10th August 1999.
Time 3 hours.

QUESTION ONE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) Let $z = \dfrac{1-i}{2+i}$.

(i) Show that $z + \dfrac{1}{z} = \dfrac{3+2i}{3-i}$.

(ii) Hence find:

(α) $\overline{\left(z+\dfrac{1}{z}\right)}$, in the form $a+bi$, where a and b are real,

(β) $\operatorname{Im}\left(z+\dfrac{1}{z}\right)$.

$\boxed{4}$ (b) (i) Express $-\sqrt{27}-3i$ in modulus-argument form.

(ii) Hence find $(-\sqrt{27}-3i)^6$, giving your answer in the form $a+bi$, where a and b are real.

$\boxed{4}$ (c) Sketch on separate Argand diagrams the locus of z defined as follows:

(i) $\arg(z-1) = \frac{3\pi}{4}$,

(ii) $\operatorname{Re}\left(z(\bar{z}+2)\right) = 3$.

$\boxed{3}$ (d) If z is a complex number such that $z = k(\cos\phi + i\sin\phi)$, where k is real, show that $\arg(z+k) = \frac{1}{2}\phi$.

QUESTION TWO (Start a new answer booklet)

Marks

$\boxed{3}$ (a) Find $\displaystyle\int x^2 e^{-2x}\,dx$.

$\boxed{4}$ (b) (i) Resolve $\dfrac{9+x-2x^2}{(1-x)(3+x^2)}$ into partial fractions.

(ii) Hence find $\displaystyle\int \frac{9+x-2x^2}{(1-x)(3+x^2)}\,dx$.

$\boxed{8}$ (c) Evaluate each of the following:

(i) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$,

(ii) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^4\theta\tan\theta\,d\theta$,

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$. (Hint: Use the substitution $t = \tan\frac{\theta}{2}$.)

Exam continues next page ...

Ti

QUESTION THREE (Start a new answer booklet)

Marks

$\boxed{6}$ (a) Consider the function $y = \ln(\ln x)$.

(i) State the domain of the function.

(ii) Prove that the function is increasing at all points in its domain.

(iii) On separate number planes, sketch the following, clearly labelling all axial intercepts and asymptotes:

(α) $y = \ln(\ln x)$,

(β) $y = \ln(\ln|x|)$,

(γ) $y = \ln|\ln x|$.

$\boxed{3}$ (b) Find a cubic equation with roots α, β and γ such that:

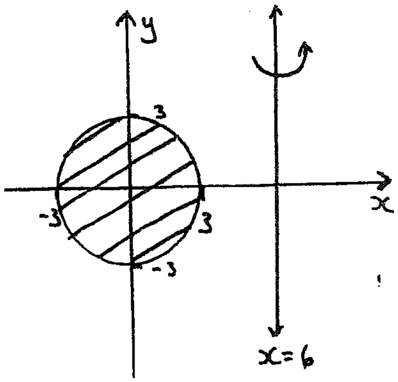
$$\alpha\beta\gamma = 5, \quad \text{and}$$
$$\alpha+\beta+\gamma = 7, \quad \text{and}$$
$$\alpha^2+\beta^2+\gamma^2 = 29.$$

$\boxed{3}$ (c) If α, β and γ are the roots of the equation $8x^3-4x^2+6x-1=0$, find the equation whose roots are $\dfrac{1}{1-\alpha}$, $\dfrac{1}{1-\beta}$ and $\dfrac{1}{1-\gamma}$.

$\boxed{3}$ (d) If the equation $x^3+3kx+\ell=0$ has a double root, where k and ℓ are real, prove that $\ell^2 = -4k^3$.

rks

(a) (i) Prove that the function $f(x) = x\sqrt{a^2 - x^2}$ is odd.



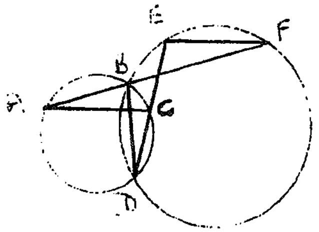
(ii) The diagram shows the region $x^2 + y^2 \leq 9$ and the line $x = 6$. Copy the diagram into your answer booklet.

(iii) Use the method of cylindrical shells to show that if the region $x^2 + y^2 \leq 9$ is rotated about the line $x = 6$ the volume V of the torus formed is given by

$$V = 24\pi \int_{-3}^{3} \sqrt{9 - x^2}\,dx - 4\pi \int_{-3}^{3} x\sqrt{9 - x^2}\,dx\,.$$

(iv) Hence find the volume of the torus.

$\boxed{3}$ (b)



In the diagram above, ABF and DCE are straight lines.

(i) Copy the diagram into your answer booklet.

(ii) Prove that AC is parallel to EF.

Exam continues next page ...

of 9^a to the horizontal, at a constant speed of 70 km/h. Take the acceleration due to gravity to be 10 m/s^2.

(i) Draw a diagram showing all the forces acting on the car.

(ii) By resolving forces vertically and horizontally, calculate the frictional force between the road surface and the wheels, to the nearest Newton.

(iii) What speed (to the nearest km/h) must the driver maintain in order for the car to experience no sideways frictional force?

QUESTION FIVE (Start a new answer booklet)

Marks

$\boxed{6}$ (a) A particle of mass m projected vertically upwards with initial speed u metres per second experiences a resistance of magnitude kmv Newtons when the speed is v metres per second where k is a positive constant. After T seconds the particle attains its maximun height h. Let the acceleration due to gravity be g m/s^2.

(i) Show that the acceleration of the particle is given by

$$\ddot{x} = -(g + kv)\,,$$

where x is the height of the particle t seconds after the launch.

(ii) Prove that T is given by

$$T = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right) \text{ seconds.}$$

(iii) Prove that h is given by

$$h = \frac{u - gT}{k} \text{ metres.}$$

$\boxed{9}$ (b) Let A and B be the points $(1, 1)$ and $(b, \frac{1}{b})$ respectively, where $b > 1$.

