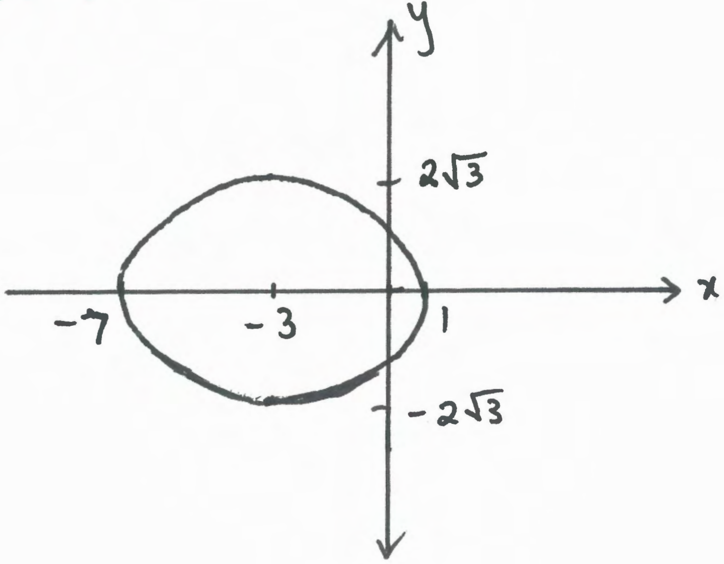
$$3x^2 + 18x + 4y^2 = 21$$

$$3(x+3)^2 + 4y^2 = 21 + 27$$

$$\frac{(x+3)^2}{16} + \frac{y^2}{12} = 1 \qquad \therefore \text{ Centre} = (-3, 0)$$

A number of students used $ae = -1$ and $\frac{a}{e} = 5$. However, this works only when the centre is (0, 0).

(ii)



D: $-7 \le x \le 1$

R: $-2\sqrt{3} \le y \le 2\sqrt{3}$

Students had some problems with the range. Also, the shape of the ellipse can be improved for some students.

Question 12 (15 Marks)

a) i) $$\frac{5x^2-11}{(x+2)(x-1)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$5x^2-11 \equiv A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

when $x=-2 \Rightarrow A=1$
when $x=1 \Rightarrow C=-2$
when $x=0 \Rightarrow B=4$.

$$\therefore \frac{5x^2-11}{(x+2)(x-1)^2} \equiv \frac{1}{(x+2)} + \frac{4}{(x-1)} - \frac{2}{(x-1)^2}.$$

ii) $$\int \frac{5x^2-11}{(x+2)(x-1)^2}\, dx$$

$$= \int \left[\frac{1}{(x+2)} + \frac{4}{(x-1)} - \frac{2}{(x-1)^2}\right] dx$$

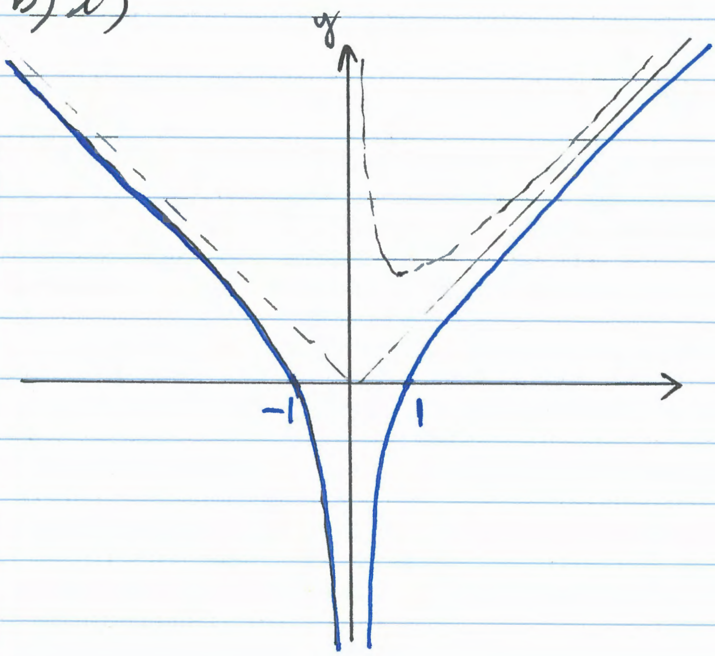
$$= \ln|x+2| + 4\ln|x-1| - 2\int (x-1)^{-2}\, dx$$

$$= \ln|x+2| + 4\ln|x-1| + \frac{2}{(x-1)} + C$$

★ Most errors that were made in this part was incorrect values for A, B and C.

