Sydney Grammar School

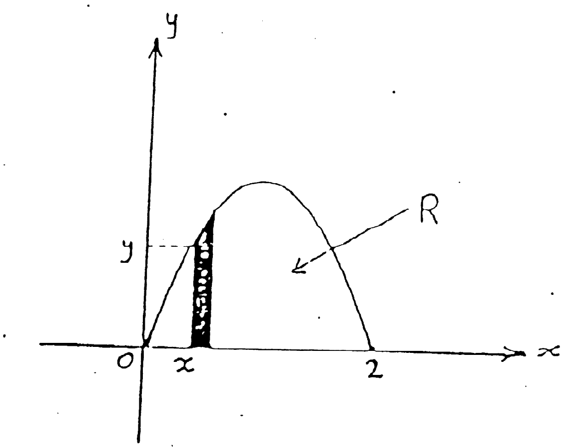
4 unit mathematics

Trial HSC Examination 1992

1. (a) Find $\int \frac{dx}{\sqrt{x^2+4x+8}}$
(b) (i) Use partial fractions to show that $\int_0^1 \frac{dx}{(x+2)(2x+1)} = \frac{1}{3}\ln 2$
(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{3}{4+5\sin x}\, dx$ using the substitution $t = \tan\frac{x}{2}$
(c) (i) Given $I_n = \int_0^1 x^n e^{2x}\, dx$, where n is a positive integer, use integration by parts to show that $I_n = \frac{1}{2}(e^2 - nI_{n-1})$
(ii) Hence evaluate $\int_0^1 x^4 e^{2x}\, dx$

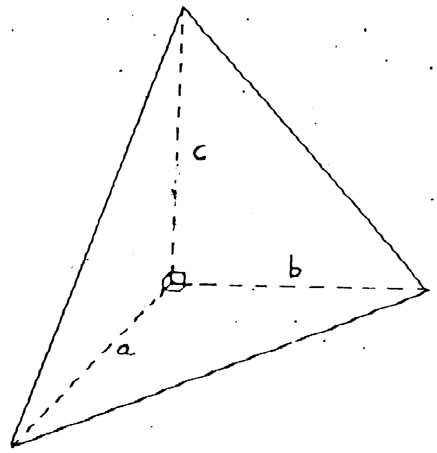
2. (a) (i) Write the complex number $-\sqrt{3}+i$ in modulus-argument form.
(ii) Hence use de Moivre's theorem to find $(-\sqrt{3}+i)^{10}$ in the form $a+bi$, where $a, b \in \mathbb{R}$.
(b) Sketch each of the following regions in separate Argand diagrams:
(i) $-1 < \Re(z) < 2$ and $0 < \Im(z) < 3$
(ii) $z\overline{z} - (1-i)z - (1+i)\overline{z} < 2$
(iii) $0 < \arg|(1-i)z| < \frac{\pi}{6}$
(c) (i) Find the square roots of the complex number $-3+4i$
(ii) Find the roots of the quadratic equation $x^2 - (4-2i)x + (6-8i) = 0$

3. (a)



The diagram shows the region R bounded by the curve $y = 2x - x^2$ and the x-axis. The typical strip shaded has width δx.
(i) Show that, when this strip is rotated about the y-axis, the cylindrical shell formed has approproximate volume $2\pi xy\delta x$.

(ii) Hence determine the volume of the solid formed when the region R is rotated about the y-axis.
(b)



By taking triangular slices parallel to the base, show that the tetrahedron (i.e., triangular pyramid) sketched above has volume $\frac{1}{6}abc$ cubic units.

4. **(a)** A particle of mass $m$kg is moving along the x-axis under the influence of a propelling force of $\frac{P}{v}$ Newtons (whose P is a positive constant and v is the speed of the particle in metres per second), and experiences a resistance of Kv^2 Newtons (where K is a positive constant).
(i) Show that $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$, where t is time in seconds.
(ii) If the magnitude of the propelling force is equal to the magnitude of the resistance at speed u metres per second, show that $K = \frac{P}{u^3}$.
(iii) Show that $\frac{dv}{dx} = \frac{P}{m}(\frac{1}{v^2} - \frac{v}{u^3})$.
(iv) Suppose the particle has initial speed $\frac{u}{3}$ metres per second. Show that the distance travelled in accelerating to a speed of $\frac{2u}{3}$ metres per second is $\frac{mu^3}{3P}\ln(\frac{26}{19})$ metres.
(b) A car travels at 54km/h around a banked circular bend of radius 90 metres.
(i) Draw a diagram showing the weight, the normal reaction and the sideways frictional force acting on the car.
(ii) Show that the road is banked at an angle of approximately 14° to the horizontal if there is no tendency for this car to slip sideways. (Take $g = 10\text{m/s}^2$).
(iii) Find, in Newtons correct to two significant figures, the sideways frictional force exerted by the road on the wheels of a second car of mass 1.2 tonnes which travels the same bend at 72km/h. (Take $g = 10\text{m/s}^2$).