(i) The tangents to the curve $y = \dfrac{1}{x}$ at A and B intersect at $C(\alpha, \beta)$. Show that $\alpha = \dfrac{2b}{b+1}$ and $\beta = \dfrac{2}{b+1}$.

(ii) Let A', B' and C' be the points $(1, 0)$, $(b, 0)$ and $(\alpha, 0)$ respectively.

(α) Draw a diagram that represents the information above.

(β) Obtain an expression for the sum of the areas of the quadrilaterals $ACC'A'$ and $CBB'C'$.

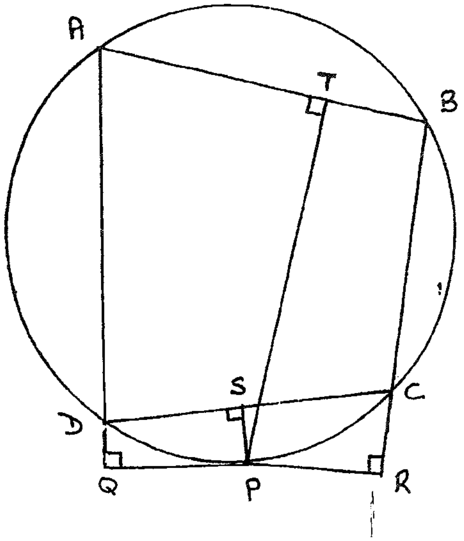
(γ) Hence or otherwise prove that for $u > 0$,

$$\frac{2u}{2 + u} < \ln(1 + u) < u\,.$$

QUESTION SIX (Start a new answer booklet)

Marks

$\boxed{7}$ (a)



In the diagram above, $ABCD$ is a cyclic quadrilateral. P is a point on the circle through A, B, C and D.
PQ, PR, PS and PT are the perpendiculars from P to AD produced, BC produced, CD and AB respectively.

(i) Copy the diagram into your answer booklet.

(ii) Explain why $SPRC$ and $AQPT$ are cyclic quadrilaterals.

(iii) Hence show that $\angle SPR = \angle QPT$ and $\angle PRS = \angle PTQ$.

(iv) Prove that ΔSPR is similar to ΔQPT.

(v) Hence show that:

(α) $PS \times PT = PQ \times PR$,

(β) $\dfrac{PS \times PR}{PQ \times PT} = \dfrac{SR^2}{QT^2}$.

Q7b

(iii) Use the cosine rule to show that

$$\Delta^2 = \tfrac{1}{16}(a^2 - (b-c)^2)((b+c)^2 - a^2).$$

Hence deduce that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(iv) A hole in the shape of the triangle ABC is cut in the top of a level table. A sphere of radius R rests in the hole. Find the height of the centre of the sphere above the level of the table-top, expressing your answer in terms of a, b, c, s and R.

Piii

$\boxed{8}$ (b) The function f is given by

$$f(x) = e^{x/(1+kx)}, \text{ where } k \text{ is a positive constant.}$$

(i) Find $f'(x)$ and $f''(x)$.

(ii) Show $f(x)$ has a point of inflexion at $\left(\frac{1}{2k^2} - \frac{1}{k}, e^{\frac{1}{k}-2}\right)$.

(iii) Show that the tangent to $y = f(x)$ at $x = a$ passes through the origin if and only if

$$k^2a^2 + (2k-1)a + 1 = 0.$$

(iv) Hence show that no tangents to $y = f(x)$ pass through the origin if and only if $k > \frac{1}{4}$.

QUESTION SEVEN (Start a new answer booklet)

Marks

$\boxed{8}$ (a) Let $P(z) = z^7 - 1$.

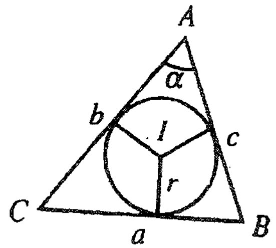
(i) Solve the equation $P(z) = 0$, displaying your seven solutions on an Argand diagram.

(ii) Show that $P(z) = z^3(z-1)\left(\left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 - 2\left(z + \frac{1}{z}\right) - 1\right)$.

(iii) Hence solve the equation $x^3 + x^2 - 2x - 1 = 0$.

(iv) Hence prove that $\operatorname{cosec}\frac{\pi}{14}\operatorname{cosec}\frac{3\pi}{14}\operatorname{cosec}\frac{5\pi}{14} = 8$.

$\boxed{7}$ (b)



The diagram above shows a circle, centre I and radius r, touching the three sides of a triangle ABC. Denote AB by c, BC by a and AC by b.
Let $\angle BAC = \alpha$, $s = \frac{1}{2}(a+b+c)$ and $\Delta =$ the area of triangle ABC.

(i) By considering the area of the triangles AIB, BIC and CIA, or otherwise, show that $\Delta = rs$.

(ii) By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that

$$\Delta^2 = \tfrac{1}{16}(4b^2c^2 - (2bc\cos\alpha)^2).$$

← iii & iv

(i) Evaluate I_1.

(ii) Show that, for $r \geq 1$:

(α) $I_{2r+1} - I_{2r-1} = \dfrac{1-(-1)^r}{2r}$,

(β) $I_{2r} - I_{2r-2} = \dfrac{1}{2r-1}$.

(iii) Hence evaluate I_8 and I_9.

] (b) The Bernoulli polynomials $B_n(x)$, are defined by $B_0(x) = 1$ and, for $n = 1, 2, 3, \ldots$,

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \text{ and}$$

$$\int_0^1 B_n(x)\,dx = 0.$$

Thus

$$B_1(x) = x - \tfrac{1}{2},$$
$$B_2(x) = x^2 - x + \tfrac{1}{6},$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.$$

(i) Show that $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$.

(ii) Show that, for $n \geq 2$, $B_n(1) - B_n(0) = 0$.

(iii) Show, by mathematical induction, that for $n \geq 1$;

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

(iv) Hence show that for $n \geq 1$ and any positive integer k;

$$n\sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n(0).$$

(v) Hence deduce that $\displaystyle\sum_{m=0}^{135} m^4 = 9\,134\,962\,308$.

