SYDNEY GRAMMAR SCHOOL TRIAL EXAMINATION 2003

FORM VI MATHEMATICS EXTENSION 2

Time allowed: 3 hours (plus 5 minutes reading time) **Exam date:** 6th August 2003

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

The writing booklets will be collected in one bundle.
Start each question in a new writing booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each booklet.

Checklist:

SGS Writing Booklets required — eight booklets per boy.
Candidature: 54 boys.

QUESTION ONE (Start a new writing booklet)

Marks

(a) Find $\displaystyle\int \frac{\sin x}{\cos^5 x}\,dx$. **1**

(b) Use completion of squares to evaluate $\displaystyle\int_{-2}^{-1} \frac{5}{x^2+4x+5}\,dx$. **3**

(c) (i) Find the real numbers A, B and C such that **3**

$$\frac{3x^2-x+8}{(1-x)(x^2+1)} \equiv \frac{A}{1-x} + \frac{Bx+C}{x^2+1}.$$

(ii) Hence find $\displaystyle\int \frac{3x^2-x+8}{(1-x)(x^2+1)}\,dx$. **2**

(d) Use integration by parts to show that $\displaystyle\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx = 4(2\ln 2 - 1)$. **3**

(e) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\displaystyle\int \frac{1}{1+\cos\theta}\,d\theta$. **3**

QUESTION TWO (Start a new writing booklet)

Marks

(a) Find the square roots of $9 - 40i$. Give your answers in the form $a + ib$. **3**

(b) Sketch on the Argand diagram the locus $|z - 1| = |z + i|$. **1**

(c) Sketch the region in the Argand diagram that satisfies both the conditions **2**

$$-\tfrac{\pi}{2} \le \arg(z - 2) \le 0 \quad \text{and} \quad \operatorname{Im}(z) \le -1.$$

(d) Let $z = 1 - i$ and $w = -1 + i\sqrt{3}$.

(i) Find $\arg z$ and $\arg w$. **1**

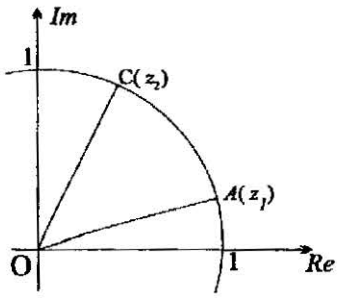
(ii) Hence find $\arg(wz)$. **1**

(iii) Hence prove that $\sin \frac{5\pi}{12} = \dfrac{\sqrt{2}(1 + \sqrt{3})}{4}$. **2**

(e) (i) Let $z = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$. Find z^9. **1**

(ii) On the Argand diagram, plot and label all complex numbers that satisfy both the conditions $z^9 = -1$ and $\operatorname{Re}(z) \le 0$. **2**

(f)



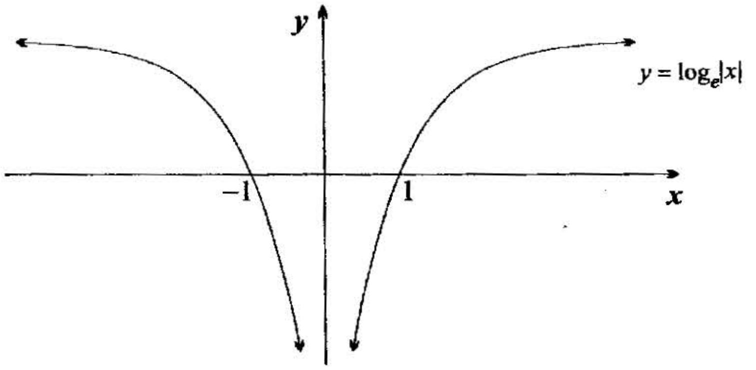
In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin and radius 1. They represent the complex numbers z_1 and z_2 respectively.

(i) Copy the diagram into your answer booklet. Then mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. **1**

(ii) Explain why AC is perpendicular to OB. **1**

QUESTION THREE (Start a new writing booklet)

(a)



The graph above shows the function $y = f(x) = \log_e |x|$.

Marks

(i) Use half a page to sketch on a number plane the graph $y = f\left(\frac{x}{2}\right)$. **1**

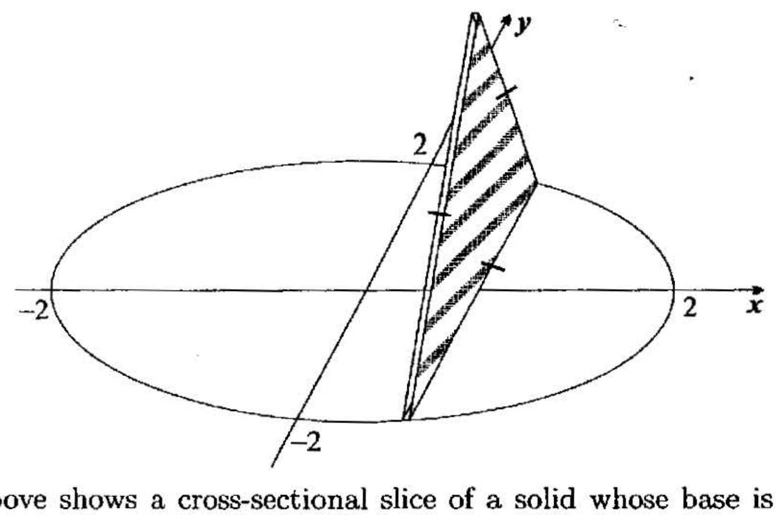
(ii) Use half a page to sketch on a number plane the graph $y = \dfrac{1}{f(x)}$. **2**

(iii) Use half a page to sketch on a number plane the graph $y^2 = f(x)$. **2**

(iv) Use half a page to sketch on a number plane the graph $y = e^{f(x)}$. **1**

(b) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = x^2 + 3$ and the x-axis between the lines $x = 0$ and $x = 3$ is rotated about the y-axis. **3**

(c) **3**



The diagram above shows a cross-sectional slice of a solid whose base is the region enclosed by the circle $x^2 + y^2 = 4$. Each such cross-section of the solid is an equilateral triangle. Find the volume of the solid.

