# 4 UNIT MATHEMATICS FORM VI

**Time allowed:** 3 hours (plus 5 minutes reading) **Exam date:** 1st August, 2000

**Instructions:**

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

**Collection:**

Each question will be collected separately.
Start each question in a new 8-leaf answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

$\boxed{2}$ (a) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$.

$\boxed{3}$ (b) The integral $I_n$ is defined by $I_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$.

(i) Show that $I_n = n\,I_{n-1} - e^{-1}$.

(ii) Hence show that $I_3 = 6 - 16e^{-1}$.

$\boxed{4}$ (c) Use partial fractions to find $\displaystyle\int \frac{2y+3}{(y-2)(y^2+3)}\,dy$.

$\boxed{2}$ (d) Use integration by parts to find $\displaystyle\int \tan^{-1} x\,dx$.

$\boxed{4}$ (e) (i) Find $\displaystyle\int \frac{dx}{x^2+2x+5}$.

(ii) Hence find $\displaystyle\int \frac{x^2}{x^2+2x+5}\,dx$.

Exam continues next page ...

QUESTION TWO (Start a new answer booklet)

Marks

2 (a) Simplify $(2-3i)^2$.

3 (b) On an Argand diagram, sketch the region specified by both the conditions

$$|z+3-4i| \leq 5 \quad \text{and} \quad \operatorname{Re}(z) \leq 1.$$

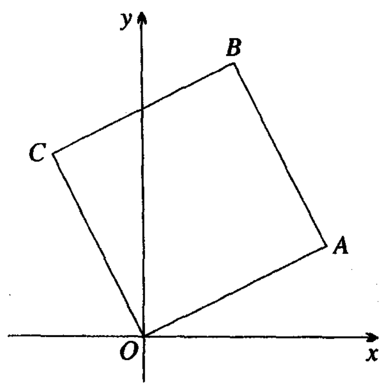
You must show intercepts with the axes, but you do not need to find other points of intersection.

3 (c) (i) Determine the modulus and argument of $-1+i$.

(ii) Hence find the least positive integer value of $n$ for which $(-1+i)^n$ is real.

4 (d) (i) Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in the Argand diagram. Show that multiplication of $z$ by $i$ rotates $z$ by $\frac{\pi}{2}$ anticlockwise about the origin.

(ii)




In the square $OABC$, shown above, the point $A$ represents $2+i$. What complex numbers do the points $B$ and $C$ represent?

3 (e) Let $z = a + ib$, where $a$ and $b$ are both real.

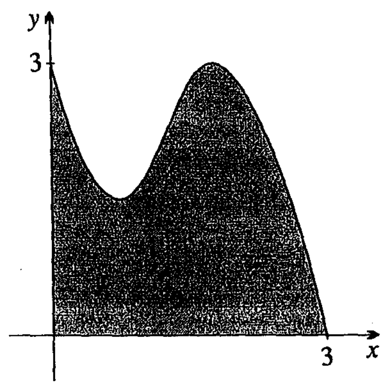
(i) For what values of $a$ and $b$ will $z + \dfrac{1}{z}$ be purely real?

(ii) Is it possible for $z + \dfrac{1}{z}$ to be purely imaginary?

QUESTION THREE (Start a new answer booklet)

Marks

$\boxed{4}$ (a)




In the diagram above, the region under the curve $y = 3 - 4x + 4x^2 - x^3$ in the first quadrant is shaded. A solid of revolution is formed by rotating this region about the $y$-axis. Use the method of cylindrical shells to find the volume of this solid.

$\boxed{5}$ (b) Suppose the cubic equation $x^3 - 2x + 4 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

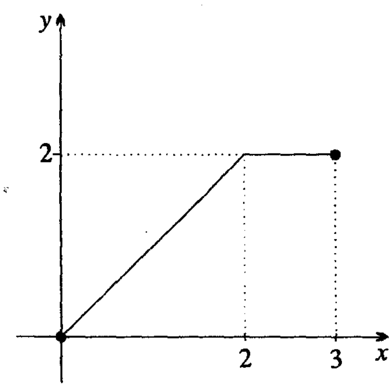
(i) Use the substitution $x^2 = y$ to show that a cubic equation which has roots $\alpha^2$, $\beta^2$ and $\gamma^2$ is:

$$y^3 - 4y^2 + 4y - 16 = 0.$$

(ii) Factorise this new cubic into linear factors by initially grouping in pairs.

(iii) Hence show that the original equation has only one real root.

$\boxed{6}$ (c)




The graph of $y = f(x)$ is shown above. Sketch graphs of:

(i) $y = f(3 - x)$,

(ii) $y = f(|x|)$,

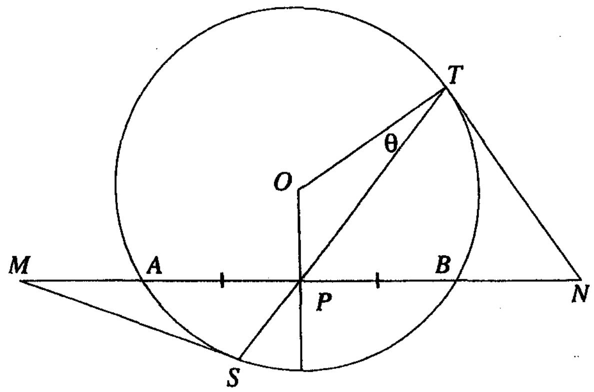
(iii) $y = \dfrac{1}{f(x)}$,

(iv) $y^2 = f(x)$.

QUESTION FOUR (Start a new answer booklet)

Marks

6 (a)




In the diagram above, $P$ is the midpoint of the chord $AB$ in the circle with centre $O$. A second chord $ST$ passes through $P$, and the tangents at the endpoints meet $AB$ produced at $M$ and $N$ respectively.

(i) Explain why $OPNT$ is a cyclic quadrilateral.

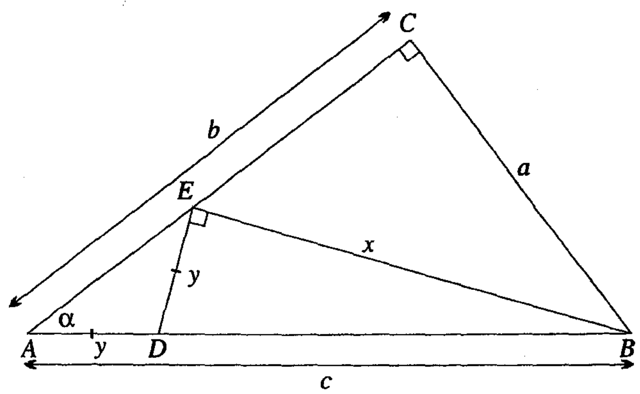
(ii) Explain why $OPSM$ is also cyclic.