REP

~~For $n = 0, 1, 2, 3 \ldots$, define~~

$$I_n = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2nx}{\sin 2x}\, dx .$$

~~(i) Evaluate I_1 .~~

~~(ii) Show that, for $r \geq 1$:~~

~~(α) $I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}$~~

~~(β) $I_{2r} - I_{2r-2} = \frac{1}{2r-1}$.~~

~~(iii) Hence evaluate I_8 and I_9 .~~

~~b) The Bernoulli polynomials $B_n(x)$, are defined by $B_0(x) = 1$ and, for $n = 1, 2, 3, \ldots$,~~

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x) , \text{ and}$$

$$\int_0^1 B_n(x)\, dx = 0 .$$

~~Thus~~

$$B_1(x) = x - \tfrac{1}{2} ,$$
$$B_2(x) = x^2 - x + \tfrac{1}{6} ,$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x .$$

~~(i) Show that $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$.~~

~~(ii) Show that, for $n \geq 2$, $B_n(1) - B_n(0) = 0$.~~

~~(iii) Show, by mathematical induction, that for $n > 1$,~~

$$B_n(x+1) - B_n(x) = nx^{n-1} .$$

~~(iv) Hence show that for $n \geq 1$ and any positive integer k ;~~

$$n \sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n(0) .$$

~~(v) Hence deduce that $\sum_{m=0}^{135} m^4 = 9\ 134\ 962\ 308$.~~

REP

1. (a)(i) $z + \frac{1}{z}$

$= \frac{3+2i}{3-i} + \frac{3-i}{3+2i}$

$= \frac{(3+2i)^2 + (3-i)^2}{(3-i)(3+2i)}$.

$= \frac{3+2i}{3-i}$ ✓

(ii) $= \frac{(3+2i)(3+i)}{10}$

$= \frac{7}{10} + \frac{9}{10} i$ ✓

(α) $\overline{\left(z + \frac{1}{z}\right)} = \frac{7}{10} - \frac{9}{10} i$ ✓

(β) $\mathrm{Im}\left(z + \frac{1}{z}\right) = \frac{9}{10}$ ✓

(b)(i) $-(\sqrt{27} + 3i) = -\sqrt{27} - 3i$

$\Rightarrow$ modulus $= 6$; $\arg z = -\frac{5}{6}\pi$ ✓

$\therefore -(\sqrt{27}+3i) = 6\cos\left(-\frac{5}{6}\pi\right) + 6i\sin\left(-\frac{5}{6}\pi\right)$ ✓

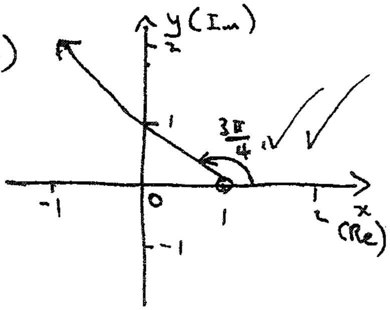
(ii) $(-6\sqrt{27} - 3i)^6$

$= 6^6\left(\cos(-5\pi) + i\sin(-5\pi)\right)$ ✓

But $\cos(-5\pi) = -1$ & $\sin(-5\pi) = 0$ ✓

$= -46656 + 0i$ ✓

(c)(i)



✓ ✓

IV

(ii) Let $z = x + iy$

$\therefore z(\bar{z} + 2)$

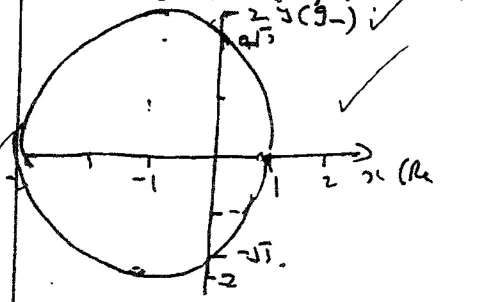
$= x^2 + y^2 + 2x + 2iy$

$\Rightarrow \mathrm{Re}(z\bar{z} + 3) = 3$

$\Rightarrow x^2 + y^2 + 2x = 3$

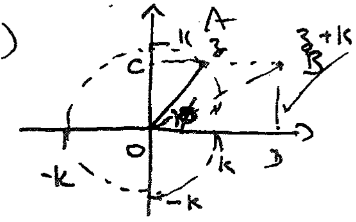
ie $(x+1)^2 + y^2 = 4$

ie Circle centre $(-1, 0)$ rad. 2. ✓



✓

(d)



$z + k = k(1+\cos\phi) + ik\sin\phi$

$= k\left(1 + 2\cos^2\frac{\phi}{2} - 1\right) + 2ik\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$= 2k\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2} + i\sin\phi/2\right)$

✓ ✓

$\therefore \arg(z+k) = \frac{\phi}{2}$

or $\angle CAO = \phi$ (alt $\angle$'s)

But Δ OAB is isosceles.

$\therefore \angle AOB = \phi/2$ (Ext. $\angle$)

ie $\angle DOB = \arg(z+k) = \phi/2$ //

15

Q2. (a) $\int x^2 e^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} + \frac{1}{2}\int 2x\,e^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}x\,e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)e^{-2x} + c$

(b) (i)

$\frac{9+x-2x^2}{(1-x)(3+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{3+x^2}$

$\therefore A = \frac{9+1-2}{3+1} = 2$

$\therefore 9+x^2-2x^2 = 2(3+x^2)+(1-x)(Bx+C)$

$\therefore 9 = 6 + C \Rightarrow C = 3$

& $-2 = 2 - B \Rightarrow B = 4$

i.e. $\frac{9+x-2x^2}{(1-x)(3+x^2)} = \frac{2}{1-x} + \frac{4x+3}{3+x^2}$

(ii) $\int \frac{9+x^2-2x^2}{(1-x)(3+x^2)}\,dx$

$= \int \frac{2}{1-x} + \frac{4x}{3+x^2} + \frac{3}{3+x^2}\,dx$

$= -2\ln|1-x| + 2\ln(3+x^2)$

$+ \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

(c) (i) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$

$= \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}0$

$= \frac{\pi}{3}$.