(d) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated around the line $y = 1$. Using a slicing technique find the volume formed. **3**

QUESTION FOUR (Start a new writing booklet)

Marks

(a) A mass of 2 kg, on the end of a string 0·6 metres long, is rotating as a conical pendulum, with angular velocity 3π radians per second. Take the acceleration due to gravity to be $10\,\text{m/s}^2$.

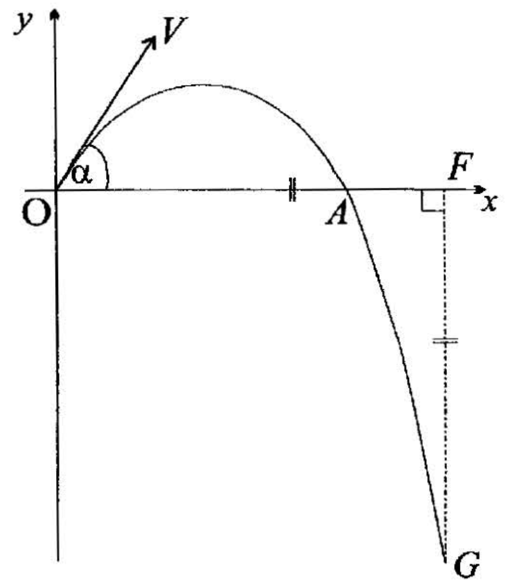
Let θ be the angle that the string makes with the vertical.

(i) Draw a diagram showing all forces acting on the mass. **1**

(ii) By resolving forces, find the tension in the string. **3**

(iii) Find θ correct to the nearest degree. **1**

(b)



$OF = FG$

In the diagram above, a projectile is fired from a point O at the top of a vertical cliff. Its initial speed is V m/s and its angle of elevation is α. Let the acceleration due to gravity be $g\,\text{m/s}^2$.

(i) By using the equations of motion $\ddot{x} = 0$ and $\ddot{y} = -g$, derive expressions for the horizontal and vertical displacements after t seconds. **2**

(ii) Let G be the point on the projectile's path where the distance below the origin equals the distance to the right of the origin. That is, $OF = FG$ on the diagram above.

(α) Prove that the time taken for the projectile to reach G is **2**

$$\frac{2V(\sin\alpha + \cos\alpha)}{g} \text{ seconds}.$$

(β) Show that $OF = \dfrac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1)$ metres. **2**

(γ) Let A be the point on the projectile's path where it is level with the point of projection. If $OF = \frac{4}{3}OA$, find α, correct to the nearest degree. **4**

You may assume that the distance OA is given by $OA = \dfrac{V^2 \sin 2\alpha}{g}$ metres.

QUESTION FIVE (Start a new writing booklet)

Marks

(a) (i) Find the general solution of $\tan 4\alpha = 1$. **1**

(ii) Use the binomial theorem and de Moivre's theorem to show that **4**

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}.$$

(iii) Hence solve the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. **3**

(iv) Hence show that **3**

$$\tan^2 \tfrac{\pi}{16} + \tan^2 \tfrac{3\pi}{16} + \tan^2 \tfrac{5\pi}{16} + \tan^2 \tfrac{7\pi}{16} = 28.$$

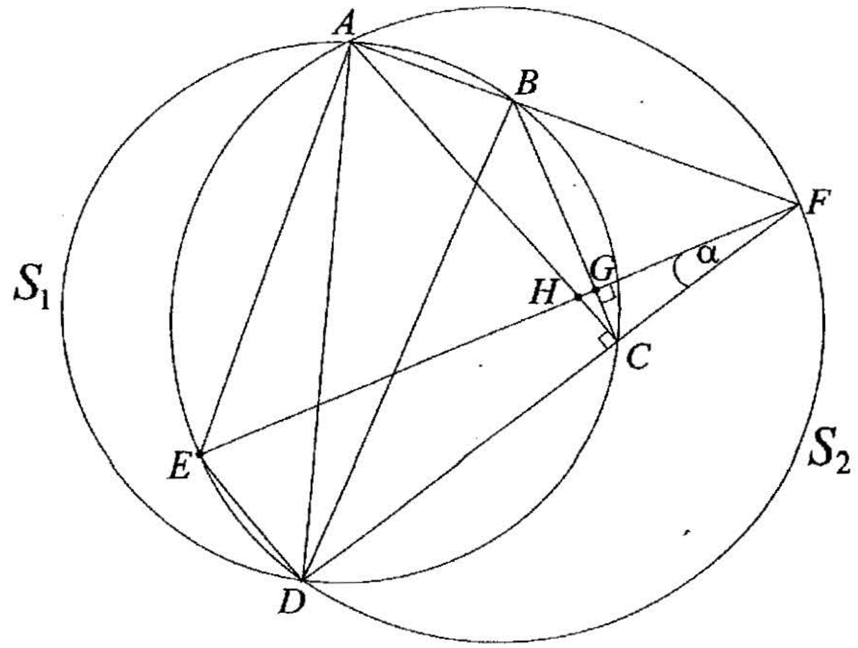
(b) Let α, β and γ be the roots of the equation $x^3 + px^2 + qx + r = 0$.

(i) Show that if the roots form an arithmetic sequence, then $2p^3 - 9pq + 27r = 0$. **2**
HINT: If α, β and γ form an arithmetic sequence, then $\alpha + \beta + \gamma = 3\beta$.

(ii) Find a similar identity involving p, q and r that holds if the roots form a geometric sequence. **2**

QUESTION SIX (Start a new writing booklet)

(a)



In the diagram above, $ABCD$ is a cyclic quadrilateral inscribed in the circle S_1, and $AC \perp DC$.

The chords AB and DC produced intersect at F, and S_2 is the circle through A, D and F.

The line through F perpendicular to BC meets BC at G, meets AC at H and meets the circle S_2 at E.

Let $\angle DFE = \alpha$.

Marks

(i) Prove that $\angle HCG = \alpha$. **1**

(ii) Prove that $AB \perp DB$. **1**

(iii) Prove that $AE \parallel BD$. **2**

(iv) Prove that E, A, B and G are concyclic. **1**

(b) Let ω be one of the non–real cube roots of 1.

(i) Show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Hence find the value of $(2 - \omega)(2 - \omega^2)(2 - \omega^4)(2 - \omega^5)$. **2**

(c) An object of mass 20 kg is dropped in a medium where the resistance at speed v m/s has a magnitude of $2v$ newtons. The acceleration due to gravity is $10\,\text{m/s}^2$.

