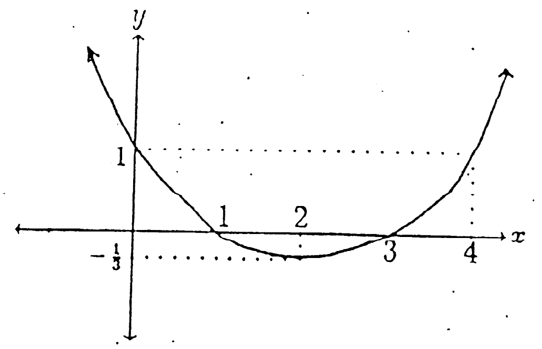
Sydney Grammar School

4 unit mathematics

Trial HSC Examination 1993

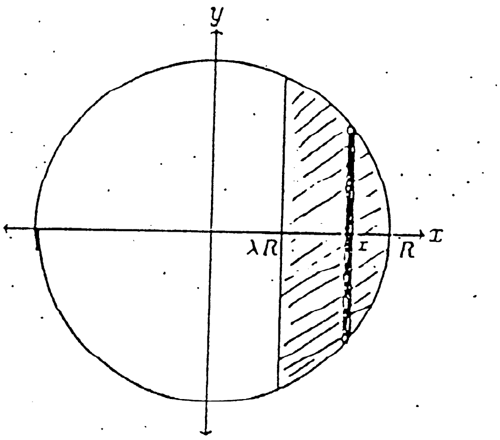
1. **(a)** Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x \cos x \; dx$.
(b) Evaluate $\int_0^1 \frac{x^2}{x+1} \; dx$.
(c) **(i)** Express $\frac{1}{16-u^2}$ in the form $\frac{A}{4-u} + \frac{B}{4+u}$
(ii) Use the substitution $u = \sqrt{16-x}$ to evaluate the integral $\int_7^{12} \frac{4}{x\sqrt{16-x}} \; dx$.
(d) Let $I_n = \int_0^1 x^n e^x \; dx$.
(i) Evaluate I_0.
(ii) Show $I_n = e - nI_{n-1}$, for $n > 1$.
(iii) Hence evaluate $I_4 = \int_0^1 x^4 e^x \; dx$.

2. **(a)**



The sketch above shows the parabola $y = f(x)$, where $f(x)$ is the quadratic $f(x) = \frac{1}{3}(x-1)(x-3)$. Without any use of Calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes and turning points.
(i) $y = \frac{1}{f(x)}$, **(ii)** $y = (f(x))^2$, **(iii)** $y = \tan^{-1}(f(x))$, **(iv)** $y = f(\ln x)$

(b)



The shaded area in the diagram above is rotated about the y-axis, and the resulting solid is a sphere of radius R with a cylindrical hole of radius λR through the middle, where $0 < \lambda < 1$. The solid is sliced into cylindrical shells. A typical cylindrical slice results from rotating about the y-axis the vertical strip shown in the diagram above.

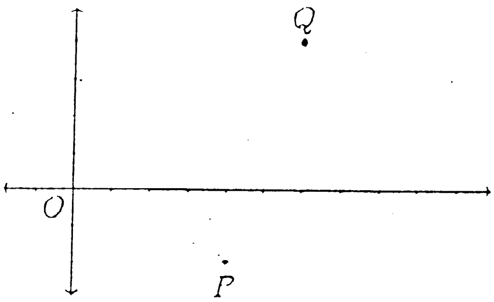
(i) If this vertical strip has width dx, and lies x units to the right of the origin, explain why the volume of the cylindrical shell it generates is $4\pi x\sqrt{R^2 - x^2}\, dx$.

(ii) Find the volume of the solid, and show that $\frac{\text{volume of solid}}{\text{volume of sphere}} = (1 - \lambda^2)^{\frac{3}{2}}$.

3. (a) On separate Argand diagrams, shade the regions:

(i) $-2 < \Im(z) \leq 5$, **(ii)** $|z| < 6$, **(iii)** $2 < z + \overline{z} < 10$, **(iv)** $\arg(z^2) = \frac{2\pi}{3}$

(b)



The diagram above shows the Argand diagram, with the points P and Q representing the complex numbers $4 - 2i$ and $6 + 4i$ respectively.

(i) The points P, O, Q and R (named in cyclic order) form a parallelogram. Find the complex number represented by R.

(ii) Find, in the form $|z - a| - r$, the locus of the circle with diameter PQ.