(ii) $\int_0^{\pi/3} \sec^4\theta\,\tan\theta\,d\theta$

$= \frac{1}{4}\sec^4\theta\Big]_0^{\pi/3}$

$= \frac{1}{4}\left(2^4 - 1\right)$

$= \frac{15}{4} \quad (= 3.75)$

(iii) $\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$

Let $t = \tan\frac{\theta}{2} \Rightarrow \frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$= \frac{1}{2}(1+t^2)$

$\Rightarrow d\theta = \frac{2}{1+t^2}\,dt$

θ	0	$\frac{\pi}{2}$
x	0	1

$= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{2}{1+t^2+2t+1-t^2}\,dt$

$= \int_0^1 \frac{1}{1+t}\,dt$

$= \ln|1+t|\Big|_0^1$

$= \ln 2$

15

(ii) $y = \ln(\ln x)$

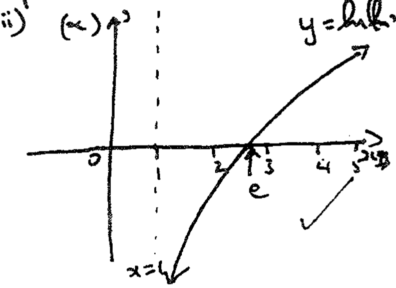
$\Rightarrow \frac{dy}{dx} = \frac{1}{x\ln x}$

For $x>1$, $\ln x>0 \Rightarrow x\ln x>0$

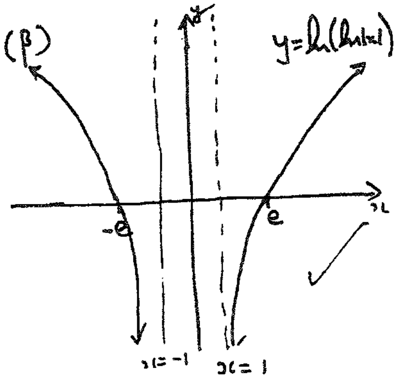
$\Rightarrow \frac{dy}{dx}>0$, for $x>1$

i.e. The function is increasing at all points in its domain.

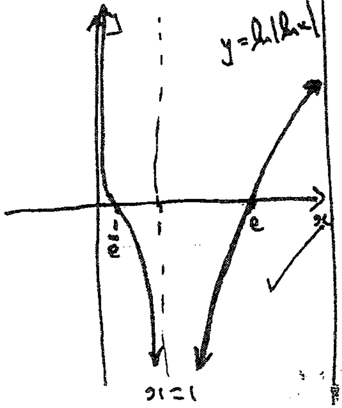
(iii) (α)



(β)



(γ)



(b) Let an equation be

$x^3+bx^2+cx+d=0$

$d = -\alpha\beta\gamma = -5$

$b = -(\alpha+\beta+\gamma) = -7$

& $c = \alpha\beta+\alpha\gamma+\beta\gamma$

Now, $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\Rightarrow \alpha\beta+\alpha\gamma+\beta\gamma = \frac{7^2-29}{2}$

$\Rightarrow c = 10$

Hence an equation is

$x^3-7x^2+10x-5=0$.

(or any multiple of this equation).

(c) Let $y = \frac{1}{1-x}$

$\Rightarrow x = 1-\frac{1}{y} = \frac{y-1}{y}$

Hence the required equation is:

$8\left(\frac{y-1}{y}\right)^3 - 4\left(\frac{y-1}{y}\right)^2 + 6\left(\frac{y-1}{y}\right) - 1 = 0$

ie $8(y-1)^3 - 4y(y-1)^2 + 6y^2(y-1) - y^3 = 0$

i.e. $9y^3 - 22y^2 + 20y - 8 = 0$

(or any multiple!)

(d) Let the double root be α

$\therefore \frac{d}{dx}\left(x^3+3kx+l\right)\Big|_{x=\alpha} = 0$

i.e $3\alpha^2 + 3K = 0$

$\Rightarrow \alpha = (-K)^{\frac{1}{2}}$

$\therefore (-k)^{3/2} + 3k(-k)^{\frac{1}{2}} + l = 0$

i.e. $-K(-k)^{\frac{1}{2}} + 3k(-k)^{\frac{1}{2}} = -l$

so $2k(-k)^{\frac{1}{2}} = -l$

$\therefore -4k^3 = l^2$

15

4(a)(i) f(x)

$\therefore f(\alpha) = \alpha\sqrt{a^2-\alpha^2}$

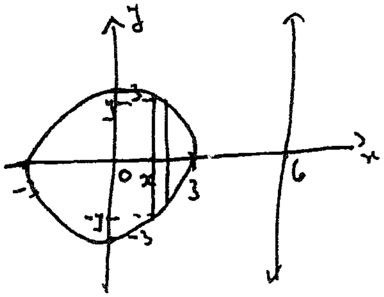
$+ f(-\alpha) = -\alpha\sqrt{a^2-(-\alpha)^2}$

$= -\alpha\sqrt{a^2-\alpha^2}$

ie $f(\alpha) = -f(\alpha)$

ie $f(x)$ is an odd function.

ii)



iii) Consider a typical slice of width δx, x units from O. When this is rotated about $x=6$ it generates a cylindrical shell of radius $6-x$, height $2y$ and thickness δx.

$\therefore \delta V \doteqdot 2\pi(6-x)\,2y\,\delta x$

$= 4\pi(6-x)\sqrt{9-x^2}\,\delta x$

$\therefore V = 4\pi\int_{-3}^{3}(6-x)\sqrt{9-x^2}\,dx$

$= 24\pi\int_{-3}^{3}\sqrt{9-x^2}\,dx - 4\pi\int_{-3}^{3}x\sqrt{9-x^2}\,dx$

(iv) By (i) $\int_{-3}^{3}x\sqrt{9-x^2}\,dx = 0$

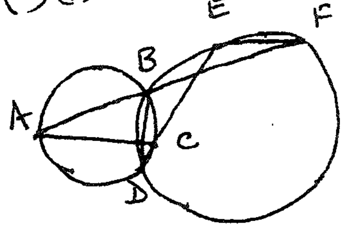
as $x\sqrt{9-x^2}$ is odd.