(i) Draw a diagram to show the forces on the object and show that the equation of motion is $\ddot{x} = \dfrac{100 - v}{10}$. **1**

(ii) Find an expression for the velocity at time t seconds after the object is dropped. 2

(iii) Find the terminal velocity of the object. 1

(iv) Show that the distance x metres travelled when the speed is v m/s is given by 2

$$x = 1000\log_e\left(\frac{100}{100-v}\right) - 10v\,.$$

(v) Hence find the distance the object has fallen before reaching half its terminal velocity. 1

QUESTION SEVEN (Start a new writing booklet)

(a) A straight line is drawn through a fixed point $P(a,b)$ in the first quadrant on a number plane. The line cuts the positive part of the x-axis at A and the positive part of y-axis at B. Let $\angle OAB = \theta$.

Marks

(i) Prove that the length of AB is given by 2

$$AB = a\sec\theta + b\operatorname{cosec}\theta\,.$$

(ii) Show that the length of AB will be a minimum if 3

$$\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

(iii) Show that the minimum length of AB is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. 2

(b) (i) On the same number plane, sketch the graphs $y = \pi\sin x$ and $y = x$, for $0 \le x \le \pi$. 1

(ii) Explain why there is a number α between 0 and π such that $\pi\sin\alpha = \alpha$. Furthermore, show that $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$. Do **NOT** try to evaluate α. 1

(iii) Let $f(x) = \sqrt{\pi^2 - x^2}\cos x - x\sin x$, for $-\pi \le x \le \pi$.

(α) Prove that $f(x)$ is an even function. 1

(β) Evaluate $f(x)$ at $x = 0,\ \frac{\pi}{3},\ \frac{\pi}{2}$ and π. 1

(γ) If α is the number defined in part (ii), show that $f(\alpha) = -\pi$. 1

(δ) Show that $f'(\alpha) = 0$, and hence find 3 stationary points of $f(x)$ and determine their nature. 3

QUESTION EIGHT (Start a new writing booklet)

Marks

(a) (i) Find k in terms of n if $\sin n\theta + \sin(n-2)\theta = 2\sin k\theta\cos\theta$. **1**

(ii) If n is an integer greater than 1 and $I_n = \int \sin n\theta \sec\theta \, d\theta$, prove that **2**

$I_n + I_{n-2} = \dfrac{2\cos(n-1)\theta}{1-n} + C$, where C is a constant of integration.

(iii) Hence prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5\theta \sin\theta}{\cos\theta} \, d\theta = \frac{23}{15}$. **4**

(b) (i) Let $a_1, a_2, \ldots, a_{k+1}$ be positive real numbers. Define the function $\psi(x)$ by **3**

$$\psi(x) = a_1 + a_2 + \cdots + a_k + x - (k+1)(a_1a_2\cdots a_kx)^{\frac{1}{k+1}}, \text{ for } x > 0.$$

Show that the minimum value of $\psi(x)$ occurs at $x = x_0$, where

$$x_0 = (a_1a_2\cdots a_k)^{\frac{1}{k}}.$$

(ii) Let $A_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and $G_n = \sqrt[n]{a_1a_2\cdots a_n}$. By considering $\psi(a_{k+1})$ from part (i) and using mathematical induction, prove that $A_n \geq G_n$. **5**

REP

1. (a) $\displaystyle\int \frac{\sin x}{\cos^5 x}\,dx$

$\displaystyle = \int \sin x \cos^{-5} x\,dx$

$= \frac{1}{4}\cos^{-4} x + c$

$= \frac{1}{4}\sec^4 x + c$

$\stackrel{\text{or}}{=} \dfrac{1}{4\cos^4 x} + c$ [1]

(b) $\displaystyle\int_{-2}^{-1} \frac{5}{x^2+4x+5}\,dx$

$\displaystyle = \int_{-2}^{-1} \frac{5}{(x+2)^2+1}\,dx$

$= \Big[5\tan^{-1}(x+2)\Big]_{-2}^{-1}$

$= 5(\tan^{-1} 1 - \tan^{-1} 0)$

$= \frac{5}{4}\pi$ [3]

(c) (i) $\dfrac{3x^2-x+8}{(1-x)(x^2+1)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{x^2+1}$

Hence $A = \dfrac{3-1+8}{1^2+1} = 5$

$3x^2 - x + 8 \equiv 5(x^2+1) + (Bx+C)(1-x)$

Hence $5 - B = 3$ and $5 + C = 8$

So $B = 2$ and $C = 3$ [3]

(ii) $\displaystyle\int \frac{3x^2-x+8}{(1-x)(x^2+1)}\,dx = \int \frac{5}{1-x} + \frac{2x+3}{x^2+1}\,dx$

$\displaystyle = \int \frac{5}{1-x} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx$

$= \ln\left|x^2+1\right| - 5\ln|1-x| + 3\tan^{-1} x + c$

$= \ln\left|\dfrac{x^2+1}{(1-x)^5}\right| + 3\tan^{-1} x + c$ [2]

(d) $\displaystyle\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx$

$\displaystyle = \Big[2\sqrt{x}\ln x\Big]_1^4 - 2\int_1^4 \frac{\sqrt{x}}{x}\,dx$

$\displaystyle = 4\ln 4 - 2\int_1^4 \frac{1}{\sqrt{x}}\,dx$

$= 4\ln 4 - 4\Big[\sqrt{x}\Big]_1^4$

$= 4(2\ln 2 - 1)$, as required. [3]

(e) $\displaystyle\int \frac{1}{1+\cos\theta}\,d\theta$

Let $t = \tan\frac{\theta}{2}$

Hence $\cos\theta = \dfrac{1-t^2}{1+t^2}$

Also $d\theta = \dfrac{2\,dt}{1+t^2}$

$$= \int \frac{2}{(1+t^2)\left(1+\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int \frac{2}{1+t^2+1-t^2}\,dt$$

$$= \int dt$$

$$= t + c$$

$$= \tan\tfrac{\theta}{2} + c \qquad \boxed{3}$$

2. (a) Let $z = x + iy$, hence $z^2 = 9 - 40i = (x+iy)^2$

So $x^2 - y^2 + 2ixy = 9 - 40i$

Equate real and imaginary parts.