Question 12 (continued)

b) i)

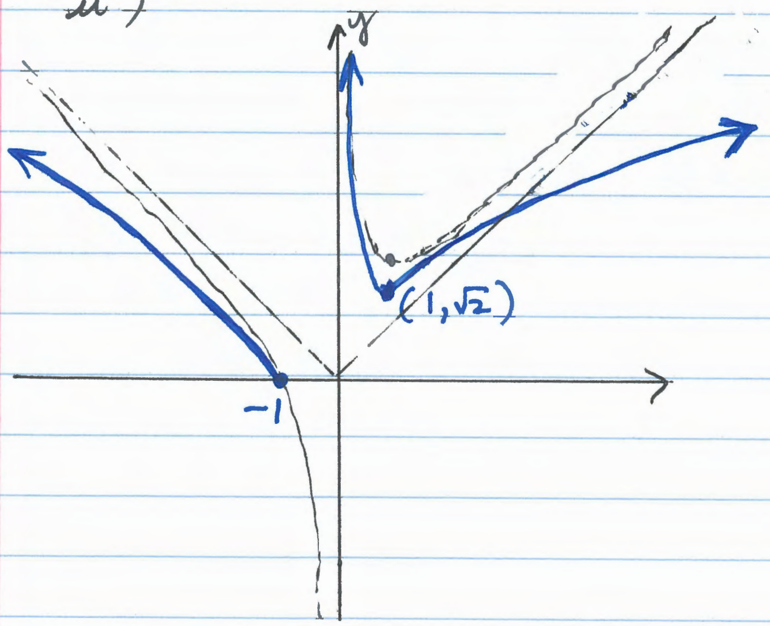


$y = f(-|x|)$

* Commonly mistaken for $y = -f|x|$

* x is all reals
$\therefore f(-|x|)$ consists of the graph for $x \leq 0$ and its reflection in the y-axis.

ii)

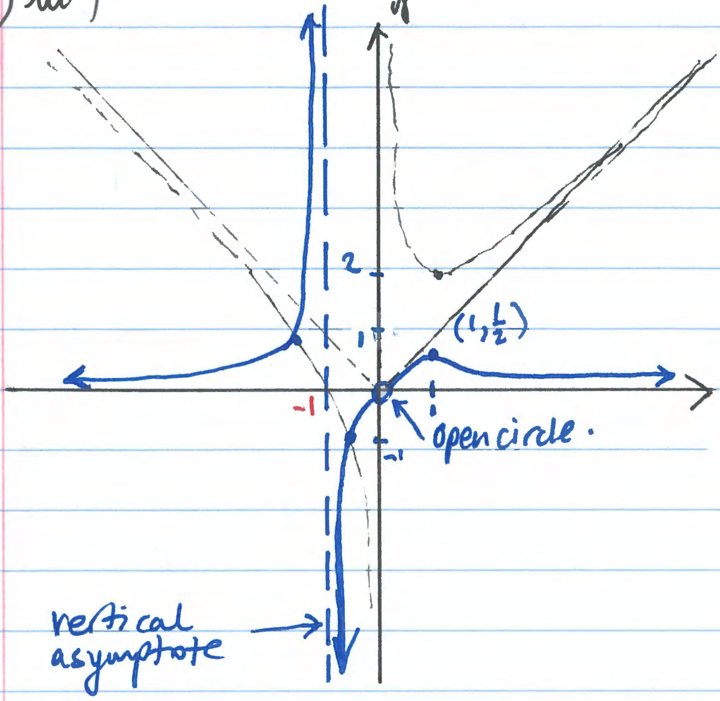


$y = \sqrt{f(x)}$

* Most students lost 2 marks for incorrectly sketching part i). Possibly due to not really thinking about the function given and just rushing to put pen to paper.