$\int_{-3}^{3}\sqrt{9-x^2}\,dx$ = Area of sem-circle radius $3 = \frac{1}{2}\pi\times3\times3$

$\therefore V = 24\pi \times \frac{9}{2}\pi$

ie $V = 108\pi^2$ units^3.

(b)(i)



(ii) ∠BAC = ∠BDC (∠'s in same

But in circle DBEF

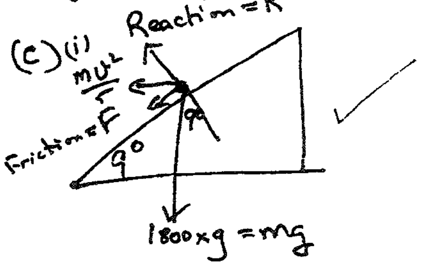
∠BDE = ∠BFE (∠'s in same

But ∠BDC = ∠BDE

$\therefore$ ∠BAC = ∠BFE

⇒ AC // EF as the alternate angles are equal.

(c)(i)



(ii) Resolve ↑:

$R\cos 9° - F\sin 9° = mg$ ①

$\rightarrow$: $R\sin 9° + F\cos 9° = \frac{mv^2}{r}$ ②

② × cos 9° − ① × sin 9° ⇒

$F = m\left(\frac{v^2}{r}\cos 9° - g\sin 9°\right)$

$= 1.8\times10^3\left(\left(\frac{70\,000}{3600}\right)^2\frac{\cos 9°}{130} - 10\sin\ldots\right)$

= 2355 N (nearest N)

(iii) No sidways friction force

ie $R\cos 9° = mg$ ①

and $R\sin 9° = \frac{mv^2}{R}$ ②

② ÷ ①: $\tan 9° = \frac{v^2}{Rg}$

$\therefore v = \sqrt{Rg\tan 9°}$

$= \sqrt{\tan 9° \times 10\times130}$ m/s

$= \sqrt{\tan 9°\times10\times130} \times \frac{3600}{1000}$ km/hr

= 51·6 . . .

= 52 km/h (nearest km/h).

IVI

15

5 (a)

(i) Take up as positive

$F = m\ddot{x} = -mg - kmv$

kmv

mg

$\therefore \ddot{x} = -(g+kv)$

(ii) $\therefore \frac{dv}{dt} = -(g+kv)$

$\therefore \int_u^0 \frac{dv}{g+kv} = -\int_0^T dt$

$\therefore -T = \frac{1}{k} \ln|g+kv| \Big|_u^0$

$= \frac{1}{k}\{\ln g - \ln(g+ku)\}$

$\therefore T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$ seconds

(iii) Now $\ddot{x} = v\frac{dv}{dx}$

so $v\frac{dv}{dx} = -(g+kv)$

$\therefore \int_0^h dx = -\int_u^0 \frac{v}{g+kv}\,dv$

$$kv+g \overline{\big)\, v \quad} \quad \frac{1}{k}$$

$$\frac{v + g/k}{-g/k}$$

$\therefore h = \int_0^u \frac{1}{k} - \frac{\frac{g}{k}}{g+kv}\,dv$

$h = \left[\frac{v}{k} - \frac{g}{k^2}\ln|g+kv|\right]_0^u$

$h = \frac{1}{k}\left[u - \frac{g}{k}\ln(g+ku) + \frac{g}{k}\ln g\right]$

$h = \frac{1}{k}\left[u - \frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right]$

$\therefore h = \frac{1}{k}\left[u - gT\right]$

i.e. $h = \frac{u-g}{k}$ metres.

(b)(i) At A $\frac{dy}{dx} = -\frac{1}{x^2}$

so gradient of tgt $= -1$.

$\therefore$ Eq. of tgt. at A is: $\frac{y-1}{x-1} = -$

i.e. $x + y - 2 = 0$

At B, gradient of tgt. $= -\frac{1}{b}$

$\therefore$ eq. of tgt. is:

$x + by - 2b = 0$

(α, β) is the sol^n of

$$\left.\begin{array}{l} x+y-2=0 \\ x+by-2b=0 \end{array}\right\}$$

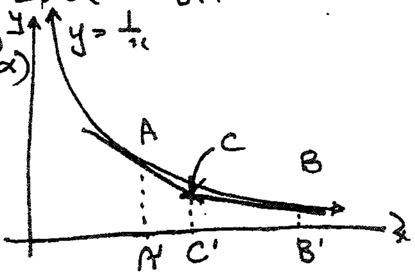
$\therefore (b^2-1)y = 2(b-1)$

$\therefore y = \frac{2}{b+1} = \beta$

& $x = 2 - y = 2 - \frac{2}{b+1}$

$\Rightarrow \alpha = \frac{2b}{b+1}$.

(ii) (α)



(β) Area ACC'A'

$= \frac{1}{2}\left(1 + \frac{2}{b+1}\right)\left(\frac{2b}{b+1} - \right.$

$= \frac{1}{2} \cdot \frac{b+3}{b+1} \cdot \frac{b-1}{b+1}$

$= \frac{(b+3)(b-1)}{2(b+1)^2}$.

Area of CBB'C'

$= \frac{1}{2}\left(\frac{2}{b+1} + \frac{1}{b}\right)\left(b - \frac{2b}{b+1}\right)$

$= \frac{1}{2}\,\frac{3b+1}{b(b+1)} \cdot \frac{b(b-1)}{b+1}$

$= \frac{(3b+1)(b-1)}{2(b+1)^2}$

$\therefore$ Sum of areas

$= \frac{1}{2(b+1)^2}\left((b+3)(b-1) + (3b+1)(b-1)\right)$

$= \frac{1}{2(b+1)^2}\left(4b^2 - 4\right)$

$= \frac{2(b-1)}{b+1}$.

