# 4 UNIT MATHEMATICS FORM VI

**Time allowed:** 3 hours (plus five minutes reading) **Exam date:** 11th August, 1998

**Instructions:**

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

**Collection:**

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

Tear out Page 11 and place it inside the answer booklet for Question Seven.

QUESTION ONE (Start a new answer booklet)

(a) Evaluate $\displaystyle\int_0^1 \frac{4}{\sqrt{3x+1}}\,dx$.

(b) Evaluate $\displaystyle\int_1^e \log_e x\,dx$.

(c) Evaluate $\displaystyle\int_0^\pi \sin^3 x\,dx$.

(d) Find $\displaystyle\int \frac{dx}{2+\sin x}$ using the substitution $t = \tan\frac{1}{2}x$. Leave your answer in terms of $t$.

(e) (i) Find constants $A$, $B$ and $C$ so that $\displaystyle\frac{x^2+x+1}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$.

(ii) Find $\displaystyle\int \frac{x^2+x+1}{x^3+3x^2}\,dx$.

QUESTION TWO (Start a new answer booklet)

(a) Simplify $(2-5i)^3$.

(b) Solve $z^2 = 5-12i$, giving your answer in the form $a+ib$.

(c) (i) Express $-1+i$ in modulus argument form.

(ii) Hence evaluate $(-1+i)^{10}$.

(d) (i) Sketch the locus of $\arg z = \frac{\pi}{4}$.

(ii) Sketch the locus of $\arg \overline{z} = \frac{\pi}{4}$.

(iii) Sketch the locus of $\arg(-z) = \frac{\pi}{4}$.

(e) The points $O$, $I$, $Z$ and $P$ on the Argand diagram represent the complex numbers 0, 1, $z$ and $z+1$ respectively, where:

$$z = \cos\theta + i\sin\theta$$

is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why $OIZP$ is a rhombus.

(ii) Show that $\displaystyle\frac{z-1}{z+1}$ is purely imaginary.

(iii) Find the modulus and argument of $z+1$.

QUESTION THREE (Start a new answer booklet)

Marks

[5] (a) Let $\omega$ be a non-real root of $z^7 - 1 = 0$.

(i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

(ii) Show that $(1+\omega)(1+\omega^2)(1+\omega^4) = 1$.

(iii) Simplify $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$.

(iv) Sketch on the Argand diagram all seven roots of $z^7 - 1 = 0$.

[4] (b) It is known that $2 + i$ is a root of the equation:

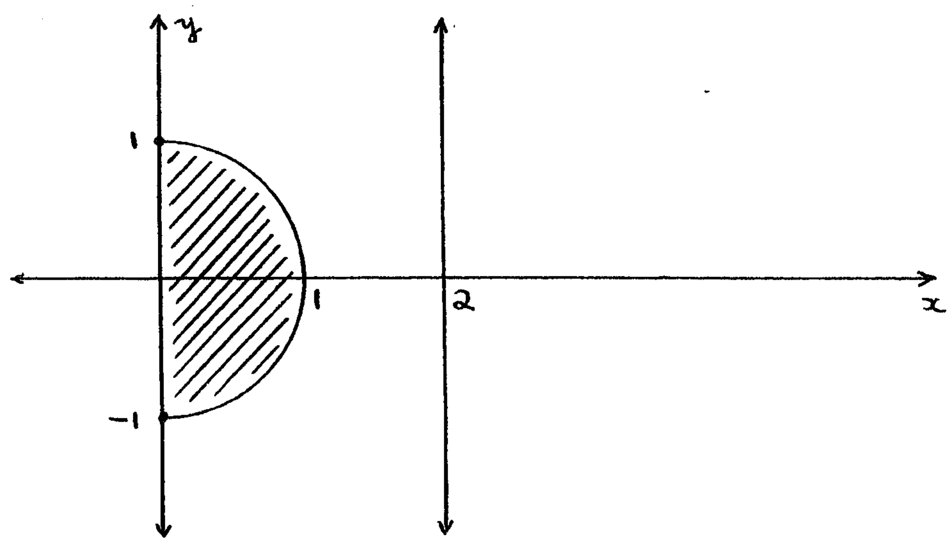
$$x^6 - 7x^4 + 31x^2 - 25 = 0.$$

(i) Give a reason why $2 - i$ is also a root of the equation.