Question 12 (continued)

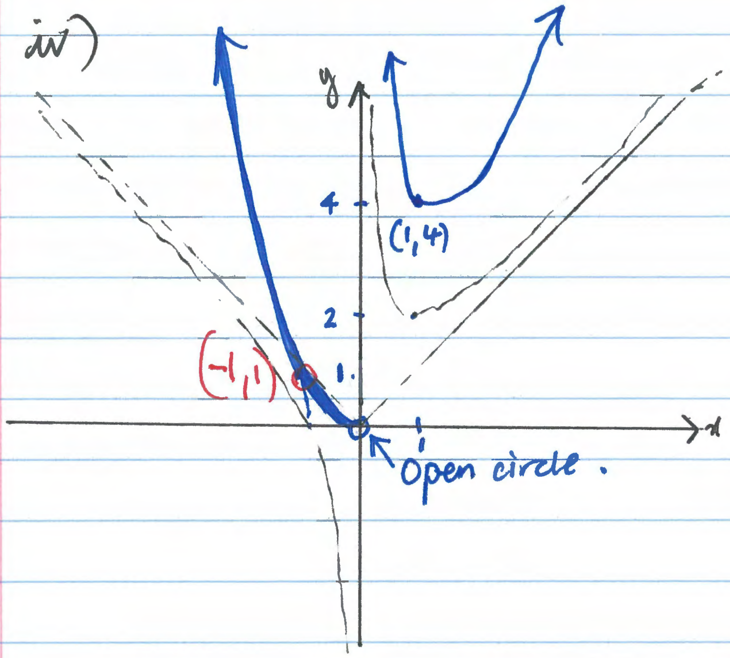
b) iii)



$y = \frac{1}{f(x)}$

- As $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- As $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$.

iv)



$y = 2^{f(x)}$

* If students did not include critical points in their sketches, they lost one mark overall.

Question 12 (continued)

c). $t = \tan\frac{x}{2}$ when $x=0$, $t=0$

$dt = \sec^2\frac{x}{2}\,dx$ when $x=\frac{\pi}{2}$, $t=1$.

$dx = \frac{2dt}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x+3\sin x}$$

$$= \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{(1+t^2)}$$

$$= \int_0^1 \frac{2dt}{5(1+t^2)+4(1-t^2)+3(2t)}$$

$$= \int_0^1 \frac{2dt}{5+5t^2+4-4t^2+6t}$$

$$= \int_0^1 \frac{2dt}{t^2+6t+9}$$

$$= \int_0^1 \frac{2dt}{(t+3)^2}$$

$$= 2\int_0^1 (t+3)^{-2}\,dt$$

$$= \left[\frac{-2}{t+3}\right]_0^1$$

$$= -\frac{2}{4}+\frac{2}{3}$$

$$= \frac{1}{6}.$$

Generally, completed well!

☆ Students lost one mark if they made an algebraic error in their solution.

13 (a) (i) $w^3 = 1$

$(w^3)^2 = 1^2$

$(w^2)^3 = 1$

$\therefore$ w^2 is the other complex root since $w \neq 1$

(iii) $1 + 2w^2 + 3w + 2w + 4w^3 + 6w^2$
$+ 3w^2 + 6w^4 + 9w^3$

$= \overset{14}{8} + 11w + 11w^2$

$= \cancel{3w + 3w^2}$ 14

$= \cancel{6 + 3w}$

(ii) $1 + w + w^2 = w^3 + w + w^2$

$= w(w^2 + 1 + w)$

$\therefore 1 + w + w^2 = 0$ since $w \neq 1$

(iii) If $\alpha = 1$, $p + q + r = 0$

" $\alpha = w$, $pw^5 + qw + r = pw^2 + qw + r = 0$

" $\alpha = w^2$, $pw^{10} + qw^2 + r = pw + qw^2 + r = 0$

$\therefore (p + q + r)(pw^2 + qw + r)(pw + qw^2 + r)$
for any root of $x^3 - 1 = 0$

Many students only considered the case of $\alpha = 1$.

