# FORM VI MATHEMATICS EXTENSION 2

**Time allowed:** 3 hours (plus 5 minutes reading) **Exam date:** 7th August 2002

**Instructions:**

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

**Collection:**

Each question will be collected separately.
Start each question in a new 4-page examination booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

**Checklist:**

SGS Examination booklets required — 8 booklets per boy.
60 Boys

QUESTION ONE (Start a new answer booklet)

Marks

(a) Find $\displaystyle\int \frac{dx}{x \log x}$. **2**

(b) Find $\displaystyle\int \frac{dx}{x^2 + 6x + 10}$. **2**

(c) Use the substitution $u = \sqrt{1-x}$ to evaluate $\displaystyle\int_0^1 x^2\sqrt{1-x}\,dx$. **3**

(d) Use integration by parts to evaluate $\displaystyle\int_0^1 \sin^{-1} x\,dx$. **3**

(e) (i) Find real numbers $a$, $b$ and $c$ such that $\displaystyle\frac{5x^2 - 5x + 14}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}$. **2**

(ii) Find $\displaystyle\int \frac{5x^2 - 5x + 14}{(x^2+4)(x-2)}\,dx$. **3**

QUESTION TWO (Start a new answer booklet)

Marks

(a) Let $z = 1 + 2i$ and $w = 3 + i$. Find $\dfrac{1}{zw}$ in the form $x + iy$. **2**

(b) (i) Express $\dfrac{1}{2}(-1 + i\sqrt{3})$ in modulus–argument form. **2**

(ii) Hence express $\dfrac{1}{16}(-1 + i\sqrt{3})^4$ in the form $x + iy$. **2**

(c) Sketch the region in the Argand plane where the inequalities **3**

$$\frac{\pi}{4} \leq \arg(z - i) \leq \frac{3\pi}{4} \qquad \text{and} \qquad |z - i| \leq 2$$

both hold simultaneously.

(d) The origin $O$ and the points $A$, $B$ and $C$ representing the complex numbers $z$, $\dfrac{1}{z}$ and $z + \dfrac{1}{z}$ respectively are joined to form a quadrilateral. Write down the condition or conditions for $z$ so that the quadrilateral $OABC$ will be

(i) a rhombus, **1**

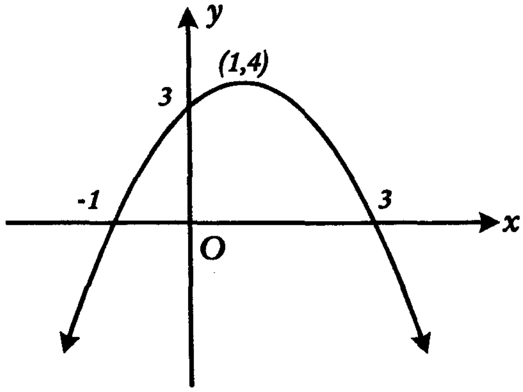
(ii) a square. **1**

(e) (i) Write down the six complex sixth roots of unity in modulus–argument form. Sketch the roots on an Argand diagram and explain why they form a regular hexagon. **2**

(ii) Factorise $z^6 - 1$ completely into real factors. **2**

QUESTION THREE (Start a new answer booklet)

(a)




Let $f(x) = -(x-3)(x+1)$. In the diagram above, the graph of $y = f(x)$ is drawn. On separate diagrams, and without the use of calculus, sketch the following graphs. Indicate clearly any asymptotes, any intercepts with the coordinate axes, and any other significant features. Each diagram should take about ten lines.

Marks

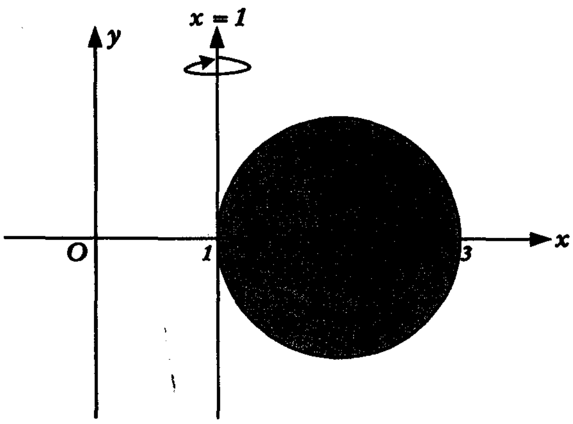
(i) $y = |f(x)|$, **2**

(ii) $y = \dfrac{1}{f(x)}$, **2**

(iii) $y = e^{f(x)}$, **2**

(iv) $y^2 = f(x)$. **2**

(b)




In the diagram above, the circle $(x-2)^2 + y^2 = 1$ is drawn. The region bounded by the circle is rotated about the line $x = 1$.

(i) Use the method of cylindrical shells to show that the volume of the solid so formed is given by **3**

$$V = 4\pi \int_1^3 (x-1)\sqrt{1-(x-2)^2}\, dx.$$

(ii) By using the substitution $x - 2 = \sin\theta$, or otherwise, evaluate the integral in part (i) to find the volume of the solid. **4**

QUESTION FOUR (Start a new answer booklet)

Marks

(a) Find all the roots of the equation $12x^3 + 44x^2 - 5x - 100 = 0$, given that two of the roots are equal. $\boxed{4}$

(b) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. ( You are NOT required to evaluate the integrals.)

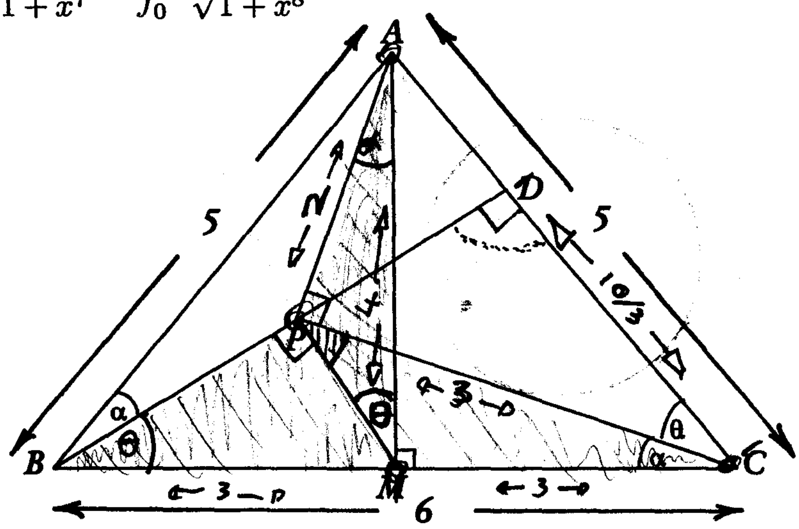
(i) $\displaystyle\int_0^1 e^{-\frac{1}{2}x^2}\,dx = 0,$ $\boxed{1}$

(ii) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7 x\,dx = 0,$ $\boxed{1}$

(iii) $\displaystyle\int_0^{\pi} \cos^9 x\,dx > 0,$ $\boxed{1}$

(iv) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^7}} > \int_0^1 \frac{dx}{\sqrt{1+x^8}}.$ $\boxed{1}$

(c)




In the diagram above, $\triangle ABC$ is isosceles with $AB = AC = 5$ and $BC = 6$ and with $AM$ perpendicular to $BC$. The point $P$ lies inside the triangle so that $AP \perp CP$ and $\angle ABP = \angle BCP = \alpha$. Let $BP$ produced meet $AC$ at $D$, and let $\theta = \angle ACP$.