So $x^2 - y^2 = 9$ and $xy = -20$

Hence $x^2 - \frac{400}{x^2} = 9$

So $x^4 - 9x^2 - 400 = 0$

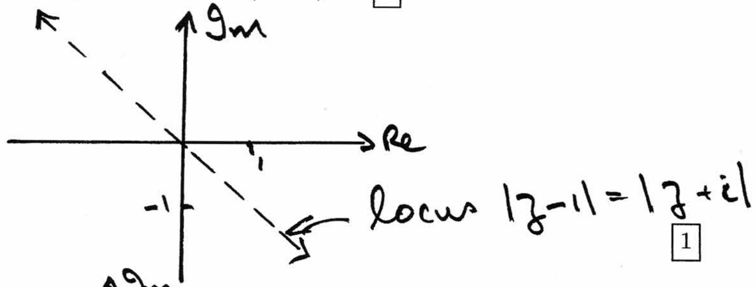
$(x^2 - 25)(x^2 + 16) = 0$

But $x \in \mathbf{R}$, so $x = \pm 5$

$x = \pm 5$ yields $y = \mp 4$

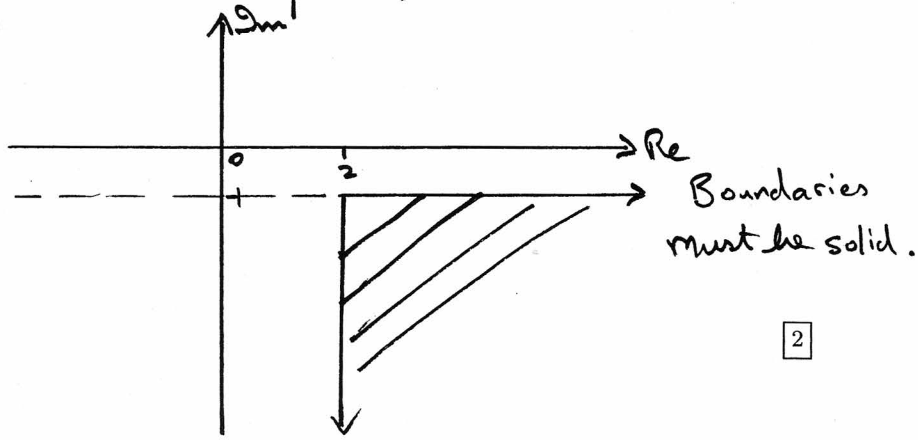
Hence the square roots are $\pm(5 - 4i)$ $\boxed{3}$

(b)



$\boxed{1}$

(c)



$\boxed{2}$

(d) (i) $\arg z = -\frac{\pi}{4}$ and $\arg w = \frac{2\pi}{3}$ [1]

(ii) $\arg(wz) = \arg w + \arg z = \frac{5\pi}{12}$ [1]

(iii) Now $wz = \sqrt{3} - 1 + i(\sqrt{3}+1)$

Hence $\sin\frac{5\pi}{12} = \dfrac{\text{Im}(wz)}{|wz|} = \dfrac{\text{Im}(wz)}{|w||z|}$

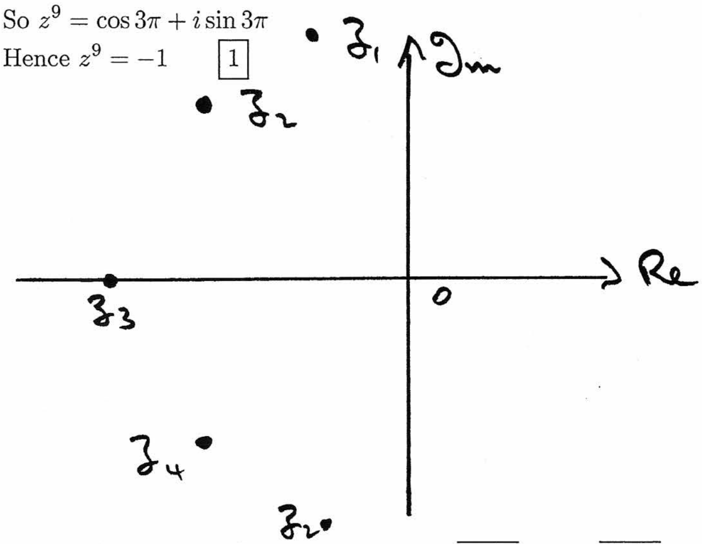
So $\sin\frac{5\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \dfrac{\sqrt{2}(1+\sqrt{3})}{4}$ as required. [2]

(e) (i) $z^9 = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^9$

So $z^9 = \cos 3\pi + i\sin 3\pi$

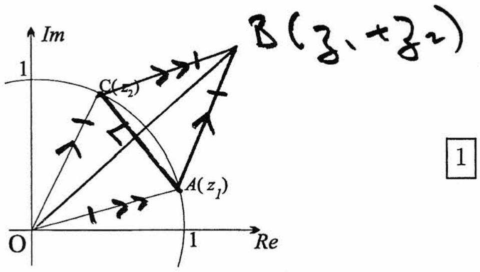
Hence $z^9 = -1$ [1]

(ii)



$z_1 = \text{cis}\,\frac{5\pi}{9}$, $z_2 = \text{cis}\,\frac{7\pi}{9}$, $z_3 = -1$, $z_4 = \overline{\text{cis}\,\frac{7\pi}{9}}$, $z_5 = \overline{\text{cis}\,\frac{5\pi}{9}}$. [2]

(f) (i)

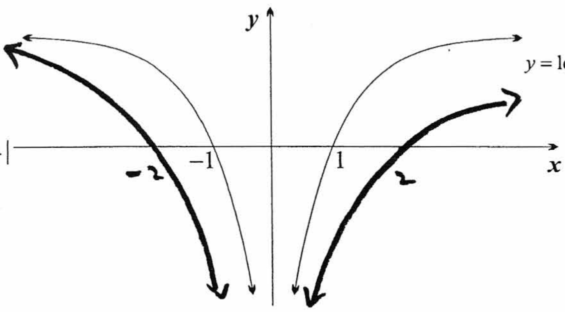


[1]

(ii) $OABC$ is a rhombus and hence the diagonals are perpendicular. [1]

3. (a) (i)