(iii) Let $\angle OTS = \theta$. Show that $\angle ONP = \angle OMP = \theta$.

(iv) Hence prove that $AM = BN$.

QUESTION FOUR (Continued)

6 (b)




In the diagram above, $\triangle ABC$ is right-angled at $C$, with $a < b < c$. The points $E$ on $AC$ and $D$ on $AB$ are constructed so that $\angle BED$ is a right angle and $\triangle ADE$ is isosceles with $AD = DE$.

Let $EB = x$, let $AD = DE = y$, and let $\angle CAB = \alpha$.

(i) Prove that $\triangle ABC \,|||\, \triangle BEC$.

(ii) Show that $x = \dfrac{ca}{b}$.

(iii) Explain why $\angle BDE = 2\alpha$.

(iv) Hence show that $y = \dfrac{c(b^2 - a^2)}{2b^2}$.

3 (c) Suppose the function $f(x)$ may be written as $f(x) = g(x) + h(x)$, where $g(x)$ is even and $h(x)$ is odd.

(i) Find an expression for $g(x)$ in terms of $f(x)$ alone.

(ii) Hence write down $g(x)$ for the function $f(x) = e^x$.

Exam continues next page ...

QUESTION FIVE (Start a new answer booklet)

Marks

$\boxed{8}$ (a) An object of mass $m$ kg is projected vertically upwards from ground level with an initial speed $U$ m/s. Its characteristic shape results in air resistance which is proportional to the square of its velocity, that is, $mkv^2$ for some constant $k$. The only other force acting on the body is that due to gravity.

Take upwards as the positive direction for displacement $x$. Take ground level as the origin of displacement.

(i) ($\alpha$) Show that $\ddot{x} = -(g + kv^2)$.

($\beta$) Use $\ddot{x} = \dfrac{dv}{dt}$ to show that the time $T_u$ taken to reach the highest point of its flight is

$$T_u = \frac{1}{\sqrt{gk}} \tan^{-1}\left(U\sqrt{\frac{k}{g}}\right).$$

(ii) Let $T_d$ be the time taken for the object to fall back down to the ground, and for convenience let $w = U\sqrt{\frac{k}{g}}$. It can be shown that $\sqrt{gk}\,(T_d - T_u)$ simplifies to the function

$$f(w) = \log_e\left(w + \sqrt{w^2+1}\right) - \tan^{-1} w.$$

($\alpha$) Evaluate $f(0)$.

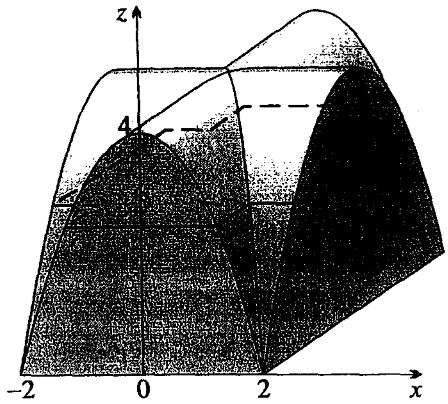
($\beta$) Determine $f'(w)$, and show that $f'(w) > 0$ for $w > 0$.

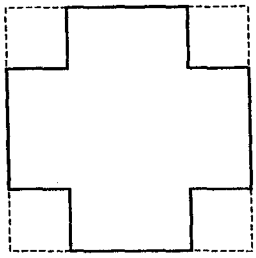
($\gamma$) Hence show that it takes longer for the object to fall back to the ground than it does to reach its highest point.

Exam continues overleaf ...

QUESTION FIVE (Continued)

7 (b)



A sandstone cap on the corner of a fence is shown above, formed in the shape of two intersecting parabolic cylinders.

On the front face, the equation of the parabola is $z = 4 - x^2$, where $x$ is the horizontal distance measured from the mid-point of the base of the front face, and $z$ is the height.

The shape of a horizontal slice of thickness $dz$ taken at height $z$ is also shown. It is a square with four smaller squares removed, one from each corner.

(i) Find $x$ in terms of $z$.

(ii) Show that $V = \displaystyle\int_0^4 \left(4^2 - 4(2 - \sqrt{4 - z}\,)^2\right) dz$.

(iii) Hence find the volume of stone in the cap.

Exam continues next page ...

QUESTION SIX (Start a new answer booklet)

Marks

$\boxed{8}$ (a) Let $P(z) = z^7 - 1$.

(i) Use de Moivre's theorem to find the roots of $P(z)$.

(ii) Hence write $P(z)$ as a product of:

($\alpha$) real and complex linear factors,

($\beta$) real linear and irreducible quadratic factors.

(iii) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

(iv) ($\alpha$) Write down the quotient when $P(z)$ is divided by $z - 1$.

($\beta$) Hence show that $(1 - \cos\frac{2\pi}{7})(1 - \cos\frac{4\pi}{7})(1 - \cos\frac{6\pi}{7}) = \frac{7}{8}$.

$\boxed{7}$ (b) For $n = 0, 1, 2, \ldots$, the integral $G_n$ is defined by

$$G_n = \int_0^{\pi} \frac{\sin nx}{3 - 2\cos x}\,dx.$$

(i) Find $G_0$ and show that $G_1 = \frac{1}{2}\log 5$.

(ii) Show that $G_{n+1} + G_{n-1} - 3G_n = \frac{1}{n}\Big((-1)^n - 1\Big)$.
[HINT: You may assume that $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$.]

(iii) Calculate $G_3$.

Exam continues overleaf ...

QUESTION SEVEN (Start a new answer booklet)

Marks

6 (a) Let $\omega$ be a non-real cube root of unity.

(i) Show that $1+\omega+\omega^2=0$.

(ii) Hence simplify $(1+\omega)^2$.

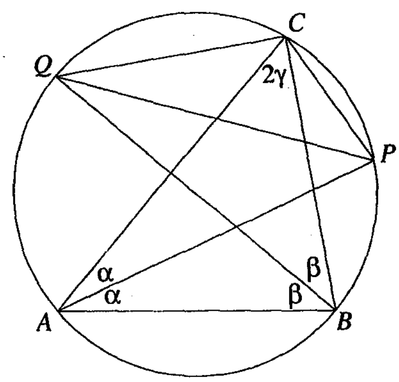
(iii) Show that $(1+\omega)^3=-1$.