(i) Explain why $A, P, M$ and $C$ are concyclic. $\boxed{1}$

(ii) Join $PM$. Give a reason why $\angle PMA = \theta$. $\boxed{1}$

(iii) Prove that $\triangle MPA \;|||\; \triangle BPC$. $\boxed{1}$

(iv) Show that $\tan\theta = \dfrac{2}{3}$. $\dfrac{AP}{PC}$ $\boxed{2}$

(v) Use the sine rule in $\triangle BDC$ to show that $DC = \dfrac{10}{3}$. $\boxed{2}$

QUESTION FIVE (Start a new answer booklet)

Marks

(a) The circular bend on a bike track has a constant radius of 20 metres and is banked at a constant angle of 30° to the horizontal. A bicycle rider can safely negotiate the bend if the maximum sideways thrust $F$, up or down the slope, is at most one-tenth of the normal reaction $N$. By resolving the forces vertically and horizontally, show that the range of speeds $V$, correct to two decimal places and in metres per second, at which the bend can be safely negotiated is **8**

$$9{\cdot}50 \le V \le 11{\cdot}99.$$

Take $g = 10\,\text{m/s}^2$.

(b) (i) Use de Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **1**

(ii) Deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \dfrac{1}{2}$. **2**

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$. **2**

(iv) Hence evaluate $\cos\dfrac{\pi}{9}\cos\dfrac{2\pi}{9}\cos\dfrac{4\pi}{9}$. **2**

QUESTION SIX (Start a new answer booklet)

Marks

(a) (i) Show that $\dfrac{d}{dx}\Big(\log_e(\sec x + \tan x)\Big) = \sec x$. **1**

(ii) Hence or otherwise show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec x\,dx = \log_e\left(\sqrt{2}+1\right)$. **2**

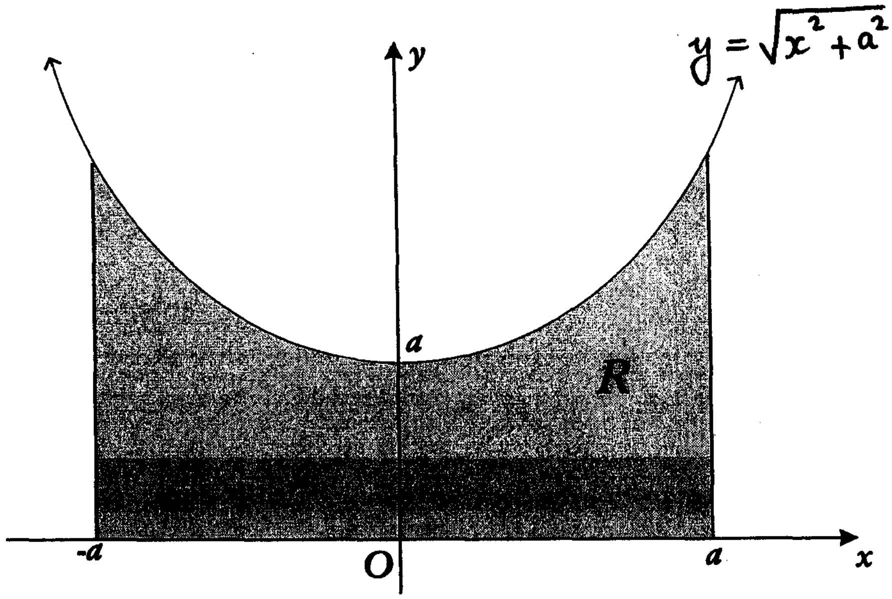
(iii) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x\,dx$. Use integration by parts to show that for $n \ge 2$, **3**

$$I_n = \frac{1}{n-1}\Big((\sqrt{2})^{n-2} + (n-2)I_{n-2}\Big).$$

(iv) Hence find $I_3$. **1**

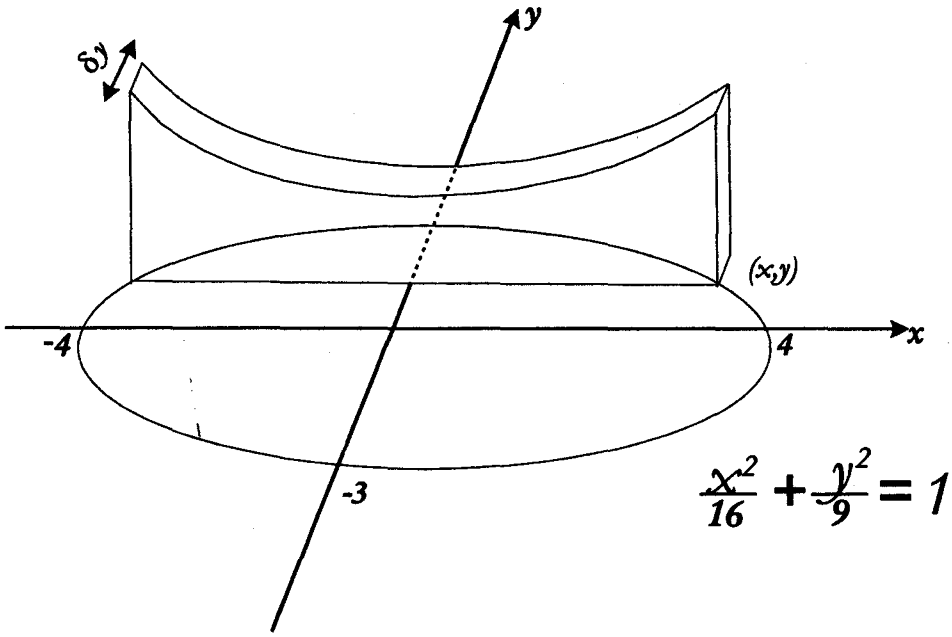
QUESTION 6 CONTINUES ON THE NEXT PAGE.