(ii) Give a reason why $-(2+i)$ is also a root of the equation.

(iii) Find the other three roots, giving reasons for each step in your argument (it is not intended that long division be used).

[6] (c)




The shaded semicircle in the diagram above is rotated about the line $x = 2$.

(i) Using the method of cylindrical shells, show that the volume $V$ of the resulting solid is given by:

$$V = \int_0^1 4\pi(2-x)\sqrt{1-x^2}\,dx.$$

(ii) Hence find the volume of the solid.

QUESTION FOUR (Start a new answer booklet)

Marks

4 (a) Give reasons why each of the following statements is true or false. You need not evaluate the integrals:

(i) $\displaystyle\int_{-1}^{1} \frac{e^x - e^{-x}}{2}\,dx = 0,$

(ii) $\displaystyle\int_0^1 x^8\,dx < \int_0^1 x^9\,dx,$

(iii) $\displaystyle\int_0^\pi \sin^8 x\,dx > \int_0^\pi \sin 8x\,dx.$

6 (b) A projectile is fired vertically upwards with initial speed $V$. It experiences air resistance proportional to the square of its speed, as well as gravitational acceleration $g$, so that in its upwards flight, the equation of motion is:

$$\ddot{x} = -g - kv^2, \text{ for some constant } k > 0.$$

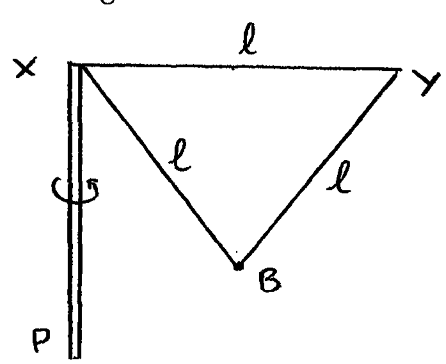
(i) By replacing $\ddot{x}$ by $\dfrac{dv}{dt}$ and integrating, find the time $t$ when the velocity is $v$.

(ii) Hence find the time $T$ taken to reach maximum height.

(iii) By replacing $\ddot{x}$ by $v\,\dfrac{dv}{dx}$ and integrating, find the height $x$ when the velocity is $v$.

(iv) Hence find the maximum height $H$.

5 (c)




The diagram above shows a rod $XY$ of length $\ell$ metres projecting horizontally from a rotating vertical pole $XP$. A basket $B$ of mass $m$ kg is held by two strings $XB$ and $YB$, each of length $\ell$. The pole, rod and basket are rotating with angular speed $\omega$.

Let $T$ be the tension in the string $XB$, and let $U$ be the tension in the string $YB$.

(i) Show that the radius of the basket's circular motion about the pole is $\frac{1}{2}\ell$.

(ii) Resolve forces at $B$ radially and vertically.

(iii) Show that the string $BY$ will remain taut provided $\omega^2 \le \dfrac{2g}{\ell\sqrt{3}}$.

(iv) Find the ratio of the tensions in the two strings, assuming they are both taut.

QUESTION FIVE (Start a new answer booklet)

Marks

8 (a) Sketch, showing the exact values of all turning points, intercepts and endpoints:

(i) $y = \cos \pi x$, for $-1 \le x \le 3$,

(ii) $y^2 = \cos \pi x$, for $-1 \le x \le 3$,

(iii) $y = e^{\cos \pi x}$, for $0 \le x \le 2$ (you need not find inflexions here),

(iv) $y = \cos \pi e^x$, for $x \le 1$ (again, you need not find inflexions).