(iv) Use part (iii) to simplify $(1+\omega)^{3n}$ and hence show that

$${}^{3n}C_0 - \tfrac{1}{2}({}^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 - \tfrac{1}{2}({}^{3n}C_4 + {}^{3n}C_5) + {}^{3n}C_6 - \cdots + {}^{3n}C_{3n} = (-1)^n.$$

[HINT: You may assume that $\mathrm{Re}(\omega)=-\frac{1}{2}$ and that $\mathrm{Re}(\omega^2)=-\frac{1}{2}$.]

5 (b)




In the diagram above, $AB$ is a fixed chord of a circle and $C$ is a variable point on the major arc $AB$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at $P$ and $Q$ respectively.

Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$.

(i) Show that $\angle PCQ = \alpha+\beta+2\gamma$.

(ii) Hence explain why the length of $PQ$ is constant.

(iii) Use the sine rule to show that $\dfrac{AB}{PQ} = 2\sin\gamma$.

4 (c) (i) Show that $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$.

(ii) Hence show that

$$\tan^2\left(\tfrac{\alpha+\beta}{2}\right) - \tan\alpha\tan\beta = \frac{\cos(\alpha+\beta)\big(1-\cos(\alpha-\beta)\big)}{\cos\alpha\cos\beta\big(1+\cos(\alpha+\beta)\big)}.$$

(iii) Hence show that for $0<\alpha<\frac{\pi}{4}$ and $0<\beta<\frac{\pi}{4}$,

$$\sqrt{\tan\alpha\tan\beta} \le \tan\left(\tfrac{\alpha+\beta}{2}\right).$$

Exam continues next page ...

QUESTION EIGHT (Start a new answer booklet)

Marks

5 (a) Let $y = uv$ be the product of $u$ and $v$, where $u$ and $v$ are functions of $x$.

(i) Show that $y'' = u\,v'' + 2u'v' + u''v$.

(ii) Develop similar expressions for $y'''$, $y''''$ and $y'''''$.

(iii) Hence, or otherwise, find and simplify $\dfrac{d^5}{dx^5}\left((1-x^2)e^{-x}\right)$.

10 (b) The *Bernstein polynomial* $B_{n,k}(t)$ of degree $n$ and order $k$ is defined by:

$$B_{n,k}(t) = {}^n\mathrm{C}_k\, t^k(1-t)^{n-k}, \quad \text{for } 0 \le k \le n.$$

(i) Write down the three Bernstein polynomials of degree 2, namely $B_{2,0}(t)$, $B_{2,1}(t)$ and $B_{2,2}(t)$.

(ii) The three fixed complex numbers $\alpha$, $\beta$ and $\gamma$ are represented on the Argand diagram by the points $A$, $B$ and $C$ respectively. Three other complex numbers $p$, $q$ and $r$ are represented by the points $P$, $Q$ and $R$ respectively.

The point $P$ divides the interval $AB$ in the ratio $t:(1-t)$.
The point $Q$ also divides the interval $BC$ in the ratio $t:(1-t)$.
Likewise the point $R$ divides the interval $PQ$ in the ratio $t:(1-t)$.

($\alpha$) Use the ratio division formula to find $p$ and $q$ in terms of $\alpha$, $\beta$ and $\gamma$.

($\beta$) Hence show that

$$r = \alpha\, B_{2,0}(t) + \beta\, B_{2,1}(t) + \gamma\, B_{2,2}(t).$$

($\gamma$) Given that $\alpha = 1+i$, $\beta = 2+3i$ and $\gamma = 3+i$, find the Cartesian equation of the locus of $R$ as $t$ varies.

(iii) ($\alpha$) Show that

$$\sum_{k=0}^{n} B_{n,k}(t) = 1.$$

($\beta$) Show that for $r \le k \le n$

$$\frac{{}^k\mathrm{C}_r}{{}^n\mathrm{C}_r}\, B_{n,k}(t) = {}^{n-r}\mathrm{C}_{k-r}\, t^k(1-t)^{n-k}.$$

($\gamma$) Using the previous two parts, or otherwise, show that

$$\sum_{k=2}^{5} \frac{{}^k\mathrm{C}_2}{{}^5\mathrm{C}_2}\, B_{5,k}(t) = t^2.$$

DNW

QUESTION ONE

Marks

$\boxed{2}$ (a) $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = \Big[\sqrt{x^2+1}\Big]_0^{\sqrt{3}}$ $\boxed{\checkmark}$

$= \sqrt{4} - \sqrt{1}$

$= 1.$ $\boxed{\checkmark}$

$\boxed{3}$ (b) (i) $\displaystyle I_n = \int_0^1 x^n e^{-x}\,dx$

$\displaystyle = \Big[-x^n e^{-x}\Big]_0^1 - \int_0^1 -nx^{n-1}e^{-x}\,dx$ $\boxed{\checkmark}$

$\displaystyle = -e^{-1} + n\int_0^1 x^{n-1}e^{-x}\,dx$

$= n\,I_{n-1} - e^{-1}.$ $\boxed{\checkmark}$

(ii) $I_3 = 3\,I_2 - e^{-1}$

$= 6\,I_1 - 4e^{-1}$

$= 6\,I_0 - 10e^{-1}$

$= 6 - 16e^{-1}.$ $\boxed{\checkmark}$

$\boxed{4}$ (c) Put $\displaystyle\frac{2y+3}{(y-2)(y^2+3)} = \frac{A}{y-2} + \frac{By+C}{y^2+3}$ $\boxed{\checkmark}$

so $2y+3 = A(y^2+3) + By(y-2) + C(y-2).$

Now when $y=2$ $7 = 7A$

so $A = 1.$

When $y=0$ $3 = 3A - 2C$

so $C = 0.$

Finally, when $y=1$ $5 = 4A - B$

so $B = -1.$ $\boxed{\checkmark\checkmark\text{(-1 per error)}}$

Thus $\displaystyle\int \frac{2y+3}{(y-2)(y^2+3)}\,dx = \int\left(\frac{1}{y-2} - \frac{y}{y^2+3}\right)dy$