(b) **4**




In the diagram above, the shaded region $\mathcal{R}$ is bounded by the upper branch of the hyperbola $y = \sqrt{x^2 + a^2}$, the lines $x = -a$ and $x = a$, and the $x$-axis, where $a$ is positive. Show that the area of this region is given by $a^2\Big(\sqrt{2} + \log_e(\sqrt{2} + 1)\Big)$. You may use the results of part (a).

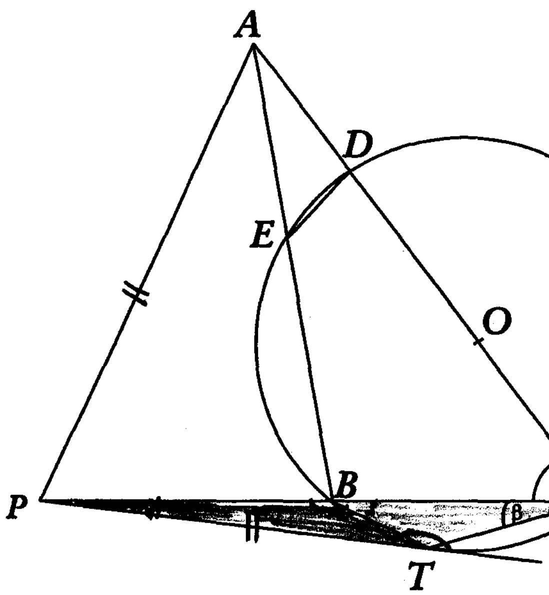
(c) **4**




In the diagram above, a solid is constructed with base the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. Each cross-section perpendicular to the $y$-axis is a plane figure that is similar to the region $\mathcal{R}$ described in part (b). Find the volume of this solid.

QUESTION SEVEN (Start a new answer booklet)

(a)




In the diagram above, $O$ is the centre of the circle, $PT$ is a tangent to the circle and $PT = PA$. Let $\angle ACP = \alpha$ and $\angle BCT = \beta$.

Marks

(i) Show that $\triangle PBT \parallel\!| \triangle PTC$. **2**

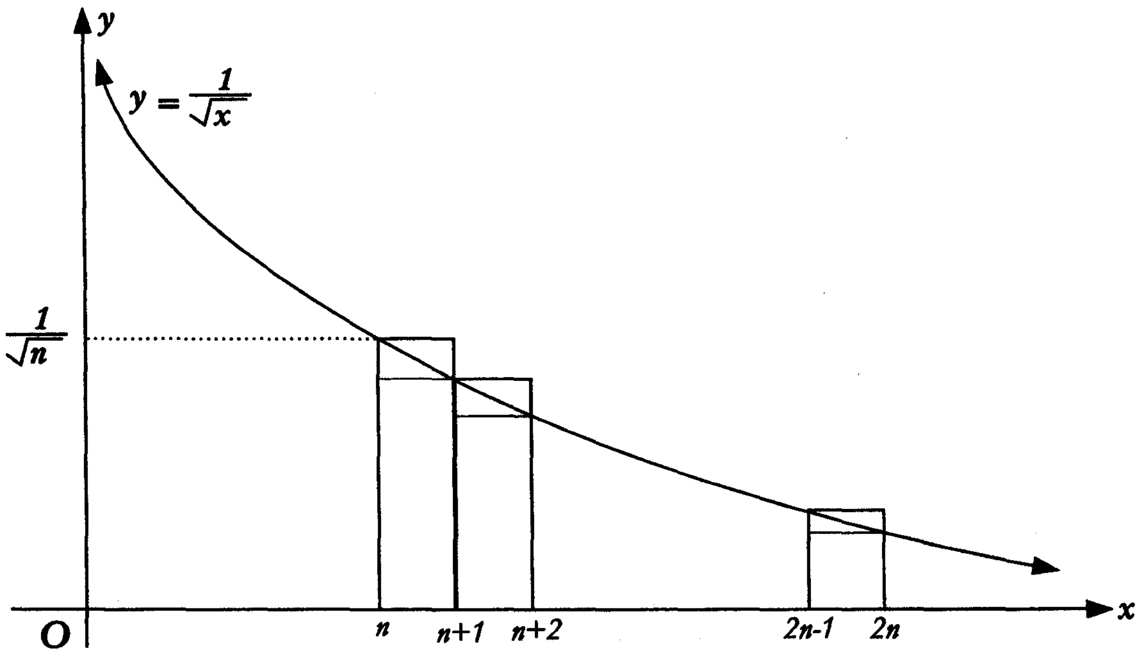
(ii) Show that $\triangle APB \parallel\!| \triangle CPA$. **3**

(iii) Show that $DE \parallel AP$. **3**

QUESTION 7 CONTINUES ON THE NEXT PAGE.

(b) (i) Show that $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = 2\sqrt{n}(\sqrt{2}-1)$. $\boxed{2}$

(ii)




In the diagram above, the graph of $y = \dfrac{1}{\sqrt{x}}$ has been drawn, and $n$ upper and lower rectangles have been constructed between $x = n$ and $x = 2n$, each of width 1 unit. Let $S_n = \dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \dfrac{1}{\sqrt{n+3}} + \cdots + \dfrac{1}{\sqrt{2n}}$.

($\alpha$) By considering the sums of areas of upper and lower rectangles, show that: $\boxed{4}$

$$2\sqrt{n}(\sqrt{2}-1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}(\sqrt{2}-1)$$

($\beta$) Hence find, correct to four decimal places, $\boxed{1}$

$$\frac{1}{\sqrt{10^8+1}} + \frac{1}{\sqrt{10^8+2}} + \frac{1}{\sqrt{10^8+3}} + \cdots + \frac{1}{\sqrt{2 \times 10^8}}.$$

QUESTION EIGHT (Start a new answer booklet)

Marks

(a) (i) If $a$ and $b$ are any positive real numbers, prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$. **2**

(ii) The generalisation of the result found in part (i) states that 'the arithmetic mean of $n$ positive real numbers is always greater than or equal to their geometric mean'. That is, if $a_1, a_2, \ldots, a_n$ are positive real numbers, then **2**

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Assume this result and prove that $n! \leq \left(\dfrac{n+1}{2}\right)^n$, for any positive integer $n \geq 1$. (There is no need to use mathematical induction.)

(b) A particle $P$ of mass $m$ slides smoothly in a horizontal circle on the inner surface of a hemi-spherical shell with centre $O$ and radius $r$. The interval $OP$ makes an angle of $\theta$ with the vertical through $O$.