(γ) Area ABB'A

$= \frac{1}{2}\left(1 + \frac{1}{b}\right)(b-1)$

$= \frac{b^2-1}{2b}$

Now for $b > 1$

$\frac{2(b-1)}{b+1} < \int_1^b \frac{1}{x}dx < \frac{b^2-1}{2b}$

$\therefore \frac{2(b-1)}{b+1} < \ln b < \frac{(b-1)(b+1)}{2b}$

Let $b = 1 + u > 1$

ie $u > 0$.

$\therefore \frac{2u}{2+u} < \ln(1+u) < \frac{u(2+u)}{2(1+u)}$

Now $\frac{2+u}{2(1+u)} < 1$, $u > 0$.

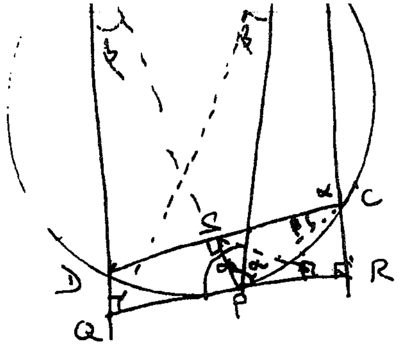
ie $\frac{u(2+u)}{2(1+u)} < u$.

IVii

so

$\frac{2u}{2+u} < \ln(1+u) < u, u >$

15



i) As $\angle DSP = \angle PRC \left(= \frac{\pi}{2}\right)$ the ext. ∠ of quad. SPRC = the int. opp. ∠ & hence PRC is a cyclic qua
As $\angle ATP = \angle AQP = \pi/2$ & hence are supplementary which ⇒ AQPT is a cyclic quad. ✓

iii) Let $\angle SPR = \alpha$.
∴ $\angle SCB = \alpha$ (Ext. ∠, cyclic quad SPRC)
∴ $\angle BAD = \alpha$ (Opp. ∠'s cyclic quad ABCD)
∴ $\angle QPT = \alpha$ (Opp ∠'s cyclic quad AQPT) ✓
ie $\angle SPR = \angle QPT$

Let $\angle PRS = \beta$
∴ $\angle PCS = \beta$ (∠'s in same seg. cyclic quad SPRC)
$\angle PAD = \beta$ (∠'s in same seg. cyclic quad ACPD
∴ $\angle PTQ = \beta$ (. . . . . AQPT)
ie $\angle PRS = \angle PTQ$ ✓ ✓

iv) In Δ's SPR & QPT
(1) $\angle SPR = \angle QPT (=\alpha)$ part (ii)
(2) $\angle PRS = \angle PTQ$ $(=\beta)$ part (iii)
∴ $\Delta SPR \parallel \Delta QPT$ (AA) ✓

v) (α) ∴ $\frac{PS}{PQ} = \frac{PR}{PT}$ (corr. sides ||| Δ's proportional

(β) Now $\frac{PS}{PQ} = \frac{PR}{PT} = \frac{SR}{QT}$ (Corr. sides prop$^{\underline{l}}$)

∴ $\frac{PS.PR}{PQ.PT} = \frac{SR}{QT} \cdot \frac{SR}{QT} = \frac{SR^2}{QT^2}$ // ✓

(b) (i) $f'(x) = e^{\frac{x}{1+kx}} \cdot \frac{1+kx-kx}{(1+kx)^2}$

$= \frac{1}{(1+kx)^2} e^{\frac{x}{1+kx}}$ ✓

$f''(x) = \frac{-2k}{(1+kx)^3} e^{\frac{x}{1+kx}} +$
$\frac{1}{(1+kx)^4} e^{\frac{x}{1+kx}}$
$= e^{\frac{x}{1+kx}} \cdot \frac{(1-2k-2k^2x)}{(1+kx)^4}$ ✓ ✓

(ii) $f''(x) = 0$ only when
$1-2k-2k^2x = 0$
ie $x = \frac{1}{2k^2} - \frac{1}{k}$ ✓

Clearly the sign of $f''(x)$ depends on the sign of $1-2k-2k^2x$ only.
This is a linear function in x.
Hence it changes sign when $x = \frac{1}{2k^2} - \frac{1}{k}$ and so $f(x)$ has a point of inflexion at
$x = \frac{1}{2k^2} - \frac{1}{k}$. ✓

∴ $f\left(\frac{1}{2k^2} - \frac{1}{k}\right) = e^{\frac{1}{k}-2}$
ie Pt. of inflexion at
$\left(\frac{1}{2k^2} - \frac{1}{k}, e^{\frac{1}{k}-2}\right)$ ✓

(iii) Tgt. at $x=a$ has grad$^{\underline{t}}$
$\frac{1}{(1+ka)^2} e^{\frac{a}{1+ka}}$

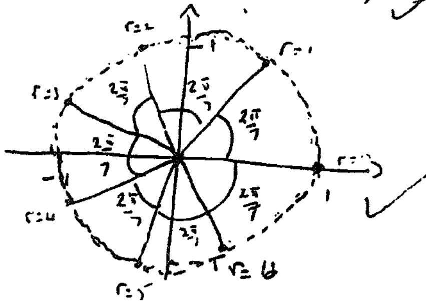
∴ Eq. of tgt. at $x=a$ is:
$y - e^{\frac{a}{1+ka}} = \frac{1}{(1+ka)^2} e^{\frac{a}{1+ka}}(x-a)$ ✓
Passes through (0,0) iff
$-e^{\frac{a}{1+ka}} = -\frac{a}{(1+ka)^2} e^{\frac{a}{1+ka}}$
i.e. $\frac{a}{(1+ka)^2} = 1$
$\Leftrightarrow a = (1+ka)^2$
$\Leftrightarrow a = 1 + 2ka + k^2a^2$
$\Leftrightarrow k^2a^2 + (2k-1)a + 1 = 0$ ✓

