4 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours (plus 5 minutes reading) **Exam date:** 4th August, 1997

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

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2 (a) If $w = 3 - 4i$ and $z = 5 + 2i$, find $z(|w| - \overline{w})$.

3 (b) Find the roots of the equation $(2+i)z^2 - 4z + (2-i) = 0$, expressing any complex roots in the form $a + bi$, where a and b are real.

3 (c) (i) Write $-\sqrt{3} + i$ in modulus-argument form.

(ii) Hence find $(-\sqrt{3} + i)^8$, giving your answer in the form $a + bi$, where a and b are real.

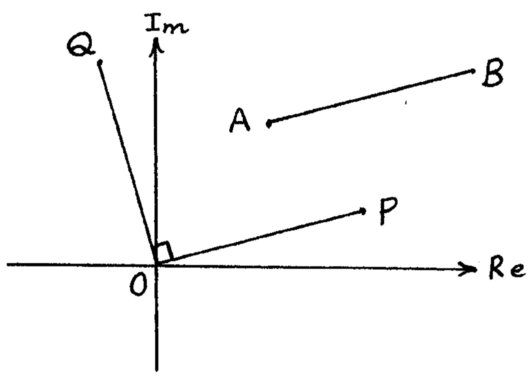
4 (d) Sketch each of the following loci on separate Argand diagrams:

(i) $\arg(z + 1 + i) = \frac{\pi}{4}$,

(ii) $|z - 1| = |z + i|$,

(iii) $z\overline{z} - i\,\overline{z} - \overline{i}\,z = 0$.

3 (e)



In the Argand diagram above, intervals AB, OP and OQ are equal in length, OP is parallel to AB and angle $POQ = \frac{\pi}{2}$. If A and B represent the complex numbers $3 + 5i$ and $9 + 8i$ respectively, find the complex number which is represented by Q.

QUESTION TWO (Start a new answer booklet)

Marks

$\boxed{4}$ (a) Find

(i) $\displaystyle\int \frac{1}{x^2 - 6x + 8}\,dx,$

(ii) $\displaystyle\int \frac{1}{\sqrt{x^2 - 6x + 8}}\,dx.$

$\boxed{3}$ (b) By using the substitution $t = \tan x$, or otherwise, show that:

$$\int \operatorname{cosec} 2x\,dx = \tfrac{1}{2}\ln|\tan x| + c.$$

[illegible] (c) (i) Show that $\displaystyle\int \frac{1-x^2}{1+x^2}\,dx = 2\tan^{-1} x - x + c.$

(ii) Use the substitution $u = \cos x$ to find the exact value of $\displaystyle\int_0^{\pi} \frac{\sin^3 x}{1+\cos^2 x}\,dx.$

$\boxed{3}$ (d) A solid has height H, and for every h between 0 and H the horizontal cross-section at height h above the base is a square, whose sides have length e^h. Find the volume of the solid.

Exam continues overleaf ...

QUESTION THREE (Start a new answer booklet)

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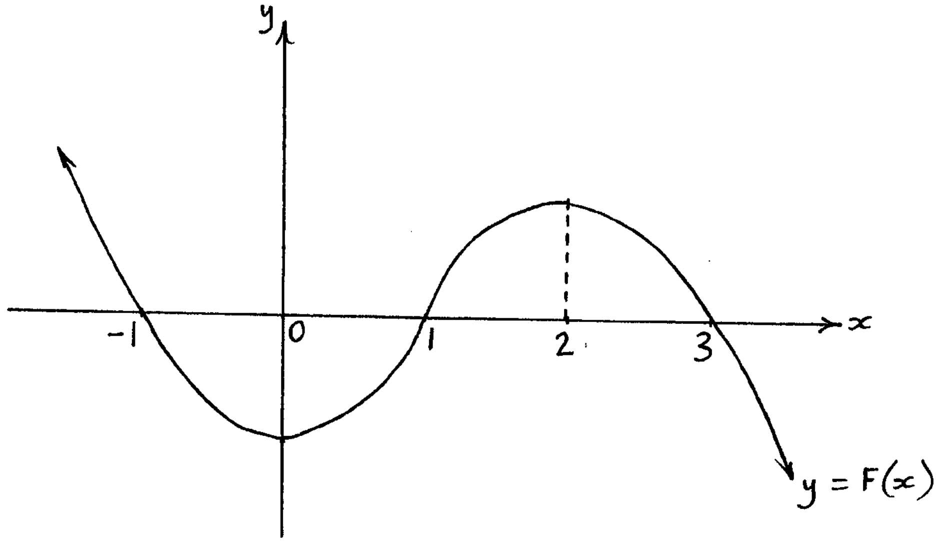
3 (a)