(i) Show that the speed $v$ of the particle is given by $v^2 = gr\sin\theta\tan\theta$, where $g$ is the acceleration due to gravity. **2**

(ii) Show that the magnitude of the force $N$ exerted by the shell on the particle is given by **3**

$$N = \frac{m}{2r}\left(v^2 + \sqrt{v^4 + 4r^2g^2}\right).$$

(c) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\, dx$.

(i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$, for $n \geq 2$. **3**

(ii) Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$. **3**

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4U TRIAL 2002

Question 1

a) $\int \frac{dx}{x \log x} = \log(\log x) + c$

b) $\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1}$

$= \tan^{-1}(x+3) + c$

c) $u = \sqrt{1-x}$ when $x=0$, $u=1$

$x = 1-u^2$ and $x=1$, $u=0$

and $dx = -2u\,du$

$\int_0^1 x^2\sqrt{1-x}\,dx = \int_1^0 (1-u^2)^2 . u . -2u\,du$

$= \int_0^1 2u^2(1-u^2)^2\,du$

$= \int_0^1 2u^2 - 4u^4 + 2u^6\,du$

$= \left[\frac{2u^3}{3} - \frac{4u^5}{5} + \frac{2u^7}{7}\right]_0^1$

$= \frac{16}{105}$

d) $\int_0^1 \sin^{-1}x\,dx =$

let $u = \sin^{-1}x$ $\quad v = x$

$du = \frac{1}{\sqrt{1-x^2}}$ $\quad dv = 1$

$= \left[x\sin^{-1}x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$= \frac{\pi}{2} + \frac{1}{\cancel{2}}\left[\cancel{2}(1-x^2)^{\frac{1}{2}}\right]_0^1$

$= \frac{\pi}{2} + [0 - 1]$

$= \frac{\pi}{2} - 1$

e) $5x^2 - 5x + 14 = (ax+b)(x-2) + c(x^2+4)$

$= ax^2 + (-2a+b)x - 2b + cx^2 + 4c$

$= (a+c)x^2 - (2a+b)x - 2b + 4c$

$$\left.\begin{aligned} a+c &= 5 \\ -2a+b &= -5 \\ -2b+4c &= 14 \end{aligned}\right\} \begin{aligned} &\text{—— } 1 \\ &\text{—— } 2 \\ &\text{—— } 3 \end{aligned}$$

From 1; $c = (5-a)$ and ③ becomes $-2b + 4(5-a) = 14$

$-b + 10 - 2a = 7$

$2a + b = 3$

$b = (3-2a)$.

substitute into ②: $-2a + (3-2a) = -5$

$\therefore a = 2, \; b = 1, \; c = 3$

(ii) $$\int \frac{5x^2 - 5x + 14}{(x^2+4)(x-2)}\,dx = \int \frac{2x+1}{x^2+4} + \frac{3}{x-2}\,dx$$

$$= \int \frac{2x}{x^2+4} + \frac{1}{x^2+4} + \frac{3}{x-2}\,dx$$

$$= \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + 3\ln(x-2) + c$$

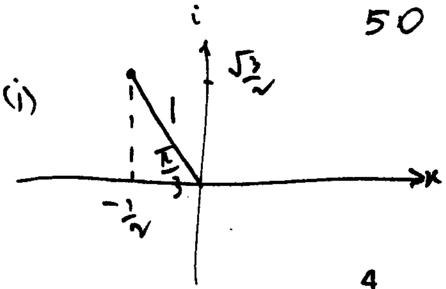
Question 2

a) $\dfrac{1}{zw} = \dfrac{1}{(1+2i)(3+i)}$

$= \dfrac{1}{1+7i} \times \dfrac{1-7i}{1-7i}$

$= \dfrac{1-7i}{50}$

b) (i)




$\frac{1}{2}\left(-1+i\sqrt{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

(ii) $\frac{1}{16}\left(-1+i\sqrt{3}\right)^4 = \left\{\frac{1}{2}\left(-1+i\sqrt{3}\right)\right\}^4$

Hence

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$

$= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \frac{1}{2}\left(-1+i\sqrt{3}\right)$

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or

otherwise

$\frac{1}{16}\left(-1+i\sqrt{3}\right)^4 = \frac{1}{16}\left((-1)^4 + 4(-1)^3 i\sqrt{3} + 6(-1)^2(i\sqrt{3})^2 + 4(-1)(i\sqrt{3})^3 + (i\sqrt{3})^4\right)$

$= \frac{1}{16}\left(1 - 4i\sqrt{3} - 6\times 3 + 4i.3\sqrt{3} + 9\right)$

$= \frac{1}{16}\left(-8 + 8i\sqrt{3}\right)$

$= \frac{8}{16}\left(-1+i\sqrt{3}\right)$

$= \frac{1}{2}\left(-1+i\sqrt{3}\right)$

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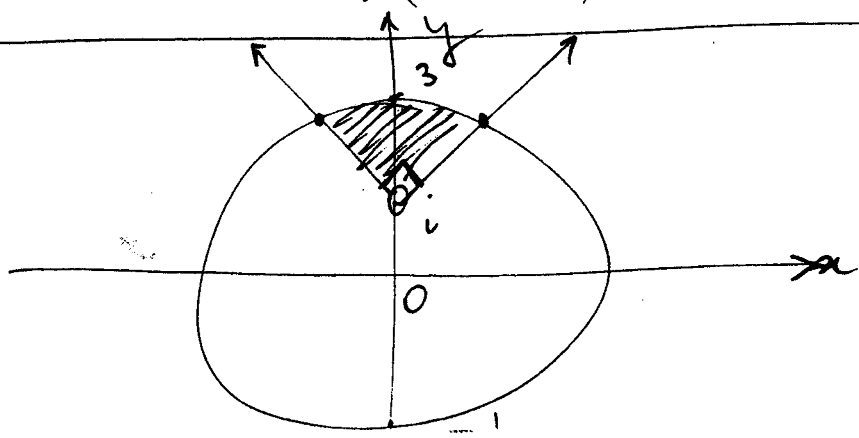
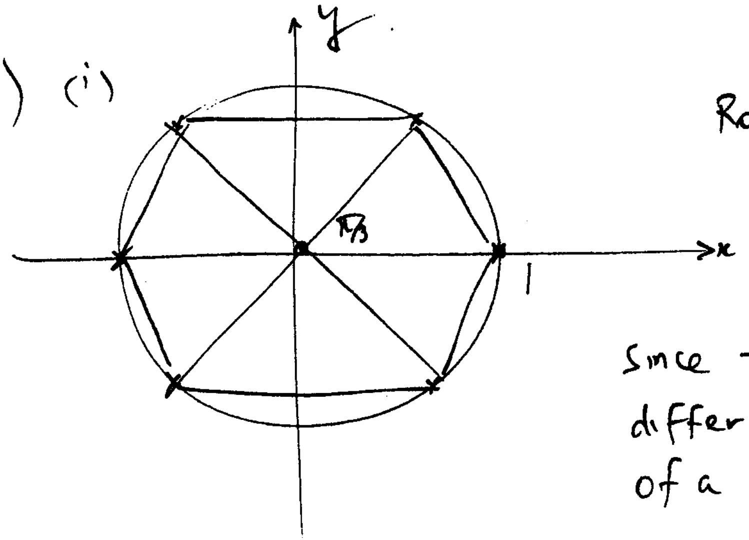
c)



d) (i) $|z| = 1$

(ii) $|z| = 1$ and $\arg z = \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$.