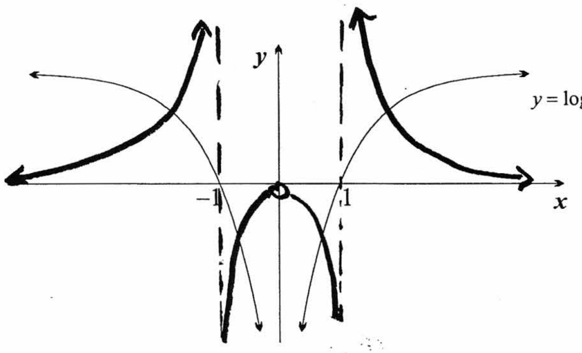
$y = f(\frac{x}{2}) = \log_e |\frac{x}{2}|$



1

(ii)

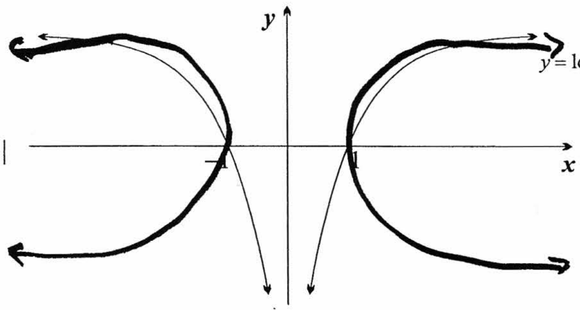
$y = \frac{1}{f(x)} = \frac{1}{\log_e |x|}$



2

(iii)

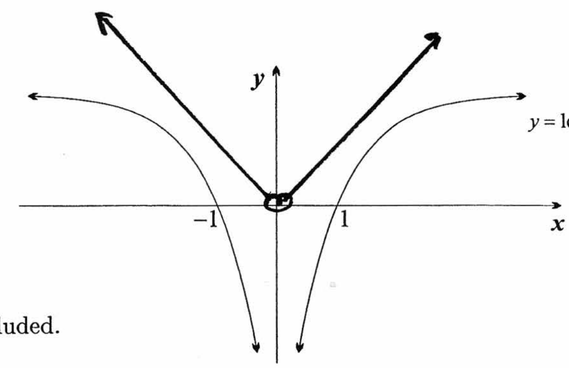
$y^2 = \log_e |x|$



2

(iv)

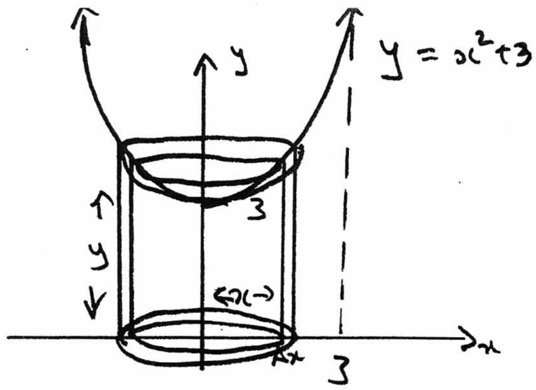
$y = e^{f(x)} = e^{\log_e |x|}$



1

Note: $(0, 0)$ is excluded.

(b)



The curved surface of each cylindrical shell is given by $SA = 2\pi xy = 2\pi(x^2+3)$.

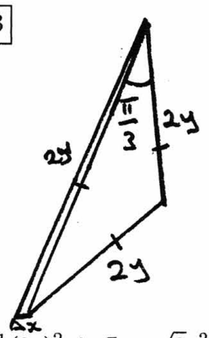
Hence the volume of a shell Δx thick is $\approx 2\pi x(x^2+3)\Delta x$.

So the volume required is $V = 2\pi \int_0^3 x^3 + 3x \, dx$.

So $V = 2\pi \left[\frac{1}{4}x^4 + \frac{3}{2}x^2\right]_0^3$

$V = \frac{135}{2}\pi \overset{\text{or}}{=} 67{\cdot}5\pi$ units^3. [3]

(c)



Area of each cross-sectional slice is $\frac{1}{2}(2y)^2 \sin\frac{\pi}{3} = \sqrt{3}y^2$

Hence the volume of a slice Δx thick is $\approx \sqrt{3}y^2\Delta x = \sqrt{3}(4-x^2)\Delta x$.

So the volume required is $\sqrt{3}\int_{-2}^{2} 4 - x^2 \, dx$.

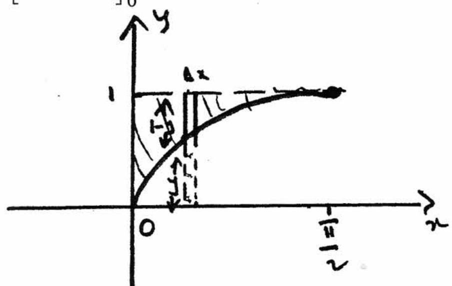
So $V = 2\sqrt{3}\int_0^2 4 - x^2 \, dx$

So $V = 2\sqrt{3}\left[4x - \frac{1}{3}x^3\right]_0^2$

$= 2\sqrt{3}(8 - \frac{8}{3})$

So $V = \dfrac{32}{3}\sqrt{3}$ units^3 [3]

(d)



The area of each slice of the solid is $\pi(1-y)^2 = \pi(1-\sin x)^2$.

If the slice is Δx thick then the volume is $\approx \pi(1-\sin x)^2\Delta x$.

$$V = \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \sin^2 x \, dx$$

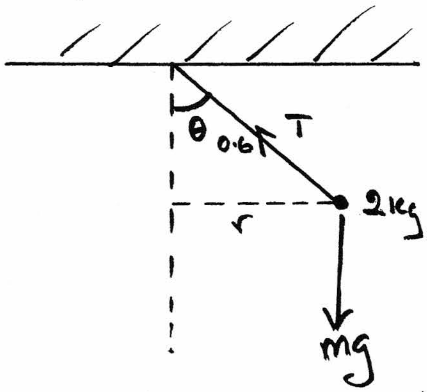
$$V = \int_0^{\frac{\pi}{2}} \tfrac{3}{2} - 2\sin x - \tfrac{1}{2}\cos 2x \, dx$$

$$V = \left[\tfrac{3}{2}x + 2\cos x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$V = \pi\left(\tfrac{3\pi}{4} - 2\right)$$

So $V = \dfrac{(3\pi - 8)\pi}{4}$ units^3 . [3]

4. (a) (i)



[1]

(ii) Resolve forces at the mass.

vert $\updownarrow$ $T\cos\theta = 2g = 20$

horoz $\leftrightarrow$ $T\sin\theta = 2r\omega^2$

Hence $T\dfrac{r}{0{\cdot}6} = 2r(3\pi)^2$

So $T = 2 \times 0{\cdot}6 \times 9\pi^2$

i.e. $T = 10{\cdot}8\pi^2 \overset{\text{or}}{\approx} 106{\cdot}6\,\text{N}$ [3]

(iii) $\cos\theta = \dfrac{20}{T}$

So $\cos\theta = \dfrac{20}{10{\cdot}8\pi^2}$

So $\theta = 79^\circ$, to nearest $^\circ$. [1]

(b) (i) $\ddot{x}(t) = 0$

Hence $\dot{x} = C_1$, a constant.