7 (b) Let $z = \cos\theta + i \sin\theta$ be any complex number of modulus 1.

(i) Show that $\dfrac{z^2 - 1}{z} = 2i \sin\theta$.

(ii) Using the formula for the sum to $n$ terms of a GP, and part (i), prove that:

$$z + z^3 + z^5 + z^7 + z^9 = \frac{\sin 10\theta + i(1 - \cos 10\theta)}{2 \sin\theta}.$$

(iii) Hence show that $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \dfrac{\sin 10\theta}{2 \sin\theta}$.

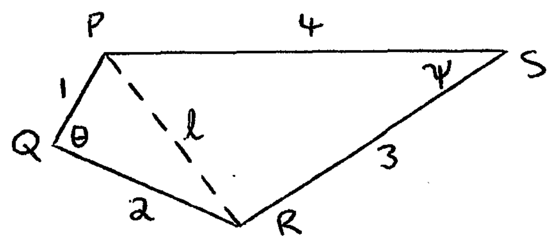
(iv) Find the first two positive solutions of:

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \tfrac{1}{2}.$$

QUESTION SIX (Start a new answer booklet)

Marks

7 (a)




The quadrilateral $PQRS$ above has sides $PQ = 1$, $QR = 2$, $RS = 3$ and $SP = 4$. Let $\theta = \angle Q$ and $\psi = \angle S$, and let $\ell = PR$.

(i) Show that the area of $PQRS$ is given by $A = \sin\theta + 6 \sin\psi$.

(ii) By equating expressions for $\ell^2$, show that $6 \cos\psi - \cos\theta = 5$.

(iii) Differentiate the identity in part (ii) implicitly to find an expression for $\dfrac{d\psi}{d\theta}$.

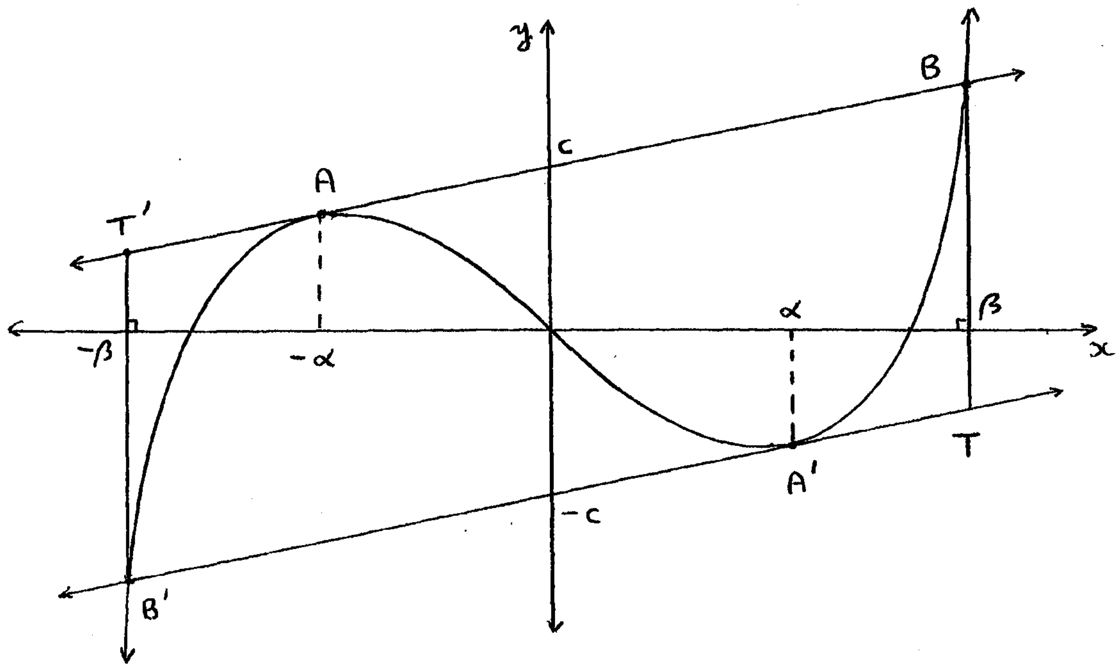
(iv) Prove that the area of the quadrilateral is maximum when it is cyclic (you need not prove that the relevant stationary point is a maximum).