e) (i)




Roots are $\pm 1$, $\text{cis}\,\frac{\pi}{3}$, $\text{cis}\,\frac{2\pi}{3}$, $\text{cis}\,\frac{4\pi}{3}$, $\text{cis}\,\frac{5\pi}{3}$

(or $\text{cis}\,\frac{k\pi}{3}$ for $k = 0, 1 \ldots 5$)

Since their moduli equal 1, their arguments differ by $\frac{\pi}{3}$ they form the vertices of a regular hexagon on the unit circle.

(ii) $z^6 - 1 = (z^3)^2 - 1$

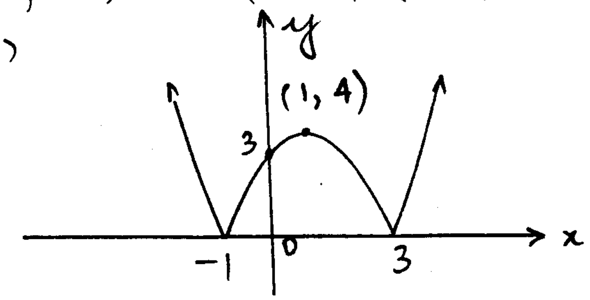
$= (z^3 + 1)(z^3 - 1)$

$= (z+1)(z^2 - z + 1)(z-1)(z^2 + z + 1)$

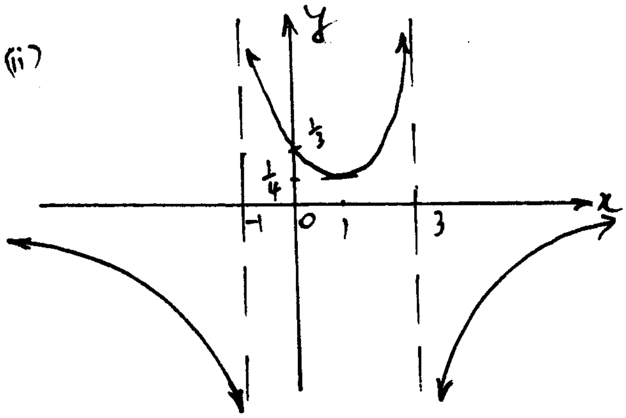
QUESTION 3

a) $f(x) = -(x-3)(x+1)$

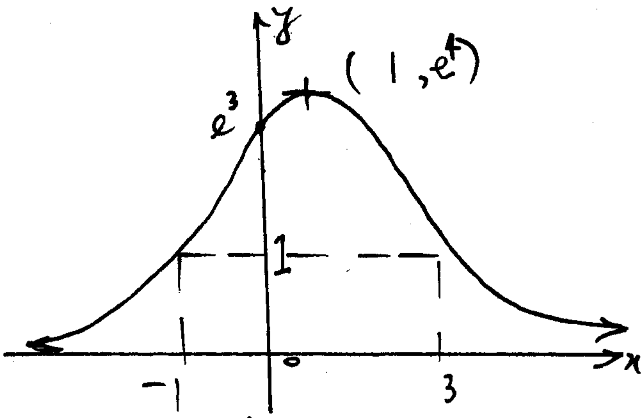
(i)



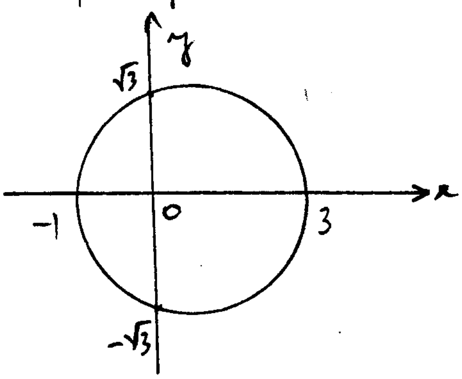

(ii)



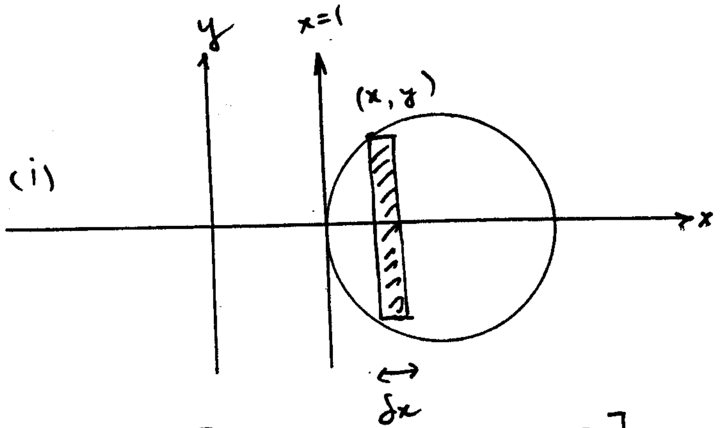

(iii)




(iv)



b) (i)




$$\delta V = \pi\left[(x+\delta x-1)^2-(x-1)^2\right]2y$$

$$= 2\pi y\left[(x+\delta x-1+x-1)(x+\delta x-1-x+1)\right]$$

$$= 2\pi y\,(2(x-1)+\delta x)\,\delta x$$

$$= 4\pi(x-1)\,y\,\delta x \qquad (\text{since } \delta x^2 \text{ terms can be ignored})$$

$$= 4\pi(x-1)\sqrt{1-(x-2)^2}\,\delta x$$

$$V = \lim_{\delta x\to 0} 4\pi\sum_{x=1}^{3}(x-1)\sqrt{1-(x-2)^2}\,\delta x$$

$$= 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$

(ii) Let $(x-2)=\sin\theta$ then $dx=\cos\theta\,d\theta$.