But $\dot{x}(0) = V\cos\alpha = C_1$.

Hence $\dot{x}(t) = V\cos\alpha$.

So $x(t) = V\cos\alpha t + C_2$,

where C_2 is a constant.

But $x(0) = 0 = C_2$.

Hence $x(t) = V\cos\alpha t$.

Also $\ddot{y} = -g$.

So $\dot{y} = -gt + C_3$,

where C_3 is a constant.

But $\dot{y}(0) = V\sin\alpha$,

Hence $C_3 = V\sin\alpha$.

So $\dot{y} = V\sin\alpha - gt$.

So $y = V\sin\alpha t - \frac{1}{2}gt^2 + C_4$,

where C_4 is a constant.

$y(0) = 0 = C_4$.

So $y(t) = V\sin\alpha t - \frac{1}{2}gt^2$.

↖ [2] ↗

(ii) (α) $OF = FG$ hence

$V\sin\alpha t - \frac{1}{2}gt^2 = -V\cos\alpha t$

So $\frac{1}{2}gt = V\sin\alpha + V\cos\alpha$, $(t \neq 0)$

So $t = \dfrac{2V(\sin\alpha + \cos\alpha)}{g}$ seconds. [2]

(β) $OF = V\cos\alpha t$

So $OF = V\cos\alpha \dfrac{2V}{g}(\sin\alpha + \cos\alpha)$

So $OF = \dfrac{V^2}{g}(2\sin\alpha\cos\alpha + 2\cos^2\alpha)$

So $OF = \dfrac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1)$ m. [2]

(NOTE: Numerous solutions possible. The most common are below.)

(γ) $OF = \frac{4}{3}OA$, so $\dfrac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1) = \dfrac{4}{3}\dfrac{V^2}{g}\sin 2\alpha$

So $3\sin 2\alpha + 3\cos 2\alpha + 3 = 4\sin 2\alpha$

So $\sin 2\alpha - 3\cos 2\alpha = 3$.

$\frac{1}{\sqrt{10}}\sin 2\alpha - \frac{3}{\sqrt{10}}\cos 2\alpha = \frac{3}{\sqrt{10}}$

Hence $\sin(2\alpha - \theta) = \frac{3}{\sqrt{10}}$,

where $\cos\theta = \frac{1}{\sqrt{10}}$

and $\sin\theta = \frac{3}{\sqrt{10}}$.

If $0^o \leq \theta \leq 90^o$ then $\theta = 71^o\ 34'$ to the nearest minute.

So $2\alpha = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right) + \theta = 2\theta$

That is $\alpha = \theta$.

OR Let $t = \tan\alpha$

Hence $\sin 2\alpha = \dfrac{2t}{1+t^2}$

and $\cos 2\alpha = \dfrac{1-t^2}{1+t^2}$

So $\dfrac{2t}{1+t^2} - \dfrac{3(1-t^2)}{1+t^2} = 3$

So $2t - 3 + 3t^2 = 3 + 3t^2$

So $2t = 6$

So $\tan\alpha = 3$

So $\alpha = 72^o$ to the nearest degree. [4]

5. (a) (i) $\tan 4\alpha = 1$

So $4\alpha = n\pi + \frac{\pi}{4}$, $n \in Z$

So $\alpha = (4n+1)\frac{\pi}{16}$, $n \in Z$ $\boxed{1}$

(ii) $(\cos\alpha + i\sin\alpha)^4 = \cos 4\alpha + i\sin 4\alpha$ (de M. th$\underline{\text{m}}$).

But the binomial theorem gives

$(\cos\alpha + i\sin\alpha)^4 = \cos^4\alpha + 4i\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha$
$-4i\cos\alpha\sin^3\alpha + \sin^4\alpha$

Now equate the real and imaginary parts.

Hence $\sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha$

and $\cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

So $\tan\alpha = \dfrac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha}$

Hence $\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$, as required. $\boxed{4}$

(iii) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

So $4x - 4x^2 = x^4 - 6x^2 + 1$

i.e. $\dfrac{4x - 4x^3}{x^4 - 6x^2 + 1} = 1$

Let $x = \tan\alpha$

So $\dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha} = 1$

Hence $\tan 4\alpha = 1$

So $\alpha = (4n+1)\frac{\pi}{16}$, $n \in \mathbf{Z}$

Consider the values when $n = 0$, ± 1 and -2.

i.e. $x = \tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $-\tan\frac{3\pi}{16}$ ($\overset{\text{or}}{=} \tan\frac{13\pi}{16}$) or $-\tan\frac{7\pi}{16}$ ($\overset{\text{or}}{=} \tan\frac{9\pi}{16}$) $\boxed{4}$

(iv) $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16}$

$= \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= (-4)^2 - 2(-6)$

$= 28$, as required. $\boxed{2}$

(b) (i) $\alpha + \beta + \gamma = 3\beta$

So $\beta = -\frac{p}{3}$

$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$

So $-p^3 + 3p^2 - 9pq + 27r = 0$

i.e. $2p^3 - 9pq + 27r = 0$ $\boxed{2}$

(ii) $\alpha\beta\gamma = \beta^3$

So $\beta = \sqrt[3]{-r}$

Hence $\left(\sqrt[3]{-r}\right)^3 + p\left(\sqrt[3]{-r}\right)^2$
$+q\left(\sqrt[3]{-r}\right) + r = 0$

So $-r + pr^{\frac{2}{3}} + q(-r)^{\frac{1}{3}} + r = 0$

i.e. $pr^{\frac{2}{3}} = qr^{\frac{1}{3}}$

So $p^3r^2 = q^3r$

Hence $p^3r = q^3$ $\boxed{2}$

6. (a) (i) $\angle GCD = \frac{\pi}{2} + \angle HCG = \frac{\pi}{2} + \alpha$ (Ext. $\angle \triangle CGF =$ sum of the int. opp.$\angle$'s)

Hence $\angle HCG = \alpha$, as required. $\boxed{1}$

(ii) $\angle ABD = \angle ACD = \frac{\pi}{2}$ ($\angle$'s in the same segment)

Hence $AB \perp DB$, as required. $\boxed{1}$

(iii) $\angle EAD = \alpha$ ($\angle$'s in the same segment)

$\angle ADB = \alpha$ ($\angle$'s in the same segment)

So $\angle BAD = \frac{\pi}{2} - \alpha$ ($\angle$ sum $\triangle BAD = \pi$)

Hence $\angle BAE = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$.