$= \log|y-2| - \tfrac{1}{2}\log(y^2+3) + C.$ $\boxed{\checkmark}$

$\boxed{2}$ (d) $\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{x^2+1}\,dx$ $\boxed{\checkmark}$

$= x\tan^{-1}x - \tfrac{1}{2}\log(x^2+1) + C.$ $\boxed{\checkmark}$

$\boxed{4}$ (e) (i) $\displaystyle\int\frac{dx}{x^2+2x+5} = \int\frac{dx}{(x+1)^2+2^2}$ $\boxed{\checkmark}$

$\displaystyle = \tfrac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$ $\boxed{\checkmark}$

(ii) $\displaystyle\int\frac{x^2}{x^2+2x+5}\,dx = \int\frac{x^2+2x+5}{x^2+2x+5}\,dx - \int\frac{2x+2}{x^2+2x+5}\,dx - \int\frac{3}{x^2+2x+5}\,dx$ $\boxed{\checkmark}$

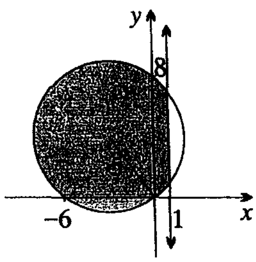
$\displaystyle = x - \log(x^2+2x+5) - \tfrac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$ $\boxed{\checkmark}$

QUESTION TWO

Marks

$\boxed{2}$ (a) $(2-3i)^2 = 4 - 12i - 9$ $\boxed{\surd}$
$= -5 - 12i$ $\boxed{\surd}$

$\boxed{3}$ (b)




$\boxed{\surd \text{ circle } \surd \text{ line } \surd \text{ shading}}$

$\boxed{3}$ (c) (i) $|-1+i| = \sqrt{2}$ $\boxed{\surd}$
and $\arg(-1+i) = \frac{3\pi}{4}$. $\boxed{\surd}$

(ii) That is, find $n$ so that $\frac{3n\pi}{4}$ is an integer multiple of $\pi$. Hence $n = 4$. $\boxed{\surd}$

$\boxed{4}$ (d) (i)
$$iz = r(-\sin\theta + i\cos\theta)$$
$$= r\big(\cos(\tfrac{\pi}{2}+\theta) + i\sin(\tfrac{\pi}{2}+\theta)\big) \quad \boxed{\surd}$$
so $\arg(iz) = \frac{\pi}{2} + \theta$
$= \frac{\pi}{2} + \arg z$. $\boxed{\surd}$

(ii) $C$ is the complex number $\gamma = i(2+i)$
$= -1 + 2i$. $\boxed{\surd}$
$B$ is the complex number $\beta = (2+i) + (-1+2i)$
$= 1 + 3i$. $\boxed{\surd}$

$\boxed{3}$ (e) $z + \dfrac{1}{z} = \dfrac{1}{a^2+b^2}\big(a(a^2+b^2+1) + ib(a^2+b^2-1)\big)$. $\boxed{\surd}$

(i) The result is real if $b(a^2+b^2-1) = 0$
thus $a^2 + b^2 = 1$
or $b = 0$. $\boxed{\surd}$
That is, on the unit circle or the real axis $(z \neq 0)$.

(ii) The result is pure imaginary if $a(a^2+b^2+1) = 0$
hence $a = 0$. $\boxed{\surd}$
That is, on the imaginary axis $(z \neq 0)$.

## QUESTION THREE

Marks

$\boxed{4}$ (a) Slice the region parallel with the $y$-axis. When rotated about the $y$-axis, a typical slice generates a cylindrical shell of radius $x$, height $y$ and thickness $dx$, so

$$dV = 2\pi xy\,dx.$$

Hence the volume is
$$\begin{aligned} V &= \pi \int_0^3 2xy\,dx \quad \boxed{\surd} \\ &= \pi \int_0^3 6x - 8x^2 + 8x^3 - 2x^4\,dx \quad \boxed{\surd} \\ &= \left[3x^2 - \tfrac{8}{3}x^3 + 2x^4 - \tfrac{2}{5}x^5\right]_0^3 \quad \boxed{\surd} \\ &= \tfrac{99\pi}{5}. \quad \boxed{\surd} \end{aligned}$$

$\boxed{5}$ (b) (i) Let $y = x^2$, then the roots $x = \alpha$, $\beta$ and $\gamma$ become $y = \alpha^2$, $\beta^2$ and $\gamma^2$ as required.

Rearranging the equation, $x(x^2 - 2) = -4.$ $\boxed{\surd}$

Square this to get $x^2(x^4 - 4x^2 + 4) = 16,$

so $y(y^2 - 4y + 4) = 16.$ $\boxed{\surd}$

Hence $y^3 - 4y^2 + 4y - 16 = 0.$

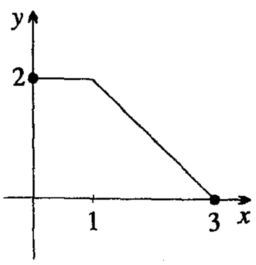
(ii) $y^2(y-4) + 4(y-4) = 0$

so $(y-4)(y^2+4) = 0$ $\boxed{\surd}$

that is $(y-4)(y-2i)(y+2i) = 0$ $\boxed{\surd}$

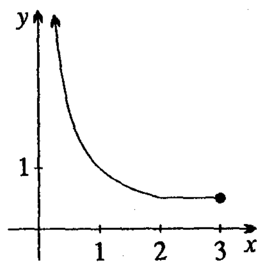
(iii) Since there is only one positive real root for $y$, it follows that there is only one real root for $x^2 = y$. (The root is actually $x = -2$.) $\boxed{\surd}$

$\boxed{6}$ (c) (i)



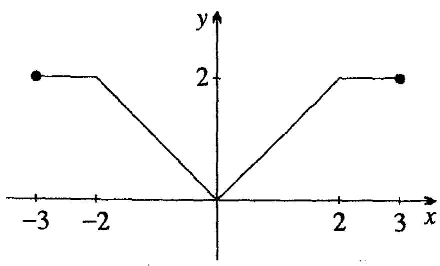

$\boxed{\surd}$

(iii)



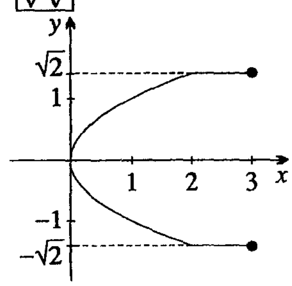

$\boxed{\surd\surd}$

(ii)




$\boxed{\surd}$

(iv)




$\boxed{\surd\surd}$

QUESTION FOUR

Marks

$\boxed{6}$ (a) (i) $\angle OTN = 90^\circ$ ($NT$ is tangent to the circle at $T$)

$\angle OPN = 90^\circ$ ($OP$ is the perpendicular bisector of chord $AB$) $\boxed{\surd}$

Hence both are angles in a semicircle with diameter $ON$.