(v) Find the area of the quadrilateral when it is cyclic.

QUESTION SIX (Continued)

2 (b) Use the table of standard integrals to find $\displaystyle\int_a^{2a} \frac{dx}{\sqrt{x^2-a^2}}$, and show that the integral is independent of $a$.

3 (c)




The diagram above shows the cubic curve $y = x^3 - x$. The tangent $AB$ with gradient $m$ touches the cubic at a point $A$ to the left of the $y$-axis and crosses the curve again at $B$.

Let the point of contact $A$ have $x$-coordinate $-\alpha$, and let $B$ have $x$-coordinate $\beta$. Let the tangent have equation $y = mx + c$.

Because the cubic is an odd function, the other tangent with this same gradient $m$ will have equation $y = mx - c$, will touch the cubic at the point $A'$ with $x$-coordinate $\alpha$, and will cross the curve at the point $B'$ with $x$-coordinate $-\beta$.

Vertical lines at $B$ and $B'$ each meet the other tangent at $T$ and $T'$ respectively.

(i) By solving simultaneously the cubic and the tangent $AB$, explain why:

$$x^3 - (m+1)x - c = 0$$

has roots $-\alpha$, $-\alpha$ and $\beta$.

(ii) Show that $\beta = 2\alpha$.

(iii) Show that $\alpha^2 = \frac{1}{3}(m+1)$.

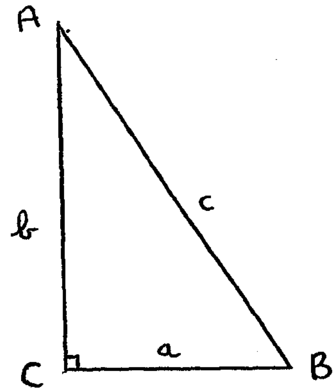
(iv) Show that the parallelogram $TBT'B'$ has area $\frac{16}{9}(m+1)^2$.

(v) Give a geometrical explanation as to why the area is zero when $m = -1$.

QUESTION SEVEN (Start a new answer booklet)

arks

4 (a)

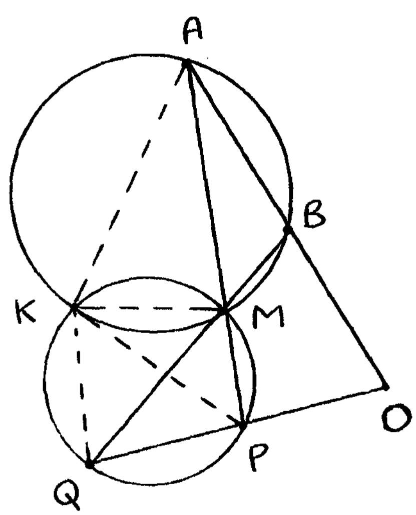



The hypotenuse of the right triangle above has length $c$, and the other two sides have lengths $a$ and $b$ respectively.

(i) If $b$ is the arithmetic mean of $a$ and $c$, find the ratio $a : c$.

(ii) If $b$ is the geometric mean of $a$ and $c$, find the ratio $a : c$.

(b) NOTE: The diagram printed below for this question has been reprinted on Page 11 so that working can be done on the diagram. Tear out Page 11, write your candidate number at the top of that sheet, and place the sheet inside your answer booklet for Question Seven.




The diagram above shows two circles intersecting at $K$ and $M$. From points $A$ and $B$ on the outer arc of one circle, lines are drawn through $M$ to meet the other circle at $P$ and $Q$ respectively. The lines $AB$ and $QP$ meet at $O$.