(iv) Only such tgts if a is real.
i.e. $\Delta < 0 \Rightarrow$ no such tgts.
$\Delta = (2k-1)^2 - 4k^2 < 0$
$\Rightarrow -4k + 1 < 0$
$\Rightarrow -4k < -1$
ie $k > \frac{1}{4}$ // ✓

15

$\therefore z^7 = 1$

$\Rightarrow z = \cos\frac{2\pi r}{7} + i\sin\frac{2\pi r}{7}$, $r = 0,1,2,3,4,5,6$ ✓



(ii) $P(z) = z^7 - 1$

$= (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$ ✓

$= z^3(z-1)\left(z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right.$

$= z^3(z-1)\left(\left(z + \frac{1}{z}\right)^3 - 3z - \frac{3}{z} + \left(z + \frac{1}{z}\right)^2 - 2\right.$

$\left. - 2\left(z + \frac{1}{z}\right) + 3z + \frac{3}{z} + 1\right)$

$= z^3(z-1)\left(\left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 - 2\left(z + \frac{1}{z}\right) - 1\right.$ ✓

(iii) Now $P(z) = 0$

$\Rightarrow \left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 - 2\left(z + \frac{1}{z}\right) - 1 = 0$

But $z + \frac{1}{z} = 2\cos\frac{2\pi r}{7}$, $r = 0, 1, \ldots, 6$ ✓

Let $z + \frac{1}{z} = x$

ie $x^3 + x^2 - 2x - 1 = 0$ has

Solutions $2\cos\frac{2\pi r}{7}$, $r = 1,2,3,4,5,6$ ✓

But $\cos\frac{2\pi}{7} = \cos\frac{12\pi}{7}$, $\cos\frac{4\pi}{7} = \cos\frac{10\pi}{7}$ + $\cos\frac{6\pi}{7} = \cos$

⇒ solution are

$2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$ & $2\cos\frac{6\pi}{7}$ ✓

Tix

(iv) Now the product of the roots of $x^3 + x^2 - 2x - 1 = 0$ is:

$8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = 1.$

$\therefore \sec\frac{2\pi}{7}\operatorname{cosec}\frac{4\pi}{7}\sec\frac{6\pi}{7} = 8$

i $\operatorname{cosec}\left(\frac{1}{2} - \frac{2}{7}\right)\pi \operatorname{cosec}\left(\frac{1}{2} - \frac{4}{7}\right)\pi \operatorname{cosec}\left(\frac{1}{2} - \frac{6}{7}\right)\pi = 8$

ie $\operatorname{cosec}\frac{3}{14}\pi \cdot \operatorname{cosec}\frac{-1}{14}\pi \cdot \operatorname{cosec}-\frac{5}{14}\pi = 8$

ie $\operatorname{cosec}\frac{3\pi}{14}\left(-\operatorname{cosec}\frac{\pi}{14}\right)\left(-\operatorname{cosec}\frac{5\pi}{14}\right) = 8$

ie $\operatorname{cosec}\frac{\pi}{14}\operatorname{cosec}\frac{3\pi}{14}\operatorname{cosec}\frac{5\pi}{14} = 8$ ✓✓

7(b)(i) Δ = Area ΔAIB + Area ΔBIC + Area ΔCIA

$= \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br$

$= r\,\frac{a+b+c}{2}$

ie $\Delta = rs$ ✓

(ii) $\Delta^2 = \frac{1}{4}b^2c^2\sin^2\alpha$

$= \frac{1}{16}\left(4b^2c^2\left(1 - \cos^2\alpha\right)\right)$

$\therefore \Delta^2 = \frac{1}{16}\left(4b^2c^2 - (2bc\cos\alpha)^2\right)$ ✓

(iii) $2bc\cos\alpha = b^2 + c^2 - a^2$ (cosine rule)

$\therefore \Delta^2 = \frac{1}{16}\left(4b^2c^2 - (b^2 + c^2 - a^2)^2\right)$ ✓

$= \frac{1}{16}\left(2bc + b^2 + c^2 - a^2\right)\left(2bc - b^2 - c^2 + a^2\right)$

$= \frac{1}{16}\left(a^2 - (b^2 - 2bc + c^2)\right)\left(b^2 + 2bc + c^2 + a^2\right.$

$= \frac{1}{16}\left(a^2 - (b-c)^2\right)\left((b+c)^2 - a^2\right)$ ✓

Now $\frac{1}{2}(a+b-c) = \frac{\quad}{2} - c = s-c$

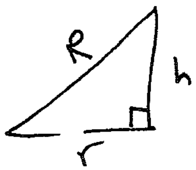
similarly $\frac{1}{2}(a-b+c) = s-b$

and $\frac{1}{2}(b+c-a) = s-a$.

$\therefore \Delta^2 = (s-c)(s-b)(s-a)s$

ie $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ ✓ $(\Delta \geqslant 0)$

(iv) Consider the cross section and let the required height = h.



$$h^2 = R^2 - r^2$$
$$= R^2 - \frac{\Delta^2}{s^2} \quad (\text{from (i)})$$
$$= R^2 - \frac{\cancel{s}(s-a)(s-b)(s-c)}{s^{\cancel{2}}}$$

ie $h = \sqrt{\frac{1}{s}\left(sR^2 - (s-a)(s-b)(s-c)\right)}$ ✓ //

15

$$= \int_0^{\pi/4} \frac{1-(1-2\sin^2 x)}{2\sin x \cos x}\,dx$$
$$= \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$
$$= -\ln|\cos x|\Big]_0^{\pi/4}$$
$$= -\ln\left(\cos\frac{\pi}{4}\right) + \ln(\cos 0)$$
$$= \ln\sqrt{2} \quad \left(= \tfrac{1}{2}\ln 2\right). ✓$$