Thus $OPNT$ is a cyclic quadrilateral. $\boxed{\surd}$

(ii) Similarly $\angle MSO = 90^\circ$ ($MS$ is tangent to the circle at $S$)

$\angle OPM = 90^\circ$ ($OP$ is the perpendicular bisector of chord $AB$)

Hence both are angles in a semicircle with diameter $OM$.

Thus $OPSM$ is a cyclic quadrilateral. $\boxed{\surd}$

(iii) $\angle ONP = \angle OTP$ (angles on the same arc $OP$ of circle $OPNT$)

$= \theta$. $\boxed{\surd}$

$\angle OST = \angle OTS$ (base angles of isosceles $\triangle OST$)

$= \theta$.

$\angle OMP = \angle OSP$ (angles on the same arc $OP$ of circle $OPSM$)

$= \theta$. $\boxed{\surd}$

(iv) The result can be obtained by proving $\triangle OPN \equiv \triangle OPM$, or the following:

$\triangle OMN$ is isosceles (base angles are equal)

$OP$ is the altitude to the base

hence $MP = PN$.

Thus $AM = PM - AP$

$= PN - PB$

$= BN$. $\boxed{\surd}$

$\boxed{6}$ (b) (i) $\angle AED = \alpha$ (base angles of isosceles $\triangle ADE$)

$\angle BEC = 180^\circ - (90^\circ + \alpha)$ (adjacent angles at $E$)

$= 90^\circ - \alpha$.

Thus $\angle CBE = \alpha$ (angle sum of $\triangle CBE$).

In $\triangle ABC$ and $\triangle BEC$

$\angle CBE = \angle DAE$ (proven)

$\angle BCE$ is common

hence $\triangle ABC \parallel\!| \triangle BEC$ (AA). $\boxed{\surd\surd}$

(ii) So using the ratio of matching sides of similar triangles

$$\frac{x}{a} = \frac{c}{b}$$

thus $x = \dfrac{ca}{b}$. $\boxed{\surd}$

(iii) $\angle BDE = 2\alpha$ (exterior angle of isosceles $\triangle ABE$). $\boxed{\surd}$

(iv) Now $\tan 2\alpha = \dfrac{x}{y}$

so
$$\begin{aligned} y &= \frac{x}{\tan 2\alpha} \\ &= \frac{ca}{b}\,\frac{1-\tan^2\alpha}{2\tan\alpha} \quad \boxed{\surd} \\ &= \frac{ca}{b}\,\frac{1-\frac{a^2}{b^2}}{2\frac{a}{b}} \\ &= \frac{c}{2}\left(1-\frac{a^2}{b^2}\right) \\ &= \frac{c(b^2-a^2)}{2b^2}. \quad \boxed{\surd} \end{aligned}$$

This can also be proven by applying Pythagoras' theorem to $\triangle BDE$.

$\boxed{\mathbf{3}}$ (c) (i)

$$f(x) = g(x) + h(x) \qquad (1)$$
$$f(-x) = g(-x) + h(-x)$$

so using symmetry, $f(-x) = g(x) - h(x)$. $\boxed{\surd}$ (2)

Adding (1) and (2),

$$f(x) + f(-x) = 2g(x)$$

so
$$g(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big). \quad \boxed{\surd}$$

(ii) Thus for the function $f(x) = e^x$ we get,

$$g(x) = \frac{e^x + e^{-x}}{2}. \quad \boxed{\surd}$$

QUESTION FIVE

Marks

$\boxed{8}$ (a) (i) ($\alpha$) Using Newton's second law $m\ddot{x} = -mg - mkv^2$

so $\ddot{x} = -(g + kv^2)$. $\boxed{\surd}$

($\beta$) $\dfrac{dv}{dt} = -(g + kv^2)$

so $\dfrac{dt}{dv} = \dfrac{-1}{g + kv^2}$.

Integrate this with respect to time to get

$t = -\frac{1}{\sqrt{gk}} \tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) + C.$ $\boxed{\surd}$

At $t = 0$, $v = U$

thus $C = \frac{1}{\sqrt{gk}} \tan^{-1}\left(U\sqrt{\frac{k}{g}}\right)$

and $t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(U\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)\right)$. $\boxed{\surd}$

The velocity is zero at the maximum height so it follows that

$$T_u = \frac{1}{\sqrt{gk}} \tan^{-1}\left(U\sqrt{\frac{k}{g}}\right). \ \boxed{\surd}$$

(ii) ($\alpha$) $f(0) = 0$. $\boxed{\surd}$

($\beta$) $f'(w) = \left(1 + \dfrac{w}{\sqrt{w^2+1}}\right)\dfrac{1}{w + \sqrt{w^2+1}} - \dfrac{1}{w^2+1}$ $\boxed{\surd}$

$= \dfrac{1}{\sqrt{w^2+1}} - \dfrac{1}{w^2+1}$ (This result may also be obtained directly from tables.)

$= \dfrac{\sqrt{w^2+1} - 1}{w^2+1}$

$> 0$ for $w > 0$, since $\sqrt{w^2+1} > 1$. $\boxed{\surd}$

($\gamma$) Hence $f(w)$ is an increasing function, increasing from 0. Thus $f(w) > 0$ for $w > 0$, and so $T_d - T_u > 0$. That is, it takes longer to fall. $\boxed{\surd}$

$\boxed{7}$ (b) (i) $x = \sqrt{4 - z}$ for $x > 0$ or $x = -\sqrt{4 - z}$. $\boxed{\surd\surd}$

(ii) Here $x$ is a distance and so the positive square root is used.

Area of the small square $= (2 - x)^2$

$= (2 - \sqrt{4 - z})^2$.

Area of the large square $= 4^2$.

Thus area of slice $= 4^2 - 4(2 - \sqrt{4 - z})^2$.