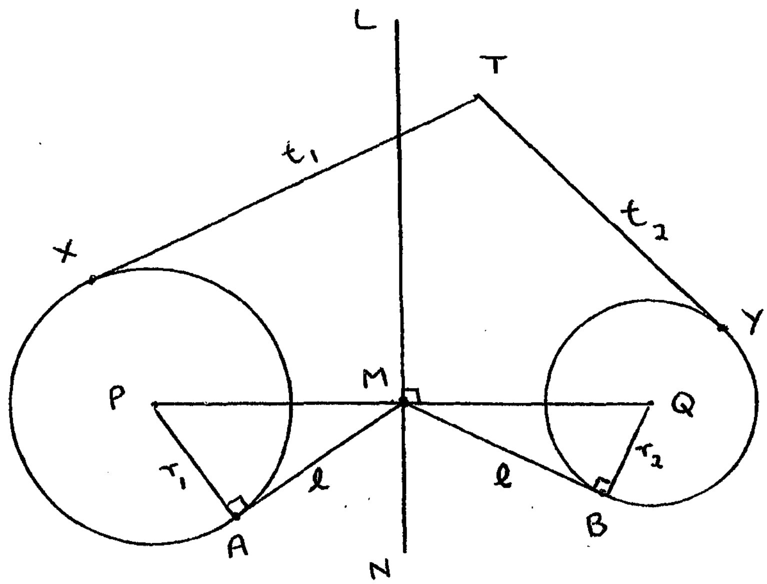
(i) Let $\theta = \angle KAB$, and give a reason why $\angle KMQ = \theta$.

(ii) Prove $AKPO$ is a cyclic quadrilateral.

(iii) Let $\psi = \angle AKM$. Show that if $OBMP$ is a cyclic quadrilateral, then the points $A$, $K$ and $Q$ are collinear.

QUESTION SEVEN (Continued)

(c) NOTE: The diagram printed below for this question has been reprinted on Page 11 so that working can be done on the diagram. Tear out Page 11, write your candidate number at the top of that sheet, and place the sheet inside your answer booklet for Question Seven.




The diagram above shows two non-intersecting circles with centres $P$ and $Q$ respectively and radii $r_1$ and $r_2$ respectively. The point $M$ is the point on $PQ$ where the tangents $MA$ and $MB$ respectively to the two circles have equal length $\ell$. The line $LMN$ is drawn through $M$ perpendicular to $PQ$.

Let $T$ be any point outside the two circles, and let the tangents $TX$ and $TY$ to the two circles have lengths $t_1$ and $t_2$ respectively.

(i) Show that $PM^2 - QM^2 = {r_1}^2 - {r_2}^2$.

(ii) Show that $TP^2 - TQ^2 = ({t_1}^2 - {t_2}^2) + ({r_1}^2 - {r_2}^2)$.

(iii) Prove that the tangents $TX$ and $TY$ are equal if and only if $T$ lies on $LMN$.

(iv) A third circle is drawn tangent to the first circle at a point $J$ and to the second circle at a point $K$. Prove that $LMN$ and the tangents at $J$ and $K$ are concurrent.

QUESTION EIGHT (Start a new answer booklet)

arks

3 (a) Solve $|x^2 - 2x - 3| < 3x - 3$.

7 (b) Consider the integral $I_n = \int_0^1 x^{2n+1}\, e^{-x^2}\, dx$.

(i) Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \geq 1$.

(ii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$.