(b) (i) $(5 + 4i - 3 + 2i)\,\text{cis}\left(-\frac{\pi}{3}\right) + 3 - 2i$

$= (2 + 6i)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 3 - 2i$

$= 1 + \sqrt{3}i + 3i + 3\sqrt{3} + 3 - 2i$

$= 4 + 3\sqrt{3} + (1 - \sqrt{3})i$

Many students forgot to add back the $3 - 2i$

(ii) $5 + 4i - 3 + 2i + 4 + 3\sqrt{3} + (1 - \sqrt{3})i - 3 + 2i + 3 - 2i$

$= 6 + 3\sqrt{3} + (7 - \sqrt{3})i$

(c) (i) $\alpha^2 = -\beta^2 = \gamma^2$

$\alpha, \beta, \gamma \neq 0$

Either $\beta^2 < 0$ or $\alpha^2, \gamma^2 < 0$ ⟸

Many students omitted this from their proof.

Since the complex roots of a real polynomial occur in conjugate pairs

$\therefore$ α and β are not real

(ii) Let $\alpha = a + ib$, $b \neq 0$

$(a + ib)^2 = -\beta^2$

$a^2 - b^2 = -\beta^2$ $2ab = 0$

$\therefore a = 0$

$\therefore$ α and γ are purely imaginary

Many students asserted rather than showed that α and γ are purely imaginary.

(iii) let $\alpha = ib$ and $\gamma = -ib$

$\alpha^2\beta^2\gamma^2 = (-8)^2$

$\alpha^2\beta^2\gamma^2 = 64$

~~$\alpha\beta\gamma =$~~

$\beta^6 = 64$

$\beta = 2$ since $\alpha\gamma = b^2$

$\beta = 2$ is most feasible

$A = -(-2 + ib - ib)$

$= 2$

$\alpha\beta + \alpha\gamma + \beta\gamma = B$

$-2ib + b^2 + 2ib = B$

$\therefore B = 4$

14 (a)(i) $x\frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$= -\frac{\frac{c}{t}}{ct}$$

$$= -\frac{1}{t^2}$$

$m_n = t^2$

$y - \frac{c}{t} = t^2(x - ct$

$ty - c = t^3x - ct^4$

$t^3x - ty = ct^4 - c$

(ii) at N, $t^3x + tx = ct^4 - c$

$x(t^3 + t) = c(t^4 - 1)$

$x = \frac{c(t^2-1)(t^2+1)}{t(t^2+1)}$ $\qquad$ $y = \frac{c(1-t^2)}{t}$

$= \frac{c(t^2-1)}{t}$

at P, eqn of tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

at T, $-x - \frac{c}{t} = -\frac{x}{t^2} + \frac{c}{t}$

$-t^2x - ct = -x + ct$

$x - t^2x = 2ct$

$x(1 - t^2) = 2ct$

$x = \frac{2ct}{1-t^2}$ $\qquad$ $y = \frac{2ct}{t^2-1}$

(iii) $OT^2 \times ON^2 = \left(\frac{4c^2t^2}{(1-t^2)^2} + \frac{4c^2t^2}{(t^2-1)^2}\right)\left(\frac{c^2(t^2-1)^2}{t^2} + \frac{c^2(1-t^2)^2}{t^2}\right)$

$$= \frac{8c^2t^2}{(1-t^2)^2} \times \frac{2c^2(1-t^2)^2}{t^2}$$

$$= 16c^4$$

$OT \times ON = 4c^4$

(b) $V_{shell} = \pi(R^2 - r^2)h$

$= \pi\left\{(3-x)^2 - (3-x-\delta x)^2\right\} y$

$= 2\pi(3-x)y\,\delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi(3-x)y\,\delta x$

$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3-x)y\,dx$

$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3-x)\cos x\,dx$ ← Many presumed incorrectly that this is an even function

$= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3\cos x - x\cos x)\,dx$

$= 4\pi \int_{0}^{\frac{\pi}{4}} 3\cos x\,dx$

$= 12\pi \left[\sin x\right]_0^{\frac{\pi}{4}}$

$= 12\pi \times \frac{1}{\sqrt{2}}$

$= 6\sqrt{2}\pi$

(c) (i) In $\triangle ABE$ and $\triangle ADC$

$B\hat{A}E = C\hat{A}D$ (given)

$A\hat{E}B = A\hat{C}D$ ($\angle$s in same segment)

$\therefore \triangle ABE \,|||\, \triangle ADC$ (equiangular)

(ii) $\frac{AB}{AD} = \frac{AE}{AC}$ (corresponding $\angle$s in similar $\triangle$s)

(iii) $AD.DE = BD.DC$ (products of intersecting chords)

$AB \times AC - BD.DC = AD.AE - AD.DE$

$= AD(AE - DE)$

$= AD^2$ Q.E.D.

Many assumed incorrectly that $A\hat{D}B = A\hat{D}C = 90°$

Question 15

(a)

$$I = \int \cos(\ln x) \times 1 \, dx \qquad \underline{u = \cos(\ln x)} \qquad v = x$$

$$I = x\cos(\ln x) + \int \sin(\ln x) \, dx \qquad u' = -\frac{1}{x}\sin(\ln x) \qquad \underline{v' = 1}$$

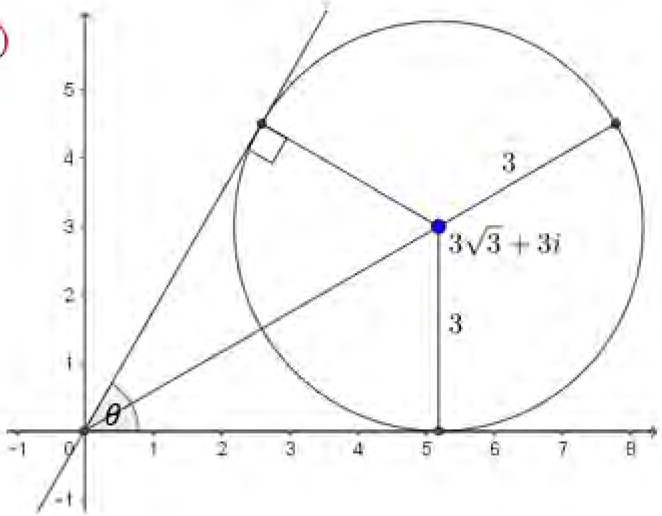
$$I = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x) \, dx \qquad \underline{u = \sin(\ln x)} \qquad v = x$$

$$I = x\cos(\ln x) + x\sin(\ln x) + 2C - I \qquad u' = \frac{1}{x}\cos(\ln x) \qquad \underline{v' = 1}$$

$$2I = x\cos(\ln x) + x\sin(\ln x) + 2C$$

$$I = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$$

(b)(i)