5. **(a)** Find any x-intercepts and stationary points on the curve $y = x^2(x-3)$. Hence sketch the curve.
(b) By considering the sketch drawn in part **(a)**, draw a sketch on separate diagrams of each of the following curves:
(i) $y = |x^2(x-3)|$,

(ii) $|y| = x^2(x-3)$,
(iii) $y = \frac{1}{x^2(x-3)}$,
(iv) $y = \frac{1}{x^2(|x|-3)}$.
(c) For what values of c does the equation $x^2(x-3) = c$ have one real root? (Give reasons for your answer).

6. (a) (i) Find, in the form $a + ib$, where a and b are real, the four fourth roots of -16.
(ii) Hence write $z^4 + 16$ as a product of two quadratic factors with real coefficients.
(iii) Let α be the fourth root of -16 whose principal argument lies between 0 and $\frac{\pi}{2}$. Show that $\alpha + \frac{\alpha^3}{4} + \frac{\alpha^5}{16} + \frac{\alpha^7}{64} = 0$.
(b) Consider the sequence defined by:

$$\begin{cases} u_1 &= 12, \\ u_2 &= 30, \\ u_n &= 5u_{n-1} - 6u_{n-2}, \text{ for } n \geq 3. \end{cases}$$

(i) Determine the values of u_3 and u_4.
(ii) Show that $u_n = 2 \times 3^n + 3 \times 2^n$ for $n = 1$ <u>and</u> $n = 2$.
(iii) If $u_k = 2 \times 3^k + 3 \times 2^k$ <u>and</u> $u_{k+1} = 2 \times 3^{k+1} + 3 \times 2^{k+1}$, where k is a positive integer, prove that $u_{k+2} = 2 \times 3^{k+2} + 3 \times 2^{k+2}$.
(iv) What conclusion may be reached as a result of parts **(ii)** and **(iii)**?

7. (a) The polynomial equation $x^5 - ax^2 + b = 0$ has a multiple root. Show that $108a^5 = 3125b^3$.
(b) (i) Write down the expansions of $\sin(A+B)$ and $\sin(A-B)$ and deduce that $2\sin B \cos A = \sin(A+B) - \sin(A-B)$.
(ii) Use the result from **(i)** to show that $2\sin x(\cos 2x + \cos 4x + \cos 6x) = \sin 7x - \sin x$.
(iii) Hence show that
(α) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$,
(β) $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$.
(c) Suppose b and c are positive integers and $a = b = c$.
(i) Use the binomial expansion of $(b+c)^n$, where n is a positive integer, to show that $a^n - b^{n-1}(b + cn)$ is divisible by c^2.
(ii) Hence show that $5^{42} - 2^{48}$ is divisible by 9.

8. (a) Let $I_1 = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$ and let $I_2 = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$.
(i) Using the substitution $u = \pi - x$, show that $I_1 = I_2$.
(ii) Show that $I_1 + I_2 = \frac{\pi^2}{2}$.
(iii) Hence evaluate I_1.
(b) (i) Show that $\frac{x^2}{x^4+x^2+1} \leq \frac{1}{3}$ for all real values of x.
(ii) Determine the range of $y = \tan^{-1}(\frac{1}{1+x^2})$, and the range of $y = \tan^{-1}(\frac{x^2}{1+x^2})$.

(iii) Show that $\tan^{-1}(\frac{1}{1+x^2}) + \tan^{-1}(\frac{x^2}{1+x^2}) = \tan^{-1}(1 + \frac{x^2}{1+x^2+x^4})$.
(iv) Hence determine the range of $y = \tan^{-1}(\frac{1}{1+x^2}) + \tan^{-1}(\frac{x^2}{1+x^2})$.
(c) The lengths of the sides of a triangle form an arithmetic progression and the largest angle of the triangle exceeds the smallest by $90°$. Find the ratio of the lengths of the sides.