Sydney Grammar School

4 unit mathematics

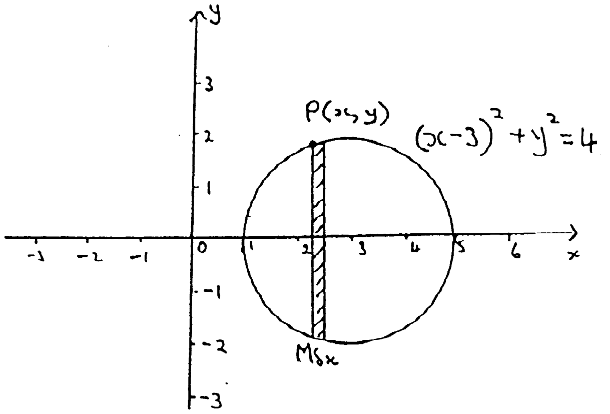
Trial HSC Examination 1995

1. **(a)** Find **(i)** $\int x^3 \ln x \; dx$ **(ii)** $\int \sin^3 \theta \; d\theta$.
(b) Find the exact value of **(i)** $\int_0^1 \frac{4x-13}{2x^2+x-6} \; dx$ **(ii)** $\int_5^7 \frac{dx}{x^2-10x+29}$
(c) Using the substitution $u = a - x$, prove that $\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx$. Hence or otherwise prove that $\int_0^1 x(3-x)^{11} \; dx = \frac{3^{12}}{52}$.

2. **(a)** Let $z = 3 - 2i$ and $u = -5 + 6i$.
(i) Find $\Im(uz)$ **(ii)** Find $|u - z|$ **(iii)** Find $\overline{-2iz}$ **(iv)** Express $\frac{u}{v}$ in the form $a + ib$, where a and b are real numbers.
(b) On separate Argand diagrams sketch:
(i) $\{z : |z - 2i| < 2\}$ **(ii)** $\{z : \arg\big(z - (1+i)\big) = -\frac{3\pi}{4}\}$.
(c) **(i)** Show that the solutions of the equation $z^3 = 1$ in the complex number system are $z = \cos\theta + i\sin\theta$ for $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.
(ii) If $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ show that $\omega^2 + \omega + 1 = 0$ and $\omega^3 - \omega^2 - \omega - 2 = 0$.
(iii) Hence or otherwise solve the cubic equation $z^3 - z^2 - z - 2 = 0$.

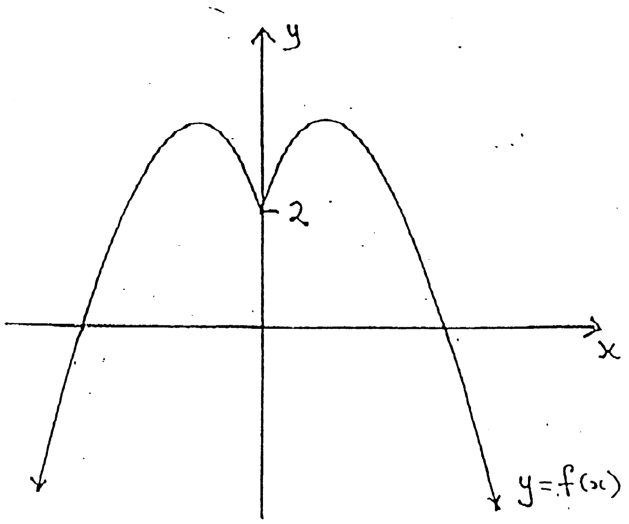
3. **(a)** Show, using the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, that $\int_0^t \sin\phi x \cos\phi(t-x) \; dx = \frac{1}{2}t\sin\phi t$.
(b) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis.
(i) Sketch the curve and the line clearly showing and labelling all the points of intersection.
(ii) By considering slices perpendicular to the x-axis, find the volume of the solid formed.

(c) The graph below is of the circle $(x-3)^2+y^2=4$. $P(x,y)$ is a point on the circumference of the circle. PM is the left-hand end of a strip of width δx which is



(i) Show, using the method of cylindrical shells, that the volume V of the doughnut-shaped solid formed when the region inside the circle is rotated about the y-axis is given by $V=4\pi\int_1^5 x\sqrt{4-(x-3)^2}\,dx$.
(ii) Hence find the volume of the doughnut by using the substitution $u=x-3$.

4. (a) The sketch is of the *even* function $y=f(x)$.



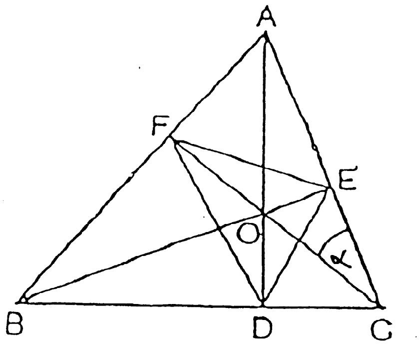
On separate number planes sketch each of the following, clearly showing all important features:
(i) **(α)** $y=f(x)-2$ **(β)** $y=f(x-2)$ **(γ)** $y=|f(x)|$ **(δ)** $y^2=f(x)$ **(ε)** $y=\frac{1}{f(x)}$.
(ii) Suppose that $f(x)$ is the function

$$f(x)=\begin{cases}\frac{1}{4}(4+x)(2-x), & \text{for } x<0,\\ \frac{1}{4}(4-x)(2+x), & \text{for } x\geq 0.\end{cases}$$

Sketch on a number plane the graph of the function $y=f'(x)$, showing all important features.