The graph of the function $y = F(x)$ is sketched above. On separate number planes sketch the graphs of:

(i) $y = F(1 - x)$,

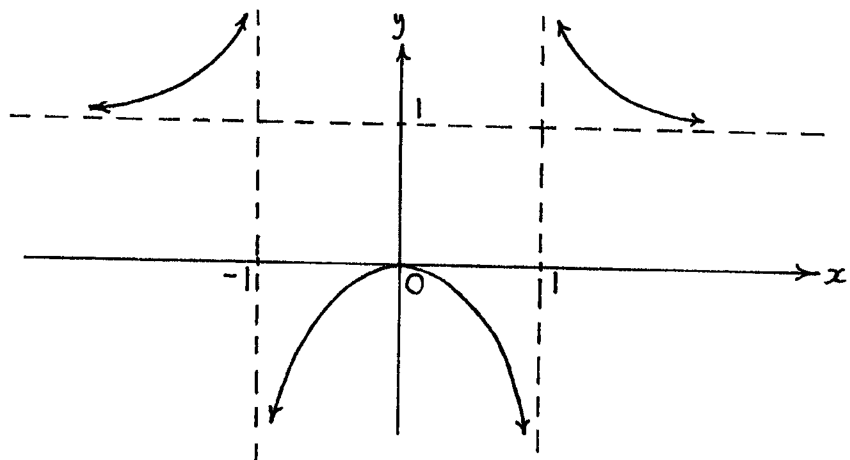
(ii) $y = F'(x)$,

(iii) $|y| = F(|x|)$.

QUESTION THREE (Continued)

5 (b) (i) Solve the equation $\dfrac{x^2}{x^2-1} = \dfrac{x^2-1}{x^2}$.

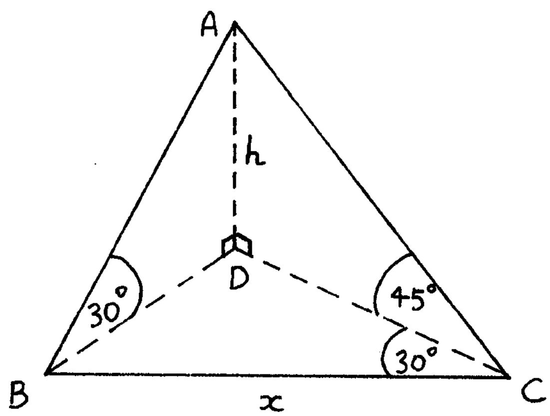
(ii)



The graph of the function $f(x) = \dfrac{x^2}{x^2-1}$ is sketched above. Neatly copy this diagram into your answer booklet, and on it sketch the curve $y = \dfrac{1}{f(x)}$, showing the coordinates of any points of intersection of the two curves.

(iii) Hence find the values of x for which $\dfrac{x^2}{x^2-1} < \dfrac{x^2-1}{x^2}$.

4 (c)



$ABCD$ is a triangular pyramid with base BCD and perpendicular height AD. Use the cosine rule in triangle BCD to show that:

$$\frac{h}{x} = \frac{\sqrt{11}-\sqrt{3}}{4}.$$

QUESTION FOUR (Start a new answer booklet)

Marks

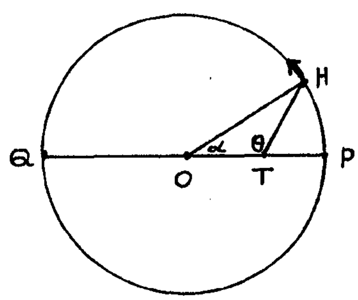
8 (a) A particle P_1 of mass m kg is dropped from point A and falls towards point B, which is directly underneath A. At the instant when P_1 is dropped, a second particle P_2, also of mass m kg, is projected upwards from B towards A with an initial velocity equal to twice the terminal velocity of P_1. Each particle experiences a resistance of magnitude mkv as it moves, where $v\,\mathrm{ms}^{-1}$ is the velocity and k is a constant.