(ii)(α) $I_{2r+1} - I_{2r-1}$

$$= \int_0^{\pi/4} \frac{1-\cos 2(2r+1) - 1 + \cos 2(2r-1)x}{\sin 2x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2(2r-1)x - \cos 2(2r+1)x}{\sin 2x}\,dx$$
$$= \int_0^{\pi/4} \frac{2\sin 4rx \sin 2x}{\sin 2x}\,dx$$
$$= 2\int_0^{\pi/4} \sin 4rx\,dx \quad ✓$$
$$= \frac{1}{2r}\Big[-\cos 4rx\Big]_0^{\pi/4}$$
$$= \frac{1}{2r}\Big[1 - \cos r\pi\Big]$$

But $\cos r\pi = \begin{cases} 1, & r \text{ even} \\ -1, & r \text{ odd}\end{cases} = (-1)^r$

$\therefore I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}$ ✓

$I_{2r} - I_{2r-2}$

$$= \int_0^{\pi/4} \frac{1-\cos 4rx - 1 + \cos(4r-4)x}{\sin 2x}\,dx$$
$$= \int_0^{\pi/4} \frac{\cos(4r-4)x - \cos 4rx}{\sin 2x}\,dx$$
$$= \int_0^{\pi/4} 2\sin(4r-2)x\,dx$$
$$= \frac{-1}{2r-1}\cos 2(2r-1)x\Big|_0^{\frac{\pi}{4}}$$
$$= \frac{-1}{2r-1}\Big[\cos(2r-1)\frac{\pi}{2} - \cos 0$$
$$= \frac{1}{2r-1}$$

$\therefore I_{2r} - I_{2r-2} = \frac{1}{2r-1}$

(iii) $I_8 = \frac{1}{7} + I_6$
$= \frac{1}{7} + \frac{1}{5} + I_4$
$= \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + I_2$
$= \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + 1 + I_0$

$I_0 = 0$

$\therefore I_8 = \frac{176}{105}\left(\doteqdot 1.676 \text{ or } 1\frac{71}{105}\right)$

$I_9 = \frac{1-(-1)^4}{8} + I_7 = 0 + \frac{1-(-1)^3}{6} + I_5$
$= \frac{1}{3} + 0 + I_3$
$= \frac{1}{3} + 1 + I_1$
$= \frac{4}{3} + \ln\sqrt{2}$ // ✓

(b)(i)

$$B_4(x) = 4\int B_3(x)\,dx$$
$$= \int 4x^3 - 6x^2 + 2x\,dx$$
$$= x^4 - 2x^3 + x^2 + C$$

Now $\int_0^1 x^4 - 2x^3 + x^2 + C\,dx =$

$\therefore \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + Cx\Big|_0^1 = 0$

$\therefore C = \frac{1}{2} - \frac{1}{5} - \frac{1}{3}$
$= -\frac{1}{30}$

$\therefore B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$
$\therefore B_4(x) = x^2(x-1)^2 - \frac{1}{30}$ ✓

(ii) $B_n(1) - B_n(0)$
$= \int_0^1 B_{n-1}(x)\,dx$
$= 0$, by definition. ✓

i.e. $B_n(1) - B_n(0) = 0$
If $n=1$: $\int_0^1 B_0(x)dx = \int_0^1 dx$
$= 0.$

(iii) Let $S(n)$ be the statement that $B_n(x+1) - B_n(x) = nx^{n-1}$ for some positive integer n.

Now $B_1(x+1) - B_1(x)$
$= x+1-\frac{1}{2} - (x - \frac{1}{2})$
$= 1$
$= 1.\,x^{1-1}$
Hence $S(1)$ is true. ✓
Let k be some positive integer for which $S(k)$ is true.
i.e. $B_k(x+1) - B_k(x) = kx^{k-1}$
Now consider
$\frac{d}{dx}\Big(B_{k+1}(x+1) - B_{k+1}(x)\Big)$
$= \frac{d\,B_{k+1}(x+1)}{d(x+1)} \cdot \frac{d(x+1)}{dx} - \frac{d}{dx}B_{k+1}(x)$
$\uparrow$
(Chain rule)

$(k+1)B_k(x+1) - (k+1)B_k(x)$
$= (k+1)\big(B_k(x+1) - B_k(x)\big)$
$= (k+1).\,k\,x^{k-1}$ (By the assume

$\therefore B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^k$
But this is true for all x.
Let $x=0$.
$\therefore B_{k+1}(1) - B_{k+1}(0) = C$
$\Rightarrow C = 0$, from (ii).
So $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)$
i.e. $S(k)$ true $\Rightarrow S(k+1)$ is true for any integer $k \geq 1$. ✓
i.e. $B_n(x+1) - B_n(x) = nx^{n-1}$, $n \geq 1$.

(iv) Now from the above:
$B_n(1) - B_n(0) = n.\,0^{n-}$
$B_n(2) - B(1) = n.\,1^{n-1}$
$B_n(3) - B_n(2) = n.\,2^{n}$
$\vdots \qquad \vdots \qquad \vdots$
$B_n(k) - B_n(k-1) = n(k-1)$
$B_n(k+1) - B_n(k) = n\,k^{n-1}$
Now Sum these equations:
Sum of LHS $= B_n(k+1) - B_n($
Sum of RHS $= n(0^{n-1} + 1^{n-1} + \ldots + k^{n}$
$= n\sum_{m=0}^{k} m^{n-1}$
i.e. $n\sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n($ ✓

Q 8 con't.

(v) Now $\sum_{m=0}^{135} m^4 = \frac{1}{5}\Big(B_5(136) - B_5(0)\Big)$

$B_5(x) = 5\int B_4(x)\,dx$
$= \int 5x^4 - 10x^3 + 5x^2 - \frac{1}{6}\,dx$
i.e. $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6} + C$ ✓

$\therefore \sum_{m=0}^{135} m^4 = \frac{1}{5}\Big(136^5 - \frac{5}{2}\times 136^4 + \frac{5}{3}\times 136^3 - \frac{1}{6}\times 136 + C - C\Big)$

$= 9\,134\,962\,208$ as required. ✓

15