(iii) Prove by mathematical induction that for all $n \geq 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2e\, I_n}{n!}.$$

(iv) It is clear that $0 \leq I_n \leq 1$, because $0 \leq x^{2n+1}\, e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.
Use this fact to deduce from the previous part that:

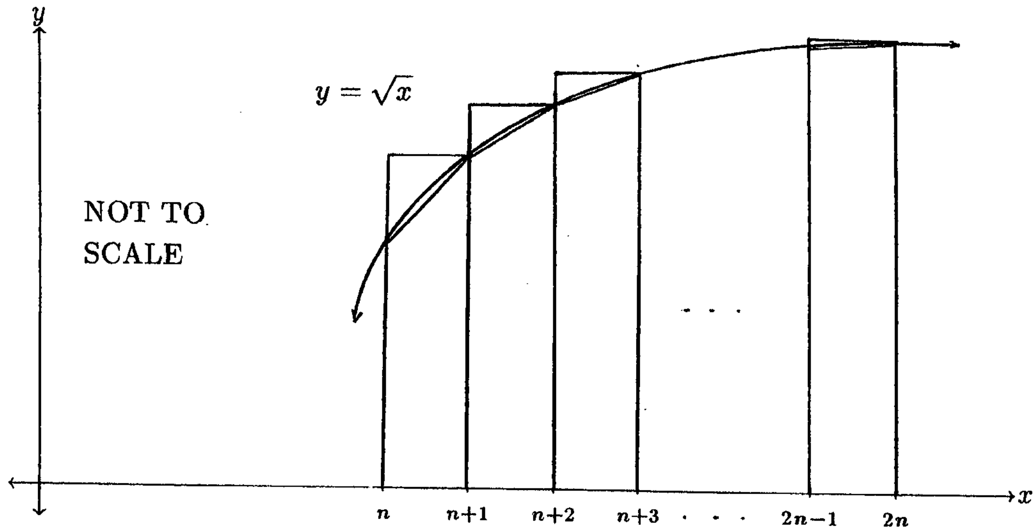
$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

CARE: QUESTION EIGHT CONTINUES OVERLEAF

QUESTION EIGHT (Continued)

(c) Show that $\int_n^{2n} \sqrt{x}\,dx = \frac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right)$, where $n$ is a positive integer.

(d)




The diagram above shows part of the graph of $y = \sqrt{x}$, drawn quite distorted so that the chords can be seen more clearly. For some positive integer $n$, ordinates have been drawn at $x = n$, $x = n+1$, $\ldots$, $x = 2n$. Upper rectangles and trapezia have been constructed as shown.

(i) Using the upper rectangles and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \cdots \sqrt{2n} \;>\; \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \sqrt{n}$$

(you may assume that $y = \sqrt{x}$ is an increasing function).

(ii) Using the trapezia and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \cdots \sqrt{2n} \;<\; \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \tfrac{1}{2}\left(\sqrt{n} + \sqrt{2n}\right)$$

(you may assume that $y = \sqrt{x}$ is concave down).

(iii) Suppose it is claimed that the average of the numbers:

$$\sqrt{1\,000\,000}, \quad \sqrt{1\,000\,001}, \quad \ldots, \quad \sqrt{2\,000\,000}$$

is about 1218·9512. If one relies on the bounds established in parts (i) and (ii), what is the maximum possible error in this claim?

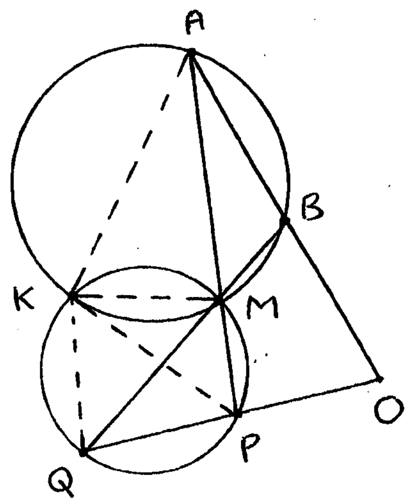
WMP

Candidate Number: ........................ ........................

Tear out this sheet of paper, write your candidate number at the top, and place the sheet inside your answer booklet for Question Seven.

QUESTION SEVEN

(b) This is a reprint of the diagram for part (b) of Question Seven.




(c) This is a reprint of the diagram for part (c) of Question Seven.