Hence $V = \displaystyle\int_0^4 (4^2 - 4(2 - \sqrt{4 - z})^2)\, dx$. $\boxed{\surd\surd}$

(iii) Expanding, $V = \displaystyle\int_0^4 (4z + 16\sqrt{4 - z} - 16)\, dz$ $\boxed{\surd}$

$= \left[2z^2 - \frac{32}{3}(4 - z)^{\frac{3}{2}} - 16z\right]_0^4$ $\boxed{\surd}$

$= 53\frac{1}{3}$ cubic units. $\boxed{\surd}$

QUESTION SIX

Marks

$\boxed{8}$ (a) (i) The roots are the solutions of $P(z) = 0$,

that is $z^7 = 1$.

Put $z = \cos\theta + i\sin\theta$,

so $\cos 7\theta + i \sin 7\theta = 1$ by de Moivre's theorem,

that is $\theta = \dfrac{2n\pi}{7}$, where $n = 0, 1, 2, 3, 4, 5$ or $6$.

Hence $z = 1, \operatorname{cis}\frac{2\pi}{7}, \operatorname{cis}\frac{4\pi}{7}, \operatorname{cis}\frac{6\pi}{7}, \operatorname{cis}\frac{8\pi}{7}, \operatorname{cis}\frac{10\pi}{7}, \operatorname{cis}\frac{12\pi}{7}$. $\boxed{\checkmark}$

(ii) ($\alpha$) Thus $P(z) = (z-1)(z-\operatorname{cis}\frac{2\pi}{7})(z-\operatorname{cis}\frac{4\pi}{7})(z-\operatorname{cis}\frac{6\pi}{7})(z-\operatorname{cis}\frac{8\pi}{7})\times$
$(z-\operatorname{cis}\frac{10\pi}{7})(z-\operatorname{cis}\frac{12\pi}{7})$. $\boxed{\checkmark}$

($\beta$) Combining conjugate pairs:

$$P(z) = (z-1)(z^2 - 2z\cos\tfrac{2\pi}{7} + 1)(z^2 - 2z\cos\tfrac{4\pi}{7} + 1)(z^2 - 2z\cos\tfrac{6\pi}{7} + 1). \quad \boxed{\checkmark}$$

(iii) In $P(z)$, the coefficient of $z^6$ is zero, hence the sum of the roots is also zero. Take the real part of this sum to get:

$$(\cos\tfrac{2\pi}{7} + \cos\tfrac{12\pi}{7}) + (\cos\tfrac{4\pi}{7} + \cos\tfrac{10\pi}{7}) + (\cos\tfrac{6\pi}{7} + \cos\tfrac{8\pi}{7}) + 1 = 0. \quad \boxed{\checkmark}$$

Thus by the symmetry of the cosine function,

$2(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}) = -1$,

so $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$. $\boxed{\checkmark}$

(iv) ($\alpha$) $P(z) = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$. $\boxed{\checkmark}$

($\beta$) Thus $(z^2 - 2z\cos\frac{2\pi}{7} + 1)(z^2 - 2z\cos\frac{4\pi}{7} + 1)(z^2 - 2z\cos\frac{6\pi}{7} + 1)$
$= z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$. $\boxed{\checkmark}$

Now put $z = 1$ to get:

$(2 - 2\cos\frac{2\pi}{7})(2 - 2\cos\frac{4\pi}{7})(2 - 2\cos\frac{6\pi}{7}) = 7$

thus $(1 - 1\cos\frac{2\pi}{7})(1 - 1\cos\frac{4\pi}{7})(1 - 1\cos\frac{6\pi}{7}) = \frac{7}{8}$. $\boxed{\checkmark}$

$\boxed{7}$ (b) (i)

$$\begin{aligned} G_0 &= \int_0^\pi 0\,dx \\ &= 0 \quad \boxed{\checkmark} \end{aligned}$$

and

$$\begin{aligned} G_1 &= \int_0^\pi \frac{\sin x}{3 - 2\cos x}\,dx \\ &= \tfrac{1}{2}\Big[\log(3 - 2\cos x)\Big]_0^\pi \\ &= \tfrac{1}{2}(\log 5 - \log 1) \\ &= \tfrac{1}{2}\log 5. \quad \boxed{\checkmark} \end{aligned}$$

(ii) Now
$$\begin{aligned} G_{n+1} + G_{n-1} &= \int_0^\pi \frac{\sin(n+1)x + \sin(n-1)x}{3 - 2\cos x}\,dx \\ &= \int_0^\pi \frac{2\sin nx \cos x}{3 - 2\cos x}\,dx \qquad \text{by the hint.} \quad \boxed{\surd} \end{aligned}$$

So
$$\begin{aligned} G_{n+1} + G_{n-1} - 3G_n &= \int \frac{2\sin nx \cos x - 3\sin nx}{3 - 2\cos x}\,dx \\ &= -\int_0^\pi \frac{\sin nx\,(3 - 2\cos x)}{3 - 2\cos x}\,dx \quad \boxed{\surd} \\ &= -\int_0^\pi \sin nx\,dx \\ &= \tfrac{1}{n}\Big[\cos nx\Big]_0^\pi \\ &= \tfrac{1}{n}\left((-1)^n - 1\right). \quad \boxed{\surd} \end{aligned}$$

(iii) Hence
$$\begin{aligned} G_2 &= 3G_1 - G_0 - 2 \\ &= \tfrac{3}{2}\log 5 - 2, \quad \boxed{\surd} \end{aligned}$$
and
$$\begin{aligned} G_3 &= 3G_2 - G_1 \\ &= 4\log 5 - 6. \quad \boxed{\surd} \end{aligned}$$

QUESTION SEVEN

Marks

$\boxed{6}$ (a) (i) Now $\omega^3 = 1$

so $\omega^3 - 1 = 0.$

Factoring, $(\omega - 1)(\omega^2 + \omega + 1) = 0$

and since $\omega \neq 1$

$$\omega^2 + \omega + 1 = 0. \quad \boxed{\surd}$$

(ii) $$\begin{aligned}(1+\omega)^2 &= 1 + 2\omega + \omega^2 \\ &= (1 + \omega + \omega^2) + \omega \\ &= \omega \quad \text{by part (i).} \quad \boxed{\surd}\end{aligned}$$

(iii) $$\begin{aligned}(1+\omega)^3 &= \omega(1+\omega) \quad \text{by part (ii),} \\ &= (1 + \omega + \omega^2) - 1 \\ &= -1 \quad \text{by part (i).} \quad \boxed{\surd}\end{aligned}$$