(i) Show that the terminal velocity of P_1 is $\dfrac{g}{k}$, where g is acceleration due to gravity.

(ii) For particle P_2, show that $t = \dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$, where $v\,\mathrm{ms}^{-1}$ is the velocity after t seconds.

(iii) Suppose the particles collide at the instant when P_1 has reached 30% of its terminal velocity. Find the velocity of P_2 when they collide. (Give your answer in terms of g and k.)

7 (b)



In the diagram above, a horse H is galloping with constant angular velocity 0.2 rad/s around a circular track. The radius OP is 50 metres, and there is a trainer standing at T on the diameter QP, where TP is 20 metres. Let $\angle HOT = \alpha$ and let $\angle HTO = \theta$.

(i) Show that $30\sin\theta = 50\sin(\alpha+\theta)$.

(ii) By differentiating both sides of (i) with respect to t, show that the angular velocity of the horse about the trainer is given by $\dfrac{\cos(\alpha+\theta)}{3\cos\theta - 5\cos(\alpha+\theta)}$.

<u>Hint</u>: $\dfrac{d}{dt}\sin(\alpha+\theta) = \left(\dfrac{d\alpha}{dt} + \dfrac{d\theta}{dt}\right)\cos(\alpha+\theta)$.

(iii) Find the angular velocity of the horse about the trainer at the instant when the horse is at Q.

QUESTION FIVE (Start a new answer booklet)

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(a) The polynomial $P(x)$ is defined by the rule $P(x) = x^4 + Ax^2 + B$, where A and B are positive real numbers.

(i) Explain why $P(x)$ has no real zeros.

(ii) If the zeros of $P(x)$ are bi, $-bi$, di and $-di$, where b and d are real, show that:

$$b^4 + d^4 = A^2 - 2B.$$

(b) (i) Use integration by parts to show that:

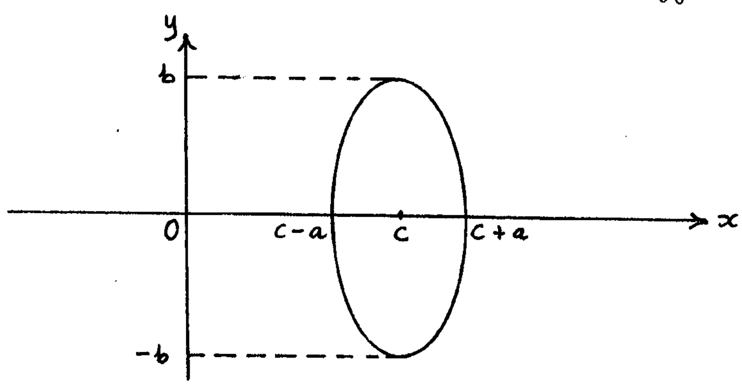
$$\int_0^1 x^{100}(1-x)^n\,dx = \frac{n}{101}\int_0^1 x^{101}(1-x)^{n-1}\,dx.$$

(ii) If $I_n = \displaystyle\int_0^1 x^{100}(1-x)^n\,dx$, use part (i) to show that $I_n = \dfrac{n}{101+n}I_{n-1}$.

(iii) Hence show that $I_n = \dfrac{n!\,100!}{(101+n)!}$.

(c) (i) Sketch the curve $x = \sqrt{b^2 - y^2}$, and hence explain why $\displaystyle\int_0^b \sqrt{b^2-y^2}\,dy = \frac{\pi b^2}{4}$.