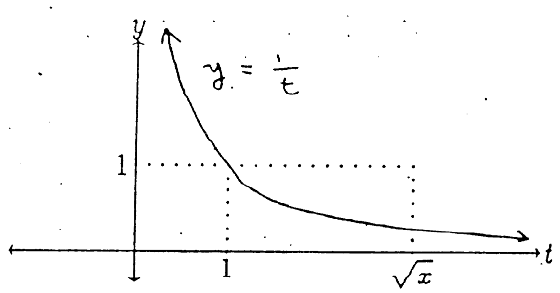
(c) **(i)** Show that when $z = r(\cos\theta + i\sin\theta)$ is multiplied by $\cos\alpha + i\sin\alpha$, its modulus remains unchanged, and its argument increses by α.

(ii) As in part **(b)**, P and Q are the points in the Argand diagram representing the complex numbrs $4 - 2i$ and $6 + 4i$ respectively.

(α) Find the complex number represented by the vector $\overrightarrow{OP}$.
(β) The points P, Q and S form an equilateral triangle. Find a possible value of the complex number represented by S.

4. (a) (i) Suppose that the real polynomial $f(x)$ can be written $f(x) = (x-\alpha)q(x)$, where α is a real number and $q(x)$ is a polynomial. Suppose also that $f'(\alpha) = 0$. Show that α is a multiple zero of $f(x)$.
(ii) Factor the polynomial $f(x) = x^6 - 7x^4 + 8x^2 + 16$ into linear factors.
(b) Let α be the complex root of the polynomial $z^7 = 1$ with smallest positive argument. Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$.
(i) Explain why $\alpha^7 = 1$ and $\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$.
(ii) Show that $\theta + \phi = -1$ and $\theta\phi = 2$, and hence write down a quadratic equation whose roots are θ and ϕ.
(iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$.
(iv) Write down α in modulus-argument form, and show that:
$\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$,
$\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$.

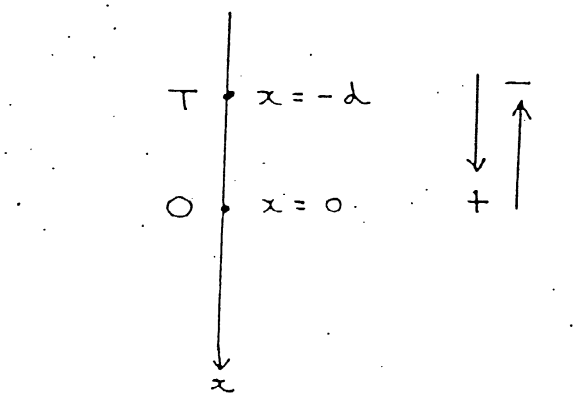
5. (a)



The diagram above shows that $0 < \int_1^{\sqrt{x}} \frac{dt}{t} < \sqrt{x}$, for all $x > 1$. Evaluate the integral, and then use this inequality to show $\lim\limits_{x\to\infty} (\frac{\ln x}{x}) = 0$.
(b) (i) Find (in exact form) all turning points and points of inflexion of $y = \frac{\ln x}{x}$, given $\frac{dy}{dx} = \frac{1-\ln x}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$.
(ii) Sketch $y = \frac{\ln x}{x}$
(iii) Show that for all $a > 1$, $\int_{\frac{1}{a}}^{a} \frac{\ln x}{x}\, dx = 0$.
(c) Suppose $f(x) = x^3 - 3ax + b$ is a cubic, where a and b are real numbers.
(i) Show that $f(x)$ has turning points if and only if $a > 0$, and find their coordinates.
(ii) Show that $f(x)$ has three distinct real zeros if and only if $b^2 < 4a^3$.

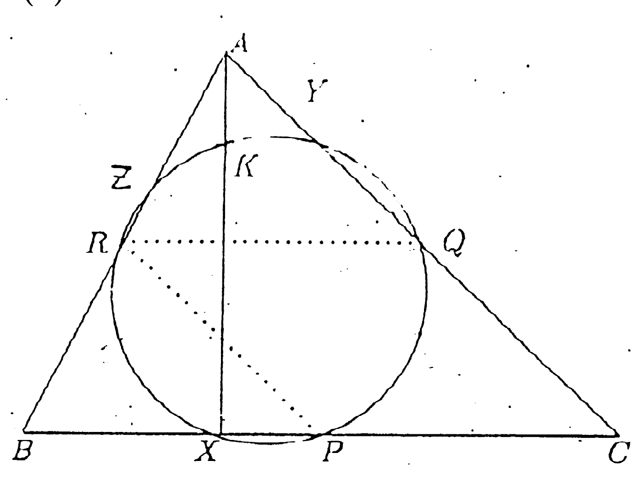
6. (a) A conical pendulum consists of a mass of $m$kg hanging on the end of a light 2 metre string from a hook on a ceiling. The mass is set rotating in a horizontal circle, and moves with a period of P seconds, with the string making a constant angle θ to the vertical.