$$\max|z| = 3 + \sqrt{\left(3\sqrt{3}\right)^2 + 3^2}$$

$$\max|z| = 3 + \sqrt{36}$$

$$\max|z| = 9$$

(ii) $y = mx \qquad (x - 3\sqrt{3})^2 + (y - 3)^2 = 9$

$$(x - 3\sqrt{3})^2 + (mx - 3)^2 = 9$$

$$x^2 - 6\sqrt{3}x + 27 + m^2x^2 - 6mx + 9 = 9$$

$$(m^2 + 1)x^2 - (6m + 6\sqrt{3})x + 27 = 0$$

$$\Delta = 0$$

$$(6m + 6\sqrt{3})^2 - 4(m^2 + 1)(27) = 0$$

$$m^2 + 2\sqrt{3}m + 3 - 3m^2 - 3 = 0$$

$$2m^2 - 2\sqrt{3}m = 0$$

$$m(m - \sqrt{3}) = 0$$

$$m = 0\,, m = \sqrt{3}$$

$$\tan\theta = 0, \sqrt{3}$$

$$\theta = 0, \frac{\pi}{3}$$

$$0 \le \theta \le \frac{\pi}{3}$$

(c)(i) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$

$$y'=\frac{b^2x}{a^2y}$$

$$m_T=\frac{b\sec\theta}{a\tan\theta}\qquad P(a\sec\theta\,,b\tan\theta)$$

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$$

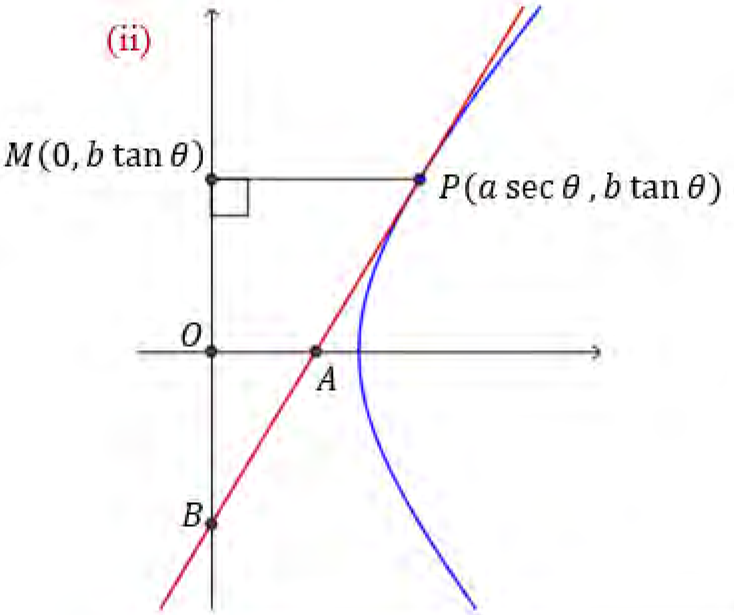
$$bx\sec\theta\ -ay\tan\theta=ab\sec^2\theta-ab\tan^2\theta$$

$$bx\sec\theta\ -ay\tan\theta=ab(1+\tan^2\theta-\tan^2\theta)$$

$$bx\sec\theta\ -ay\tan\theta=ab$$

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

(ii)



$$A\left(\frac{a}{\sec\theta},0\right)$$

$$B\left(0,-\frac{b}{\tan\theta}\right)$$

$$\frac{PA}{PB}=\frac{MO}{MB}\ (ratio\ of\ intercepts)$$

$$=\frac{b\tan\theta}{b\tan\theta+\dfrac{b}{\tan\theta}}$$

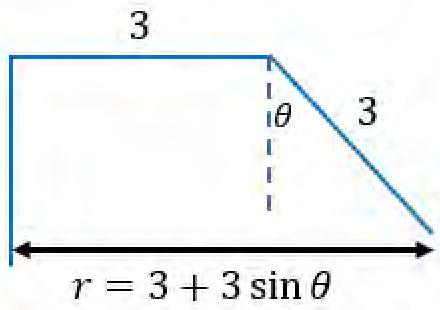
$$=\frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}}$$

$$=\frac{\sin\theta}{\cos\theta}\times\cos\theta\sin\theta$$

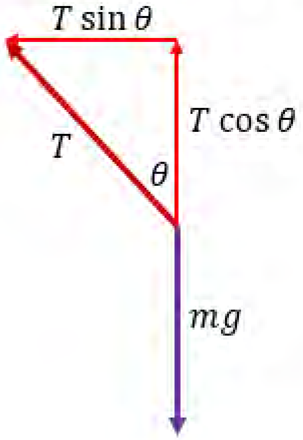
$$=\sin^2\theta$$

This was the most abandoned question. Students got themselves very confused with square root symbols. In conics questions always try to relate diagonal distances to horizontal or vertical distances.