(ii)



The ellipse $\dfrac{(x-c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a$ and $c > a$, is sketched above. The region bounded by the ellipse is rotated about the y axis to form a ring. By taking slices perpendicular to the y axis, show that the ring has volume $2abc\pi^2$.

QUESTION SIX (Start a new answer booklet)

ks

(a) (i) Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

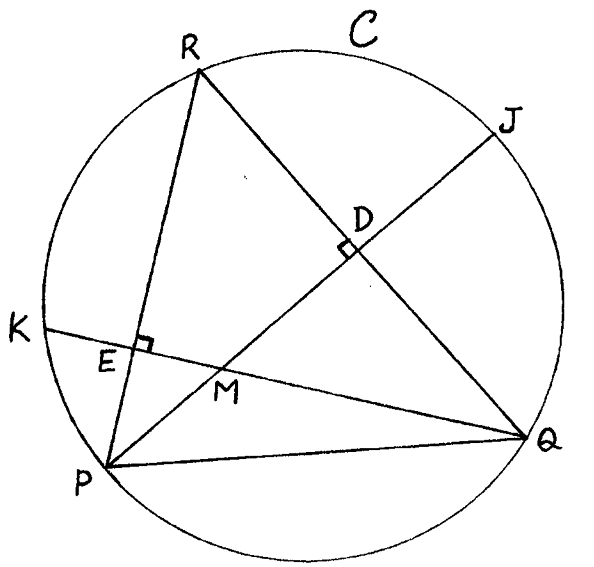
(ii) Hence show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

(iii) Given $\alpha = \tan^{-1}\frac{1}{4}$, show that $4\alpha = \tan^{-1}\frac{240}{161}$.

(iv) Write $161 + 240i$ in the form $r(\cos\theta + i\sin\theta)$, expressing θ in terms of α.

(v) Hence find, in the form $a + bi$ where a and b are integers, the four fourth roots of $161 + 240i$.

(b)



In the diagram, PQR is a triangle inscribed in the circle C. The altitude PD is produc l to meet C at J, the altitude QE is produced to meet C at K and these two altitudes intersect at M.

(i) Explain why the quadrilaterals $PQDE$ and $REMD$ are cyclic.

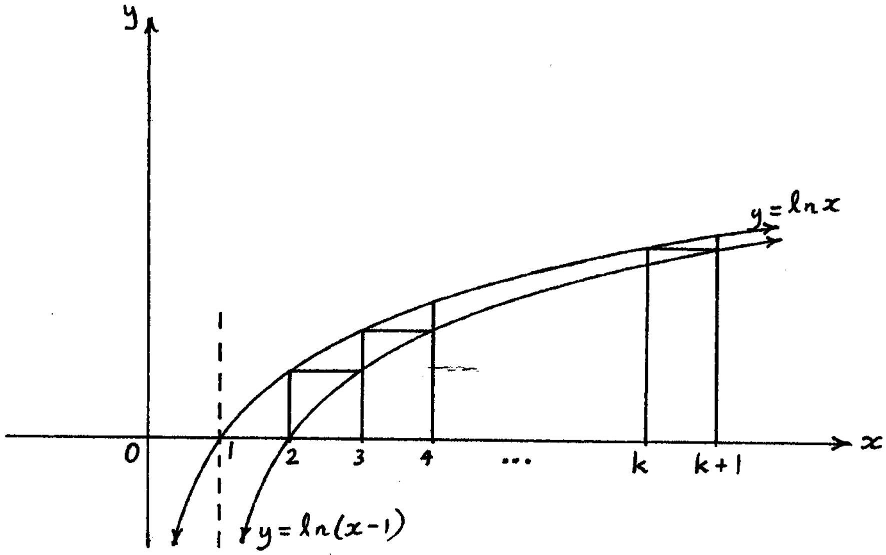
(ii) Show that PR bisects angle KRM.

(iii) Hence, or otherwise, show that $KR = JR$.

QUESTION SEVEN (Start a new answer booklet)

Marks

6 (a)



In the diagram above, the curves $y = \ln x$ and $y = \ln(x-1)$ are sketched and $k-1$ rectangles are constructed between $x = 2$ and $x = k+1$, where $k \geq 2$.
Let $S = \ln 2 + \ln 3 + \ln 4 + \cdots + \ln k$.