(i) Draw a diagram of the situation, showing the forces acting on the mass.
(ii) By resolving forces vertically and radially, express the period P as a function of the angle θ between the string and the vertical (leave your answer in terms of g).
(iii) The string can just support a stationary mass of $10m$kg hanging vertically on it, but will break under any further weight. Find the smallest period that the conical pendulum can have (leave your answer in terms of g).
(b) A particle P is thrown vertically downwards in a medium where the resistive force is proportional to the speed, so that, taking downwards as positive, the equation of motion is $\ddot{x} = g - kv$, for some $k > 0$. The initial speed is U, and the particle is thrown from a point T distant d units above a fixed point O which is taken as the origin, so that the initial conditions are when $t = 0$, $v = U$ and $x = -d$.



(i) Show that $v = \frac{g}{k} - (\frac{g-kU}{k})e^{-kt}$.
(ii) Integrate again, and show that $x = \frac{gt-kd}{k} + (\frac{g-kU}{k^2})(e^{-kt} - 1)$.
(iii) A second identical particle Q is dropped from O at the same instant that P is thrown down. Use the above results to write down expressions for v and x as functions of t for the particle Q.
(iv) The particles P and Q collide. Find when the collision occurs, and find the speed with which the particles collide.

7. (a)



In the diagram above, P, Q and R are the midpoints of the sides BC, CA and AB respectively of a triangle ABC. The circle drawn through the points P, Q and R meets the sides BC, CA and AB again at X, Y and Z respectively.
(i) Explain why $RPCQ$ is a parallelogram.
(ii) Show that $\triangle XQC$ is isosceles.
(iii) Show that $AX \perp BC$.
(iv) The lines AX, BY and CZ meet the circle again at K, J and L respectively. Show that PK, QJ and RL are concurrent.
(b) Suppose x and y are functions of t satisfying the conditions:

1) $\ddot{x} = -n^2 x$ and $\ddot{y} = -n^2 y$,
2) $x(0) = y(0)$,
3) $\dot{x}(0) = \dot{y}(0)$,

where $x(0)$ and $\dot{x}(0)$ mean the values of x and $\dot{x}$ respectively when $t = 0$.
(i) Show that $\frac{d}{dx}(\dot{x}y - x\dot{y}) = 0$, and hence that $\dot{x}y = x\dot{y}$, for all t.
(ii) Show that $\frac{d}{dt}(\frac{x}{y}) = 0$, and hence that $y = x$, for all t.
(iii) Hence show that $x = a\cos nt + b\sin nt$, where $a = x(0)$ and $b = \frac{\dot{x}(0)}{n}$.

8. (a) Show that for all real numbers A and B:
(i) $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$,
(ii) $\cos A - \cos B = 2\sin\frac{B+A}{2}\sin\frac{B-A}{2}$.
(b) Let $T = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$, where n is a positive integer and x is any real number.
(i) Use part **(a)** to simplify $T\sin\frac{1}{2}x$, and hence show that when x is not an integer multiple of 2π: $T = \frac{\sin\frac{1}{2}nx\sin\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}$.
(ii) Show that $|T| \leq |\operatorname{cosec}\frac{1}{2}x|$, for all positive integer n, and all real numbers x which are not integer multiples of 2π.
(iii) Solve $\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x = 0$, for $0 \leq x \leq 2\pi$.
(c) Consider the definite integral $D = \int_{-\pi}^{\pi}\cos\lambda x\cos nx\ dx$, where n is a positive integer, and λ is any positive real number.
(i) Show that

$$D = \begin{cases} \pi, & \text{for } \lambda = n, \\ 0, & \text{for } \lambda \text{ a positive integer, } \lambda \neq n, \\ \frac{(-1)^n 2\lambda\sin\lambda\pi}{\lambda^2 - n^2}, & \text{for } \lambda \text{ not an integer.} \end{cases}$$

(ii) Show that when $0 < \lambda < n$, $|D| < \pi$.
(iii) Show that when $\lambda > n + \frac{1}{2}$, $|D| < 3$.
(iv) Is π the maximum value of $|D|$, for all positive integers n and all positive real numbers λ?