(iv) $$\begin{aligned}(1+\omega)^{3n} &= \left((1+\omega)^3\right)^n \\ &= (-1)^n \quad \text{by part (iii).} \quad \boxed{\surd}\end{aligned}$$

Expanding the left hand side by the binomial theorem,

$$^{3n}\mathrm{C}_0 + {}^{3n}\mathrm{C}_1\omega + {}^{3n}\mathrm{C}_2\omega^2 + {}^{3n}\mathrm{C}_3\omega^3 + {}^{3n}\mathrm{C}_4\omega^4 + {}^{3n}\mathrm{C}_5\omega^5 + {}^{3n}\mathrm{C}_6\omega^6 + \ldots + {}^{3n}\mathrm{C}_{3n}\omega^{3n} = (-1)^n.$$

First simplify this using $\omega^3 = 1$.

$$^{3n}\mathrm{C}_0 + {}^{3n}\mathrm{C}_1\omega + {}^{3n}\mathrm{C}_2\omega^2 + {}^{3n}\mathrm{C}_3 + {}^{3n}\mathrm{C}_4\omega + {}^{3n}\mathrm{C}_5\omega^2 + \ldots + {}^{3n}\mathrm{C}_{3n} = (-1)^n. \quad \boxed{\surd}$$

Now take the real part of both sides and use the hint to get:

$$^{3n}\mathrm{C}_0 - {}^{3n}\mathrm{C}_1\tfrac{1}{2} - {}^{3n}\mathrm{C}_2\tfrac{1}{2} + {}^{3n}\mathrm{C}_3 - {}^{3n}\mathrm{C}_4\tfrac{1}{2} - {}^{3n}\mathrm{C}_5\tfrac{1}{2} + \ldots + {}^{3n}\mathrm{C}_{3n} = (-1)^n, \quad \boxed{\surd}$$

that is

$$^{3n}\mathrm{C}_0 - \tfrac{1}{2}({}^{3n}\mathrm{C}_1 + {}^{3n}\mathrm{C}_2) + {}^{3n}\mathrm{C}_3 - \tfrac{1}{2}({}^{3n}\mathrm{C}_4 + {}^{3n}\mathrm{C}_5) + \ldots + {}^{3n}\mathrm{C}_{3n} = (-1)^n.$$

$\boxed{5}$ (b) (i) $$\begin{aligned}\angle PCB &= \angle PAB \quad \text{(angles subtended by arc } PB) \\ &= \alpha. \\ \angle QCA &= \angle QBA \quad \text{(angles subtended by arc } QA) \\ &= \beta.\end{aligned}$$

Hence $\angle PCQ = \alpha + \beta + 2\gamma.$ $\boxed{p}$

(ii) Now $2\alpha + 2\beta + 2\gamma = 180^\circ$ (angle sum of $\triangle ABC$)

so $\alpha + \beta + \gamma = 90^\circ.$

Hence $\angle PCQ = 90^\circ + \gamma.$ $\boxed{p}$

Now $2\gamma$ is constant (angle subtended by arc $AB$)

thus $\angle PCQ$ is constant. $\boxed{p}$

So $PQ$ subtends a constant angle at the circumference, and hence has constant length.

(iii)
$$\begin{aligned}\frac{AB}{\sin 2\gamma} &= \frac{BP}{\sin\alpha} \quad \text{(applying the sine rule in } \triangle ABP\text{)}\\ &= \frac{PQ}{\sin(\alpha+\beta)} \quad \text{(applying the sine rule in } \triangle QBP\text{)}\\ &= \frac{PQ}{\sin(90^\circ-\gamma)}\\ &= \frac{PQ}{\cos\gamma}. \quad \boxed{\surd}\end{aligned}$$

So $\quad \dfrac{AB}{2\sin\gamma\cos\gamma} = \dfrac{PQ}{\cos\gamma}$

hence $\quad \dfrac{AB}{PQ} = 2\sin\gamma. \quad \boxed{\surd}$

Note: The following proof is more elegant and uses the fact that the ratio in the sine rule is the diameter of the circumcircle. Thus this ratio is the same for both $\triangle ABC$ and $\triangle PQR$.

$$\frac{PQ}{\sin(90^\circ+\gamma)} = \frac{AB}{\sin 2\gamma} \quad \text{(triangles with the same circumcircle),}$$

so $\quad \dfrac{PQ}{\cos\gamma} = \dfrac{AB}{2\cos\gamma\sin\gamma},$

hence $\quad \dfrac{AB}{PQ} = 2\sin\gamma.$

$\boxed{4}$ (c) (i) Use the $t$-formula for $\cos 2\theta$, or:

$$\begin{aligned}\text{RHS} &= \frac{1-\cos^2\theta+\sin^2\theta}{1+\cos^2\theta-\sin^2\theta}\\ &= \frac{2\sin^2\theta}{2\cos^2\theta}\\ &= \tan^2\theta\\ &= \text{LHS.} \quad \boxed{\surd}\end{aligned}$$

(ii)
$$\begin{aligned}\text{LHS} &= \frac{1-\cos(\alpha+\beta)}{1+\cos(\alpha+\beta)} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \quad \text{by part (i)}\\ &= \frac{\cos\alpha\cos\beta-\cos\alpha\cos\beta\cos(\alpha+\beta)-\sin\alpha\sin\beta-\sin\alpha\sin\beta\cos(\alpha+\beta)}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))}\\ &= \frac{(\cos\alpha\cos\beta-\sin\alpha\sin\beta)-\cos(\alpha+\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))}\\ &= \frac{\cos(\alpha+\beta)-\cos(\alpha+\beta)\cos(\alpha-\beta)}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))}\\ &= \frac{\cos(\alpha+\beta)(1-\cos(\alpha-\beta))}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))}\\ &= \text{RHS.} \quad \boxed{\surd\surd}\end{aligned}$$

(iii) Since all terms are positive it suffices to prove that

$$\tan^2(\tfrac{\alpha+\beta}{2}) - \tan\alpha\tan\beta \geq 0.$$

Since $0 < \alpha < \frac{\pi}{4}$ and $0 < \alpha < \frac{\pi}{4}$ it follows that $\cos(\alpha+\beta) > 0$ and so all terms in the equivalent fraction given in part (ii) are positive. Thus the fraction is positive, or zero when $\alpha = \beta$. Hence the inequality is proven. $\boxed{\surd}$