(e)(i)



Many students didn't use the full radius. Only using 3 sin θ or 3.



$$T\sin\theta = mr\omega^2$$
$$T\cos\theta = mg$$

$$\tan\theta = \frac{r\omega^2}{g}$$

$$\omega^2 = \frac{g\tan\theta}{r}$$

$$\omega^2 = \frac{10\tan 45°}{3 + 3\sin 45°}$$

$$\omega^2 = \frac{10}{3 + \frac{3}{\sqrt{2}}}$$

$$\omega^2 = \frac{10\sqrt{2}}{3\sqrt{2} + 3}$$

$$\omega^2 = \frac{10\sqrt{2}}{3(\sqrt{2} + 1)}$$

$$\omega^2 = \frac{10(2 - \sqrt{2})}{3}$$

$$\omega = \sqrt{\frac{10(2 - \sqrt{2})}{3}}$$

$$\omega \approx 1.397\ rad/sec$$

(ii) $$T\sin\theta = \frac{mv^2}{r}$$

$$T\cos\theta = mg$$

$$\tan\theta = \frac{v^2}{gr}$$

$$gr\tan\theta = v^2$$

$$10(3 + 3\sin\theta)\tan\theta = 16$$

$$15\tan\theta + 15\sin\theta\tan\theta - 8 = 0$$

Let $f(\theta) = 15\tan\theta + 15\sin\theta\tan\theta - 8$

$f(21^\circ) \approx -0.179$

$f(22^\circ) \approx 0.331$

Thus $21^\circ < \theta < 22^\circ$

Question 16

(a) Let $P(x) = x^4 + 2x^2 + 8x + 5$ and α be the double root.

$P(\alpha) = 0$ and $P'(\alpha) = 0$

$P'(x) = 4x^3 + 4x + 8$

$P'(\alpha) = 0 \quad \therefore \quad 4\alpha^3 + 4\alpha + 8 = 0$

$\alpha^3 + \alpha + 2 = 0$

By inspection $\alpha = -1$ is a root.

$$(\alpha+1)(\alpha^2 - \alpha + 2) = 0$$

$\downarrow$

$\Delta = 1 - 8 < 0$

$\therefore$ Complex roots

Since -1 is the only real root of $P'(x) = 0$, -1 is the double root.

$$P(x) = (x+1)^2 \times Q(x)$$

$$= (x^2 + 2x + 1)(x^2 - 2x + 5) \text{ by inspection}$$

$x^2 - 2x + 5 = 0$ when $(x-1)^2 = 1 - 5$

$x - 1 = \pm 2i$

$x = 1 \pm 2i$

$\therefore$ The roots are $-1, -1, 1+2i$ and $1-2i$.

Most students found the double root. However, Some students found the roots of $P'(x)$ not $P(x)$.

Question 16 (cont.)

(b) (i) $I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx$ for $n = 0, 1, 2 \ldots$

let $u = (1-x)^n \qquad v' = \sqrt{x}$

$u' = n(1-x)^{n-1} \times \textcircled{-1} \qquad v = \frac{2x^{\frac{3}{2}}}{3}$

$\Uparrow$ Be careful

$$I_n = \left[\frac{2x\sqrt{x}}{3}(1-x)^n\right]_0^1 - \int_0^1 -n(1-x)^{n-1} \times \frac{2}{3}x\sqrt{x}\,dx$$

$$= 0 - 0 + \frac{2n}{3}\int_0^1 x\sqrt{x}(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 (1-x-1)\sqrt{x}(1-x)^{n-1}\,dx \quad *$$

$$= -\frac{2n}{3}\int_0^1 (1-x)\sqrt{x}(1-x)^{n-1}\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx$$

$$= -\frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^n\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}(1-x)^{n-1}\,dx$$

$$I_n = -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$$

$$3I_n = -2nI_n + 2nI_{n-1}$$

$$I_n(3+2n) = 2nI_{n-1}$$

$$\therefore I_n = \frac{2n}{2n+3}I_{n-1}$$

Most students recognised the option to use integration by parts. Many students could not get to the line * (see above).

Question 16 (cont.)

(b) (ii)

$$I_3 = \frac{2\times3}{2\times3+3} I_2$$

$$= \frac{6}{9} \times \frac{2\times2}{2\times2+3} I_1$$

$$= \frac{6}{9} \times \frac{4}{7} \times \frac{2\times1}{2\times1+3} I_0$$

$$= \frac{8}{21} \times \frac{2}{5} \times \int_0^1 \sqrt{x}\, dx$$

$$= \frac{16}{105} \times \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^1$$

$$= \frac{16}{105} \left(\frac{2}{3} - 0 \right)$$

$$\therefore I_3 = \frac{32}{315}$$

Most students got this correct though

Some did not make use of I_0 to

simplify the integral as much as possible.