Hence $AB \perp AE$

So $AE \parallel BD$ (cointerior $\angle$'s are supplementary). $\boxed{2}$

(iv) $\angle BAE + \angle BGE = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ ((iii) and given $FH \perp BC$)

Hence E, A, B and G are concyclic as the opposite $\angle$'s are supplementary. $\boxed{1}$

(b) (i) $1 + \omega + \omega^2$ is a geometric series with common ratio ω.

So $1 + \omega + \omega^2 = \dfrac{\omega^3 - 1}{\omega - 1}$

But $\omega^3 = 1$

Hence $1 + \omega + \omega^2 = 0$, as required. $\boxed{1}$

(ii) $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$, as $\omega^3 = 1$

$= \left((2-\omega)(2-\omega^2)\right)^2$

$= (4 - 2\omega - 2\omega^2 + \omega^3)^2$

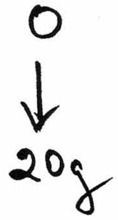
$= \left(5 - 2(\omega + \omega^2)\right)^2$

But $\omega + \omega^2 = -1$ from (i).

Hence $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5) = (5+2)^2$

i.e. $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5) = 49$ $\boxed{2}$

(c) (i)



Newton's 2 nd law gives:

$20\ddot{x} = 20g - 2v$

$\ddot{x} = 10 - \dfrac{v}{10}$

$\ddot{x} = \dfrac{100 - v}{10}$ $\boxed{1}$

(ii) $\ddot{x} = \dfrac{dv}{dt} = \dfrac{100 - v}{10}$

So $\displaystyle\int \frac{dv}{100 - v} = \tfrac{1}{10} \int dt$

So $-\ln|100 - v| = \tfrac{t}{10} + c$, for some constant c.

When $t = 0$, $v = 0$

hence $c = -\ln 100$.

So $-\dfrac{t}{10} = \ln\left|\dfrac{100 - v}{100}\right|$

So $100e^{-\frac{t}{10}} = 100 - v$

$v = 100\left(1 - e^{-\frac{t}{10}}\right)$ [2]

(iii) Terminal velocity attained when either $t \to \infty$ or $\ddot{x} = 0$

Hence the terminal velocity is 100 m/s [1]

(iv) Now $\ddot{x} = \dfrac{100 - v}{10}$

So $v\dfrac{dv}{dx} = \dfrac{100 - v}{100}$

So $\dfrac{dv}{dx} = \dfrac{100 - v}{10v}$

So $\dfrac{dx}{dv} = \dfrac{10v}{100 - v} = \dfrac{1000 - 10(100 - v)}{100 - v}$

So $\displaystyle\int dx = \int \frac{1000}{100 - v} - 10\, dv$

So $x = -1000 \ln|100 - v| - 10v + c$, for some constant c

But $x = 0$ when $v = 0$

So $c = 1000 \ln 100$ and from (iii) $v < 100$.

So $x = 1000\,(\ln 100 - \ln(100 - v)) - 10v$.

So $x = 1000 \ln\left(\dfrac{100}{100 - v}\right) - 10v$ m, as required. [2]

(v) Let $v = 50$

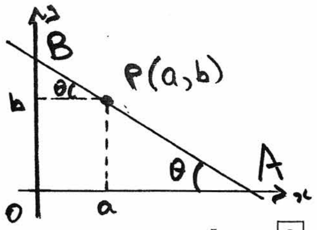
So $x = 1000 \ln 2 - 500$

So $x = 500(\ln 4 - 1)$

So $x = 193{\cdot}15$

Hence the object has fallen approximately 193·15 metres. [1]

7. (a) (i)



$AP = b \operatorname{cosec}\theta$
and $PB = a\sec\theta$.
$AB = a\sec\theta + b\operatorname{cosec}\theta$

↖ [2] ↗

(ii) $\dfrac{d}{d\theta}(AB) = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$

If $\dfrac{d}{d\theta}AB = 0$

then $a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$

So $\dfrac{\operatorname{cosec}\theta\cot\theta}{\sec\theta\tan\theta} = \dfrac{a}{b}$

So $\dfrac{\cot\theta}{\sec\theta\sin\theta\tan\theta} = \dfrac{a}{b}$

So $\dfrac{\cot\theta}{\tan^2\theta} = \dfrac{a}{b}$

So $\cot^3\theta = \dfrac{a}{b}$

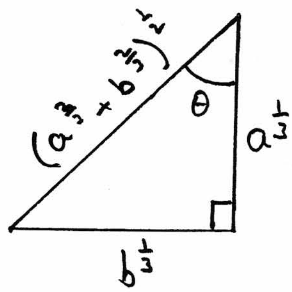
Hence $\cot\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$

Now $\dfrac{d^2}{d\theta^2}AB = a\sec\theta\tan^2\theta + a\sec^3\theta + b\operatorname{cosec}\theta\cot^2\theta + b\operatorname{cosec}^3\theta$

But $0 \le \theta \le \frac{\pi}{2}$ and hence all the trigonometric functions are positive so $\dfrac{d^2}{d\theta^2}AB > 0$.

So $\cot\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$ minimises AB. [3]

(iii) $\cot\theta = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ and as θ is acute we can represent θ as shown in the right triangle.