QUESTION EIGHT

Marks

$\boxed{5}$ (a) (i) Using the product rule $y' = u\,v' + u'v$

and again
$$\begin{aligned} y'' &= u\,v'' + u'v' + u'v' + u''v \\ &= u\,v'' + 2u'v' + u''v. \quad \boxed{\surd} \end{aligned}$$

(ii) Further applications of the product rule yield:
$$\begin{aligned} y''' &= u\,v''' + 2u'v'' + u''v' + \\ &\qquad u'v'' + 2u''v' + u'''v \\ &= u\,v''' + 3u'v'' + 3u''v' + u'''v \\ y'''' &= u\,v'''' + 3u'v''' + 3u''v'' + u'''v' + \\ &\qquad u'v''' + 3u''v'' + 3u'''v' + u''''v \\ &= u\,v'''' + 4u'v''' + 6u''v'' + 4u'''v' + u''''v \\ y''''' &= u\,v''''' + 4u'v'''' + 6u''v''' + 4u'''v'' + u''''v' + \\ &\qquad u'v'''' + 4u''v''' + 6u'''v'' + 4u''''v' + u'''''v \\ &= u\,v''''' + 5u'v'''' + 10u''v''' + 10u'''v'' + 5u''''v' + u'''''v. \quad \boxed{\surd\surd} \end{aligned}$$

It seems clear from this that the coefficients of higher order derivatives are the terms of Pascal's triangle. You may wish to prove the result by induction.

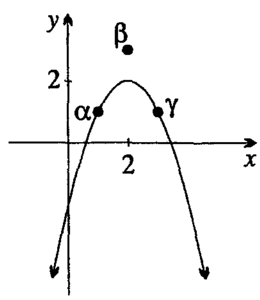
(iii) Put $u = 1 - x^2$ and note that derivatives higher than second order are zero,
hence $y''''' = (1-x^2)(-e^{-x}) + 5(-2x)e^{-x} + 10(-2)(-e^{-x})$ $\boxed{\surd}$
$= e^{-x}(x^2 - 10x + 19).$ $\boxed{\surd}$

$\boxed{10}$ (b) (i) $B_{2,0}(t) = (1-t)^2$,
$B_{2,1}(t) = 2t(1-t)$,
$B_{2,2}(t) = t^2.$ $\boxed{\surd}$

(ii) ($\alpha$) $p = \alpha(1-t) + \beta t$,
$q = \beta(1-t) + \gamma t.$ $\boxed{\surd}$

($\beta$)
$$\begin{aligned} r &= p(1-t) + qt \\ &= \alpha(1-t)^2 + \beta t(1-t) + \beta t(1-t) + \gamma t^2 \\ &= \alpha(1-t)^2 + \beta 2t(1-t) + \gamma t^2 \\ &= \alpha B_{2,0}(t) + \beta B_{2,1}(t) + \gamma B_{2,2}(t). \quad \boxed{\surd} \end{aligned}$$

($\gamma$) On the curve $r = x + iy$,
$$\begin{aligned} x &= 1 \times (1-t)^2 + 2 \times 2t(1-t) + 3 \times t^2 \\ &= 1 + 2t. \quad \boxed{\surd} \end{aligned}$$
$$\text{or } t = \tfrac{1}{2}(x-1) \tag{A}$$
$$\begin{aligned} y &= 1 \times (1-t)^2 + 3 \times 2t(1-t) + 1 \times t^2 \\ &= 1 + 4t - 4t^2. \quad \boxed{\surd} \end{aligned} \tag{B}$$




Substitute (A) into (B) for $t$ to get,
$$y = -x^2 + 4x - 2, \quad \boxed{\surd}$$
a parabola, shown in the Argand diagram on the right, through $\alpha$, $\gamma$ and $2 + 2i$.

(iii) ($\alpha$)

$$\begin{aligned}\sum_{k=0}^{n} B_{n,k}(t) &= \sum_{k=0}^{n} {}^{n}\mathrm{C}_k t^k (1-t)^{n-k} \\ &= \big(t + (1-t)\big)^n \qquad \text{by the binomial theorem} \\ &= 1^n \\ &= 1. \quad \boxed{\checkmark}\end{aligned}$$

($\beta$)

$$\begin{aligned}\frac{{}^{k}\mathrm{C}_r}{{}^{n}\mathrm{C}_r} B_{n,k}(t) &= \frac{k!}{r!(k-r)!}\,\frac{r!(n-r)!}{n!}\,\frac{n!}{k!(n-k)!} t^k (1-t)^k \\ &= \frac{(n-r)!}{(k-r)!(n-k)!} t^k (1-t)^{n-k} \\ &= {}^{n-r}\mathrm{C}_{k-r} t^k (1-t)^{n-k}. \quad \boxed{\checkmark}\end{aligned}$$

($\gamma$) Firstly, completing the above,

$$\begin{aligned}\frac{{}^{k}\mathrm{C}_r}{{}^{n}\mathrm{C}_r} B_{n,k}(t) &= {}^{n-r}\mathrm{C}_{k-r} t^{k-r} (1-t)^{n-k} t^r \\ &= t^r B_{n-r,k-r}(t). \quad \boxed{\checkmark}\end{aligned}$$

Now put $n = 5$ and $r = 2$ to get

$$\begin{aligned}\sum_{k=2}^{5} \frac{{}^{k}\mathrm{C}_2}{{}^{5}\mathrm{C}_2} B_{5,k}(t) &= \sum_{k=2}^{5} t^2 B_{3,k-2}(t) \qquad \text{from above} \\ &= t^2 B_{3,0}(t) + t^2 B_{3,1}(t) + t^2 B_{3,2}(t) + t^2 B_{3,3}(t) \\ &= t^2 \Big( B_{3,0}(t) + B_{3,1}(t) + B_{3,2}(t) + B_{3,3}(t) \Big) \\ &= t^2 \qquad \text{by part } (\alpha). \quad \boxed{\checkmark}\end{aligned}$$

Using clever manipulation of $\Sigma$-notation, the result for part ($\gamma$) can be generalised to:

$$\sum_{k=r}^{n} \frac{{}^{k}\mathrm{C}_r}{{}^{n}\mathrm{C}_r} B_{n,k}(t) = t^r.$$

Bernstein polynomials and the curves they generate, called Bézier curves, are used in computer drawing packages and were originally developed by French engineers to simplify computer aided design of automobiles.

DNW