When $x=1$, $\sin\theta=-1$ $\therefore \theta=-\frac{\pi}{2}$ and $x=3$, $\sin\theta=1$ $\theta=\frac{\pi}{2}$

So $$V = 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin\theta)\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

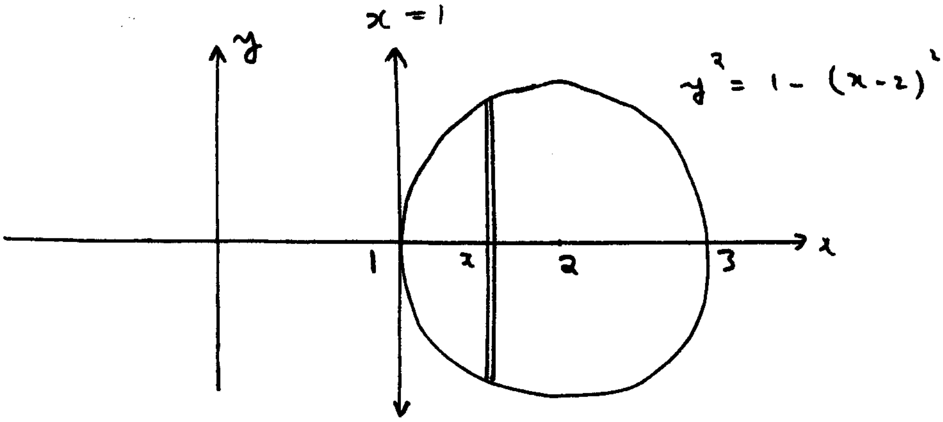
$$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin\theta)\cos^2\theta\,d\theta \quad = 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{1}{2}(1+\cos 2\theta)+\sin\theta\cos^2\theta\,d\theta$$

$$= 4\pi\left[\tfrac{1}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)-\tfrac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi\left[\tfrac{\pi}{2}+0-0\right]$$

$$= 2\pi^2$$

Q3(b) Alternative Approach




(i) Slice the region perpendicular to the $x$-axis, as shown. When the strip sketched above is rotated about the line $x=1$, it generates a cylindrical shell with radius $x-1$ and height $2y$.

Thus surface area of shell $= 2\pi(x-1) \times 2y$

$$= 4\pi(x-1)\sqrt{1-(x-2)^2}.$$

Hence volume of solid $= \int_1^3 4\pi(x-1)\sqrt{1-(x-2)^2}\,dx$.

(ii) Let $u = x-2$.

Then $du = dx$

When $x=1$, $u=-1$

When $x=3$, $u=1$

So volume $= 4\pi \int_{-1}^{1} (u+1)\sqrt{1-u^2}\,du$

$$= 4\pi \int_{-1}^{1} \sqrt{1-u^2}\,du + 4\pi \int_{-1}^{1} u\sqrt{1-u^2}\,du$$

The first integral $= 4\pi \times$ (semicircle of radius 1)

$$= 4\pi \times \frac{\pi}{2}$$

$$= 2\pi^2$$

The second integrand is odd, so the integral is zero.

Hence volume $= 2\pi^2$ cubic units.

Question 4

(a) Since $P(x)$ has a double root, then $P(x)$ and $P'(x)$ share a root.

$$P(x) = 12x^3 + 44x^2 - 5x - 100$$
$$P'(x) = 36x^2 + 88x - 5$$
$$= (18x-1)(2x+5)$$

So $P'(x)$ has roots $\frac{1}{18}$ and $\frac{-5}{2}$.

Now $P(\frac{1}{18}) \neq 0$ and $P(-\frac{5}{2}) = 0$, so $x = -\frac{5}{2}$ is the double root.

Let $\beta$ be the other root, then $P(x) = k(x+\frac{5}{2})^2(x-\beta)$.

ie/ $k(x+\frac{5}{2})^2(x-\beta) = 12x^3 + 44x^2 - 5x - 100$

$\therefore k = 12$ (comparing coefficients.

and $12(+\frac{5}{2})^2(-\beta) = -100$

$$\beta = \frac{100}{12} \times \frac{4}{25}$$
$$= \frac{4}{3}.$$

So the roots are $-\frac{5}{2}, -\frac{5}{2}, \frac{4}{3}$.

b) (i) False, since $e^{-\frac{1}{2}x^2} >$ for all $x$

(ii) True because $\tan x$ and hence $\tan^7 x$ is odd

(iii) False – the integral is zero because $\cos x$, and hence $\cos 9x$, has point symmetry in the interval, so the integral is zero.

(iv) For $0 < x < 1$,

$$0 < x^8 < x^7 < 1$$
$$1 < 1+x^8 < 1+x^7 < 2$$
$$1 < \sqrt{1+x^8} < \sqrt{1+x^7} < \sqrt{2}$$
$$1 > \frac{1}{\sqrt{1+x^8}} > \frac{1}{\sqrt{1+x^7}} > \frac{1}{\sqrt{2}}$$

so $1 > \int_0^1 \frac{dx}{\sqrt{1+x^8}} > \int_0^1 \frac{dx}{\sqrt{1+x^7}} > \frac{1}{\sqrt{2}}$ and the statement is false.

(c) (i) $\angle APC = \angle AMC = 90^\circ$ and are subtended by the chord AC and so these are angles on the circumference of a circle. Thus APMC are concyclic

(ii) $\angle PMA = \angle PCA = \theta$ (angles subtended by the chord AP at the circumference are equal)

(iii) $\angle PAM = \alpha = \angle PCM$ (angles at circumference)

$\angle PBC = \theta$ (base angles of $\Delta ABC$ are equal)

$\therefore \Delta MPA \parallel\!\!| \Delta BPC$ (A.A.)

(iv) $\frac{PA}{PC} = \tan\theta$ and $\frac{MA}{BC} = \frac{4}{6} = \frac{2}{3}$ (MA = 4 from Pythagoras' theorem)

$\frac{PA}{PC} = \frac{MA}{BC}$ (ratio of corresponding sides in similar $\Delta$)

ie, $\tan\theta = \frac{2}{3}$.