Question 16

(c) (i) $\ddot{x} = 0$

$\dot{x} = c$

when $t = 0$ $\quad \underset{\substack{\uparrow \\ \text{since travelling} \\ \leftarrow}}{\dot{x} = -V}$

$\therefore c = -V \quad \underline{\dot{x} = -V}$

$x = -Vt + C$

when $t = 0$, $x = a$

$\therefore C = a \quad \underline{x = a - Vt}$

$\ddot{y} = -g$

$\dot{y} = -gt + c$

when $t = 0$, $\dot{y} = 0$

$\therefore c = 0 \quad \underline{\dot{y} = -gt}$

$y = -\frac{gt^2}{2} + c$

when $t = 0$, $y = h$

$\therefore c = h \quad y = -\frac{gt^2}{2} + h$

i.e. $\underline{y = h - \frac{gt^2}{2}}$

The equations of motion were given and needed to be derived, starting from $\ddot{x} = 0$ and $\ddot{y} = -g$. Some students did not clearly show how the constants involved were determined. 3 marks of working were expected. Stating what is given in the question is not worthy of a mark.

(ii) Shell strikes target when $x = \frac{a}{2}$.

$\therefore \frac{a}{2} = Ut \cos\theta \quad$ i.e. when $t = \frac{a}{2U\cos\theta}$

At this time the heights of the shell and target are the same

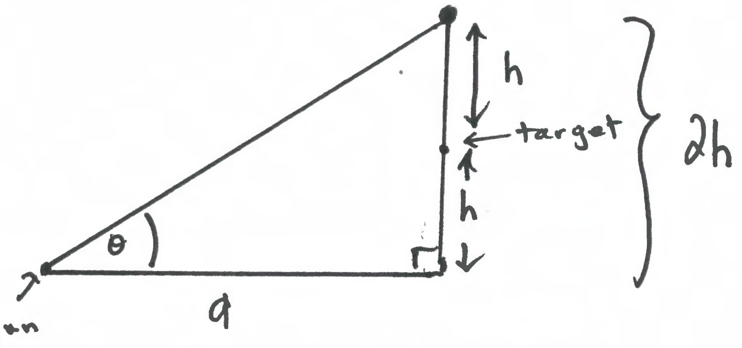
$$\therefore U\left(\frac{a}{2U\cos\theta}\right)\sin\theta - \frac{g}{2}\left(\frac{a}{2U\cos\theta}\right)^2 = h - \frac{g}{2}\left(\frac{a}{2U\cos\theta}\right)^2$$

Question 16 (continued)

(ii) continued

$$\frac{a}{2\cos\theta} \times \sin\theta = h$$

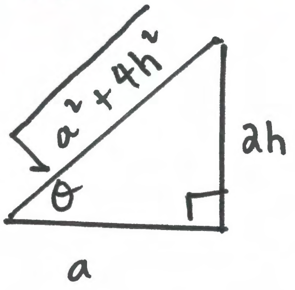
$$\tan\theta = \frac{2h}{a}$$



∴ The gun is aimed at a point h metres above the plane.

Solutions were varied and students needed to realise the significance of <u>2h</u> in their working.

(iii) Since $\tan\theta = \frac{2h}{a}$



For target, $x = \frac{a}{2}$ when $t = \frac{a}{2u\cos\theta}$

$$\frac{a}{2} = a - V\left(\frac{a}{2u\cos\theta}\right)$$

$$\frac{aV}{2u\cos\theta} = \frac{a}{2}$$

$$\frac{V}{u\cos\theta} = 1$$

From diagram above

$$\cos\theta = \frac{a}{\sqrt{a^2+4h^2}}$$

$$u = \frac{V}{\left(\frac{a}{\sqrt{a^2+4h^2}}\right)}$$

$$\therefore u = \frac{V\sqrt{a^2+4h^2}}{a}$$

Students needed to connect the part (ii) results to this question.