(b) Let α, β and γ be the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A and B are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$.
(i) Explain why β is real and α and γ are not real.
(ii) Show that α and γ are purely imaginary.
(iii) Find A and B.

5. (a) In the figure below, $\triangle ABC$ is acute angled and AD, BE and CF are altitudes concurrent at the orthocentre O. $\triangle DEF$ is called the *pedal triangle* of $\triangle ABC$.



(i) By letting $\angle OCE = \alpha$ and considering $\triangle$'s ABE and AFC prove that $\angle OCE = \angle OBF$.
(ii) Prove that O, D, C, E are concyclic.
(iii) Deduce that $\angle ODE = \angle OCE$.
(iv) Hence deduce that in an acute angled triangle the altitudes bisect the angles of the pedal triangle through which they pass.
(b) It is given that if $J_n = \int \cos^{n-1} x \sin nx \, dx$ and $n \geq 1$ then
$J_n = \frac{1}{2n-1}\big((n-1)J_{n-1} - \cos^{n-1} x \cos nx\big)$
(c) Solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x - 2)$.

6. (a) (i) A vehicle of mass m is moving with speed v around a curve of radius r banked at angle θ. If the normal reaction between the road and the vehicle is N, the lateral thrust (taken to be up the slope) is T and the acceleration due to gravity is g, draw a diagram that represents the forces on the vehicle.
(ii) Hence prove that when lateral thrust is zero, $\tan\theta = \frac{v^2}{rg}$.
(iii) A train is moving at 72 km per hour on a curve of radius 360 metres and the distance between the rails is 1.4 metres. Taking the acceleration due to gravity to be 9.8m/s^2 find how much (to the nearest centimetre) the outer rail must be raised in order that there may be no lateral thrust.
(b) (i) Prove that the acceleration of a body with displacement x and velocity v is given by $\ddot{x} = v\frac{dv}{dx}$.
(ii) A plane of mass M lands on the tarmac with speed u. When it is moving at speed v it experiences resistance forces of αv^2 due to air resistance and a constant force β due to the friction between the wheels and the tarmac, where α and β are constants
(α) Show that the equation of motion is $\frac{dv}{dx} = -\frac{1}{M}(\alpha v + \frac{\beta}{v})$.

(β) Show that the distance required to bring the plane to rest is $\frac{M}{2\alpha}\ln(1+\frac{\alpha}{\beta}u^2)$.
(γ) If it takes T seconds to bring the plane to rest show that $T = \frac{M}{\sqrt{\alpha\beta}}\tan^{-1}(u\sqrt{\frac{\alpha}{\beta}})$.

7. (a) A sequence $\{b_n\}$ is defined by $b_1 = 1$ and $b_{n+1} = b_n(b_n+1)$, for all $n \geq 1$.
(i) Evaluate b_2, b_3, b_4.
(ii) Use mathematical induction to prove that for each n: $b_{n+1} = 1 + \sum_{r=1}^{n} b_r^2$.
(iii) Show that $(2b_{n+1}+1)^2 = (2b_n+1)^2+(2b_{n+1})^2$. Hence deduce that $(2b_{n+1}+1)^2 = (2b_1+1)^2 + \sum_{r=2}^{n+1}(2b_r)^2$.
(iv) Evaluate b_5 and express it as the sum of 5 positive squares.
(v) Hence prove that $3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$.
(b) (i) Prove that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n\theta\cos n\theta$.
(ii) Hence prove that $\Re(1+i\tan\frac{\pi}{8})^8 = 64(12\sqrt{2}-17)$.

8. (a) (i) Let $f(x) = \frac{1}{1+x^2}$. **(α)** Prove that $f(x)$ is a decreasing function for all $x > 0$.
(β) Hence or otherwise prove that if $0 < x < 1$ then $\frac{1}{2} < \frac{1}{1+x^2} < 1$.
(ii) Find the sixth-degree polynomial $P(x)$ and the constant A such that $x^4(1-x)^4 \equiv (1+x^2)P(x) + A$.
(iii) Hence show that $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$.
(iv) Use **(i)** to deduce that: $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$.
(b) Let $f(x)$ be a function which satisfies the equation $f(xy) = f(x) + f(y)$ for all $x, y \neq 0$.
(i) Show that $f(1) = 0 = f(-1)$ and that $f(x)$ is an even function.
(ii) Prove that $f(x+y) - f(x) = f(1+\frac{y}{x})$ for $x, y, x+y \neq 0$.
(iii) Suppose $f(x)$ is differentiable at $x = 1$ and $f'(1) = 1$. Deduce that $f(x)$ is differentiable at any $x \neq 0$ and $f'(x) = \frac{1}{x}$.