(v) $\frac{DC}{\sin\theta} = \frac{BC}{\sin(\angle BDC)}$

$$DC = \frac{6\sin\theta}{\sin(180-(\theta+\angle DCB))}$$

$$= \frac{6\sin\theta}{\sin(\theta+\angle DCB)}$$

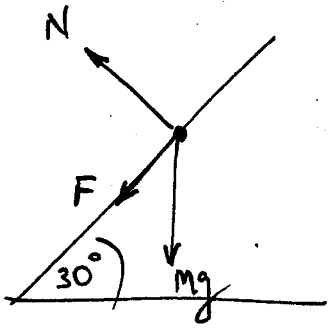
$$= \frac{6\sin\theta}{\sin\theta\cos\angle DCB + \cos\theta\sin\angle DCB}$$

$$= \frac{6}{\cos\angle DCB + \sin\angle DCB \cdot \cot\theta}$$

$$= \frac{6}{\frac{3}{5}+\left(\frac{3}{2}\right)\left(\frac{4}{5}\right)} = \frac{10}{3}.$$

Question 5

a)




$F = \frac{1}{10} N$

Vertically : $N\cos 30 - F\sin 30 = Mg$

$10F\frac{\sqrt{3}}{2} - \frac{F}{2} = Mg$

$F\left(\frac{10\sqrt{3}-1}{2}\right) = Mg$ ——— ①

Horizontally : $N\sin 30 + F\cos 30 = \frac{Mv^2}{r}$

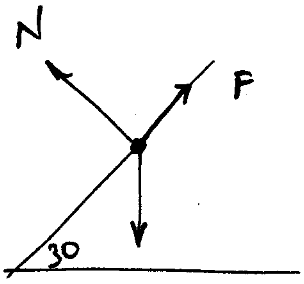
$10\times\frac{F}{2} + F\times\frac{\sqrt{3}}{2} = \frac{Mv^2}{20}$

$20F\left(5+\frac{\sqrt{3}}{2}\right) = Mv^2$ ——— ②

Dividing ② by ① : $v^2 = \frac{20Fg\left(5+\frac{\sqrt{3}}{2}\right)}{\frac{F}{2}\left(10\sqrt{3}-1\right)}$

$= \frac{20g(10+\sqrt{3})}{(10\sqrt{3}-1)}$

So $v \leqslant \sqrt{\frac{20g\,(10+\sqrt{3})}{(10\sqrt{3}-1)}}$




Vertically : $N\cos 30 + F\sin\theta = Mg$

$10F\frac{\sqrt{3}}{2} + \frac{F}{2} = Mg$

$\frac{F}{2}\left(10\sqrt{3}+1\right) = Mg$ ——— ①

Horizontally : $N\sin 30 - F\cos 30 = \frac{Mv^2}{r}$

$10F\times\frac{1}{2} - F\times\frac{\sqrt{3}}{2} = \frac{Mv^2}{20}$

$\cancel{\frac{20F}{2}}^{10}\left(10-\sqrt{3}\right) = Mv^2$ ——— ②

Divide ② by ① : $v^2 = \frac{10Fg(10-\sqrt{3})}{\frac{F}{2}(10\sqrt{3}+1)}$

$= 20g\left(\frac{10-\sqrt{3}}{10\sqrt{3}+1}\right)$

So $v \geqslant \sqrt{20g\left(\frac{10-\sqrt{3}}{10\sqrt{3}+1}\right)}$

Thus $9.50 \leqslant v \leqslant 11.99$ m/s.

$(57 \leqslant v \leqslant 72$ km/h.$)$

b) (i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

and $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

Equating coefficients gives:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= \cos^3\theta - 3(\cos\theta(1 - \cos^2\theta)$$
$$= 4\cos^3\theta - 3\cos\theta$$

(ii) Since $\cos 3\theta = \frac{1}{2}$, $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$$\Rightarrow 8\cos^3\theta - 6\cos\theta - 1 = 0$$

When $\cos\theta = x$, this equation becomes $8x^3 - 6x - 1 = 0$.

(iii) $\cos 3\theta = \frac{1}{2}$

$3\theta = 2n\pi \pm \frac{\pi}{3}$, $n = 0, \pm 1, \pm 2, \ldots$

$\therefore \theta = \frac{\pi}{9}(6n \pm 1)$

Hence the solutions are $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$.

(iv) since $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$ and $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$ we have

$$\cos\frac{\pi}{9} \cdot \cos\frac{5\pi}{9} \cdot \cos\frac{7\pi}{9} = \frac{1}{8}$$

$$\Rightarrow \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}.$$

Question 6

a) (i) $\frac{d}{dx}\log_e(\sec x + \tan x) = \frac{1}{\sec x + \tan x} \times \left[ -(\cos x)^{-2}(-\sin x) + \sec^2 x \right]$

$= \frac{1}{\sec x + \tan x} \times \left[ \frac{\sin x}{\cos^2} + \sec^2 x \right]$

$= \frac{1}{\sec x + \tan x} (\tan x \sec x + \sec^2 x)$

$= \frac{\sec x (\tan x + \sec x)}{\tan x + \sec x}$

$= \sec x$.

(ii) $\int_0^{\pi/4} \sec x \, dx = \left[ \log_e (\sec x + \tan x) \right]_0^{\pi/4}$

$= \log_e(\sqrt{2} + 1) - \log_e 1$

$= \log_e(\sqrt{2}+1)$

(iii) $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$

$= \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x \, dx$

$= \left[ \sec^{n-2} x \tan x \right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} (n-2) \sec^{n-2} x \tan^2 x \, dx$

$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} \sec^n x - \sec^{n-2} x \, dx$

$= (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2})$

$\therefore I_n (1 + (n-2)) = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$

$I_n = \frac{1}{n-1} \left( (\sqrt{2})^{n-2} + (n-2) I_{n-2} \right)$

(iv) $I_3 = \frac{1}{2}(\sqrt{2} + I_1)$

$= \frac{1}{2}(\sqrt{2} + \log(\sqrt{2}+1))$, from (ii) above.

b) $\frac{y^2}{a^2} - \frac{x^2}{a^2} = 1$

$y = \sqrt{a^2 + x^2}$

So $\text{Area} = \int_{-a}^{a} \sqrt{a^2+x^2}\, dx$.

Let $x = a\tan\theta$,

$dx = a\sec^2\theta\, d\theta$.

When $x = a$, $\tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}$

So $A = 2\int_0^{\frac{\pi}{4}} \sqrt{a^2 + a^2\tan^2\theta}\,.\, a\sec^2\theta\, d\theta$

$= 2a^2 \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$

$= 2a^2 \times \frac{1}{2}\left(\sqrt{2} + \log_e(\sqrt{2}+1)\right)$ from part (a)

$= a^2\left(\sqrt{2} + \log_e(\sqrt{2}+1)\right)$, as required

(c) $x^2 = 16\left(1 - \frac{y^2}{9}\right)$

since $x = a$, $\delta V = x^2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\delta y$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\left(1 - \frac{y^2}{9}\right)\delta y$

So $V = 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\int_{-3}^{3}\left(1 - \frac{y^2}{9}\right)dy$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\left[\left(y - \frac{y^3}{27}\right)\right]_{-3}^{3}$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)(2 + 2)$