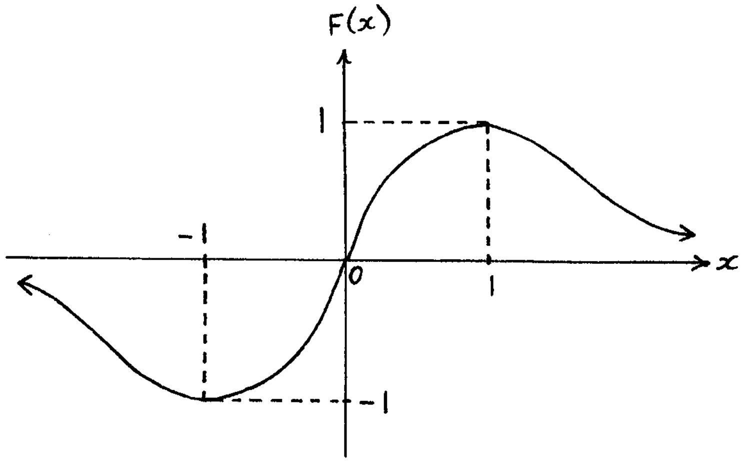
(i) Explain why S represents the sum of the areas of the $k-1$ rectangles.

(ii) Show that $\displaystyle\int_2^{k+1} \ln(x-1)\,dx = k\ln k - k + 1$.

(iii) Hence show that $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, where $k \geq 2$.

QUESTION SEVEN (Continued)

(b)



The curve $F(x) = \dfrac{2x}{1+x^2}$ is sketched above.

(i) State the range of $F(x)$.

(ii) Let x_0 be a real number not equal to 1 or -1 and consider the sequence of real numbers defined by $x_{n+1} = F(x_n)$ for $n = 0, 1, 2, \ldots$.

(α) Show that there exists a real number r such that $x_1 = G(r)$, where $G(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$.

(β) Show that $G(2x) = \dfrac{2G(x)}{1 + (G(x))^2}$.

(γ) Hence, using parts (α) and (β), show that $x_n = G(2^{n-1}r)$, for $n = 1, 2, 3, \ldots$.

QUESTION EIGHT (Start a new answer booklet)

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4 (a) (i) Suppose $f(x) = \sqrt{1+x}$. Show that $f'(x) < \frac{1}{6}$ for $x > 8$.

(ii) Using part (i), or otherwise, show that $\sqrt{1+x} \leq 3 + \dfrac{x-8}{6}$ when $x \geq 8$.

11 (b) Suppose z_1, z_2, z_3 and z_4 are distinct complex numbers. The cross-ratio $(z_1, z_2; z_3, z_4)$ of these four complex numbers is defined by:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

(i) Show that:

(α) $(z_1, z_2; z_4, z_3) = \dfrac{1}{(z_1, z_2; z_3, z_4)}$,

(β) $(z_1, z_3; z_2, z_4) = 1 - (z_1, z_2; z_3, z_4)$.

(ii) Suppose $f(z) = \dfrac{c^2}{\overline{z}}$, where c is a positive constant and z is a complex variable. Show that:

(α) $|zf(z)| = c^2$,

(β) $\arg f(z) = \arg z$.

(iii) If $f(z_1) = w_1$, $f(z_2) = w_2$, $f(z_3) = w_3$ and $f(z_4) = w_4$, then show that:

$$(w_1, w_2; w_3, w_4) = \overline{(z_1, z_2; z_3, z_4)}.$$

(iv) Suppose z_1, z_2, z_3 and z_4 represent (in cyclic order) points in the Argand diagram which lie on a circle.

(α) Show that $(z_1, z_3; z_4, z_2)$ is a negative real number.

(β) Hence deduce that $(w_1, w_2; w_3, w_4)$ is a real number greater than one.

(γ) Show that the points w_1, w_2, w_3 and w_4 are either concyclic or collinear, and find the condition on the original circle for w_1, w_2, w_3 and w_4 to be collinear.

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