$= 64\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$

QUESTION 7

a)(i) In $\Delta PBT$, $PTC$

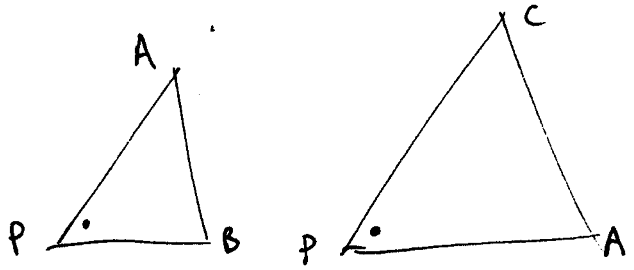
$\angle BTP = \angle PCT$ (angle in alternate segment)

$\angle BPT = \angle TPC$ (common angles)

$\therefore \Delta PBT \parallel\!| \Delta PTC$ (AA)

(ii) In $\Delta APB$, $PTC$

$\angle APB = \angle CPA$ (common)




$PB.BC = PT^2$ (secant/tangent)

$\therefore PB.BC = PA^2$ ($PA = PT$)

$$\frac{PA}{PB} = \frac{PC}{PA}$$

So $$\frac{PA}{PC} = \frac{PB}{PA}$$

$\therefore \Delta APB \parallel\!| \Delta PTC$ (SAS)

(iii) $\angle PAB = \angle PCA$ (from part (ii) – corresponding angles are equal)

$\therefore \angle PAB = \alpha$.

$\angle BED = 180 - \alpha$ (opposite angles in cyclic quadrilateral are supplementary)

$\therefore \angle AED = \alpha$ (straight angles).

So $\angle PAB = \angle AEB$ and since the angles are alternate and equal, $AP \parallel DE$.

Question 7

(b) (i) $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_n^{2n}$

$= 2\sqrt{2n} - 2\sqrt{n}$

$= 2\sqrt{n}\,(\sqrt{2}-1)$

(ii) The upper rectangles have area greater than the integral so,

$$\int_n^{2n} \frac{dx}{\sqrt{x}} < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n-1}}.$$

Adding $\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}}$ to both sides and using (i) gives

$$2\sqrt{n}\,(\sqrt{2}-1) + \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n}} \quad \text{———} \; *$$

The lower rectangles have area less than the integral so,

$$\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n}} < \int_n^{2n} \frac{dx}{\sqrt{x}} \quad \text{———} \; **$$

From * and ** we get

$$2\sqrt{n}\,(\sqrt{2}-1) + \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}} < S_n < 2\sqrt{n}\,(\sqrt{2}-1)$$

ie/ $$2\sqrt{n}\,(\sqrt{2}-1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\,(\sqrt{2}-1)$$

(iii) When $n = 10^8$, $S_n \doteqdot 8284.2712$

QUESTION 8

(a) (i) $\frac{(a+b)^2}{4} - ab = \frac{a^2+2ab+b^2}{4} - ab$

$= \frac{a^2-2ab+b^2}{4}$

$= \frac{(a-b)^2}{4}$

$\geq 0$, since $(a+b)^2 \geq 0$.

(ii) Applying the result for $n$ positive numbers, we

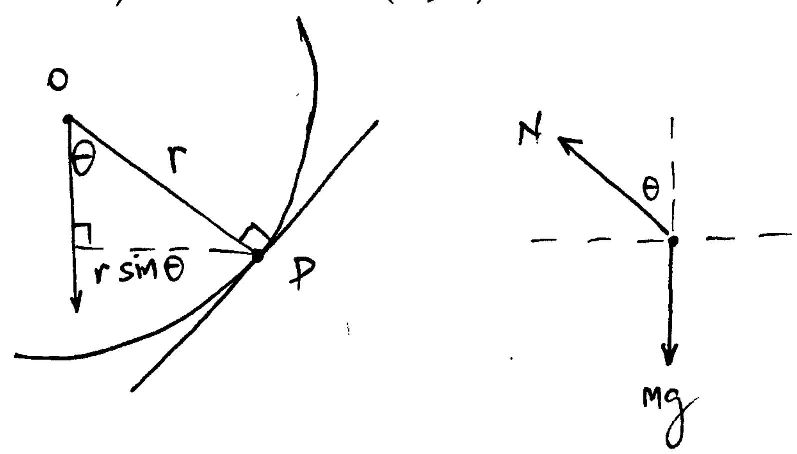
$$\frac{1+2+3+\dots+n}{n} \geq \sqrt[n]{1\times2\times3\times\dots\times n}$$

LHS $= \frac{n(n+1)}{2}$ and the RHS $= \sqrt[n]{n!}$ and thus,

$$\sqrt[n]{n!} \leq \frac{n(n+1)}{2n}$$

ie, $n! \leq \left(\frac{n+1}{2}\right)^n$.

(b)




Resolving at P:

$N\sin\theta = \frac{mv^2}{r\sin\theta}$ —— ①

$N\cos\theta - mg = 0$ —— ②

a) Dividing ① by ② $\tan\theta = \frac{\cancel{m}v^2}{r\sin\theta} \times \frac{1}{\cancel{m}g}$

$\therefore v^2 = gr\sin\theta\tan\theta$ as required.

b) (see over)

(b) (cont)

From ①, $$N\sin^2\theta = \frac{mv^2}{r}$$

From ②, $$N^2\cos^2\theta = m^2g^2$$

$$N\cos^2\theta = \frac{m^2g^2}{N}$$

Adding ① and ② $$N = \frac{mv^2}{r} + \frac{m^2g^2}{N}$$

$$N^2 - \frac{mv^2}{r}N - m^2g^2 = 0$$

$$rN^2 - mv^2N - rm^2g^2 = 0$$

Now $$\Delta = m^2v^4 + 4r^2m^2g^2$$
$$= m^2(v^4 + 4r^2g^2)$$

$$\therefore N = \frac{mv^2 + m\sqrt{v^4 + 4r^2g^2}}{2r}$$ only since $N > 0$.

$$= \frac{m}{2r}\left(v^2 + \sqrt{v^4 + 4r^2g^2}\right)$$

Question 8

a, c)(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx$

$= \left[-\sin x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$

$\therefore \quad I_n (1+(n-1)) = (n-1) I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$

(ii) $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = I_{2n}$

$= \frac{2n-1}{2n} \times I_{2n-2}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{1}{2} \times I_0$

$= \frac{2n}{2n} \times \frac{2n-1}{2n} \times \frac{2n-2}{2n-2} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{2}{2} \times \frac{1}{2} \times \int_0^{\frac{\pi}{2}} dx$

$= \frac{(2n)!}{4(n)^2 \times 4(n-1)^2 \times 4(n-2)^2 \times \cdots \times 2^2} \times \frac{\pi}{2}$

$= \frac{(2n)!}{4^n (n!)^2} \times \frac{\pi}{2}$

$= \frac{\pi (2n)!}{2^{2n+1} (n!)^2}.$