Hence $r^2 = a^{\frac{2}{3}} + b^{\frac{2}{3}}$

So $r = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}$

Hence $\sec\theta = \dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{a^{\frac{1}{3}}}$ and $\operatorname{cosec}\theta = \dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{b^{\frac{1}{3}}}$

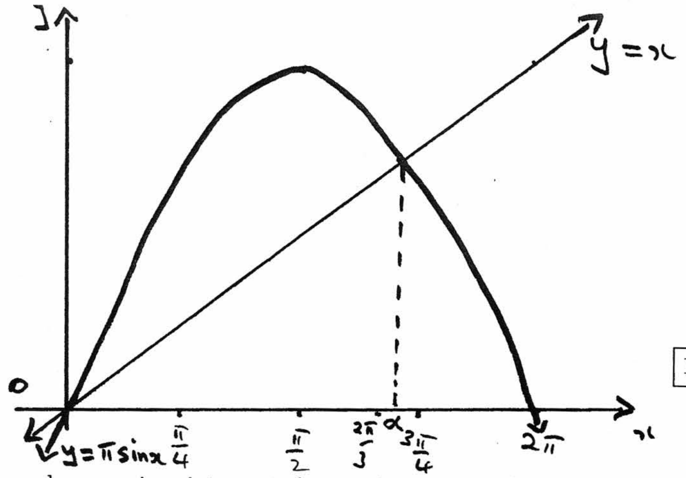
So the minimum length of AB is:

$a\dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{a^{\frac{1}{3}}} + b\dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{b^{\frac{1}{3}}}$

$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right).$

Hence the minimum length of $AB = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$, as required. [2]

(b) (i)



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(ii) As $y = x$ and $y = \pi \sin x$ intersect there exists some value, α say such that $\pi \sin \alpha = \alpha$.

Consider the function $g(x) = \pi \sin x - x$.

$g\left(\frac{2\pi}{3}\right) = \pi . \dfrac{\sqrt{3}}{2} - \frac{2\pi}{3}$

$= \dfrac{3\sqrt{3} - 4}{6}\pi \approx 0{\cdot}626 > 0$.

$g\left(\frac{3\pi}{4}\right) = \dfrac{\pi}{\sqrt{2}} - \frac{3\pi}{4}$

$= \frac{1}{4}(2\sqrt{2} - 3)\pi \approx -0{\cdot}135 < 0$.

So $g(x)$ being continuous between $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ and as $g\left(\frac{2\pi}{3}\right).g\left(\frac{3\pi}{4}\right) < 0$ there exists a zero α such that $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$. 1

(iii) (α) $f(-x) = \sqrt{\pi^2 - (-x)^2}\cos(-x) - (-x)\sin(-x)$

$= \sqrt{\pi^2 - x^2}\cos x - x \sin x$

Hence $f(-x) = f(x)$

That is $f(x)$ is even. 1

(β) $f(0) = \pi$.

$f\left(\frac{\pi}{3}\right) = \sqrt{\pi^2 - \frac{\pi^2}{9}}.\frac{1}{2} - \frac{\pi}{3}.\frac{\sqrt{3}}{2}$

$f\left(\frac{\pi}{3}\right) = \dfrac{2\sqrt{2} - \sqrt{3}}{6}\pi \approx 0{\cdot}574$.

$f\left(\frac{\pi}{2}\right) = \sqrt{\pi^2 - \frac{\pi^2}{4}}.0 - \frac{\pi}{2}.1$

$f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$.

$f(\pi) = \sqrt{\pi^2 - \pi^2}. - 1 - \pi \sin \pi$

$f(\pi) = 0$. 1

(γ) $f(\alpha) = \sqrt{\pi^2 - \alpha^2}\cos\alpha - \alpha \sin\alpha$

$= \pi\sqrt{\cos^2\alpha}\cos\alpha - \pi\sin^2\alpha$

But $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$ so $\cos\alpha < 0$ hence $\sqrt{\cos^2\alpha} = -\cos\alpha$

$= \pi(-\cos^2\alpha - \sin^2\alpha)$

So $f(\alpha) = -\pi$. 1

(δ) $f'(x) = \frac{1}{2}\frac{1}{\sqrt{\pi^2 - x^2}}. - 2x\cos x. - \sin x\sqrt{\pi^2 - x^2} - x\cos x - \sin x$

So $f'(x) = -\left(\frac{x\cos x}{\sqrt{\pi^2 - x^2}} + \sin x\sqrt{\pi^2 - x^2} + x\cos x + \sin x.\right)$

So $f'(\alpha) = \frac{-\alpha\cos\alpha}{\sqrt{\pi^2 - \alpha^2}} - \frac{\sqrt{\pi^2 - \alpha^2}\sin\alpha}{1} - \alpha\cos\alpha - \sin\alpha$

That is $f'(\alpha) = \sin\alpha + \pi\cos\alpha\sin\alpha - \pi\cos\alpha\sin\alpha - \sin\alpha$

So $f'(\alpha) = 0$.

Hence $x = \alpha$ is a stationary point.

Now $\frac{\pi}{2} < \frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$.

So $f'\left(\frac{\pi}{2}\right) = -\left(\frac{\sqrt{3}}{2}\pi + 1\right) < 0$

and $f'\left(\frac{3\pi}{4}\right) = -\left(-\frac{3}{\sqrt{14}} + \frac{\sqrt{7}}{4\sqrt{2}}\pi - \frac{3}{4\sqrt{2}}\pi + \frac{1}{\sqrt{2}}\right) \approx 0{\cdot}29 > 0$

Hence $(\alpha, -\pi)$ is a minimum.

But $f(x)$ is even so $(-\alpha, -\pi)$ is a minimum.

As $f(x)$ is continuous there must be a maximum between the two minimums above. As $f(x)$ is even the only possible maximum must occur at $x = 0$. That is there is a maximum at $(0, \pi)$.

So the turning points and their nature are:

$$\begin{cases} (-\alpha, -\pi) & \text{minimum,} \\ (0, \pi) & \text{maximum,} \\ (\alpha, -\pi) & \text{minimum.} \end{cases} \qquad \boxed{3}$$

[As a matter of interest $\alpha \approx 2{\cdot}31373413208$.]

8. (a) (i) $\sin n\theta + \sin(n-2)\theta$

$= 2\sin(n-1)\theta\cos\theta$

Hence $k = n-1$. $\boxed{1}$

(ii) $I_n + I_{n-2}$

$$= \int (\sin n\theta + \sin(n-2)\theta)\sec\theta\, d\theta$$

$$= 2\int \sin(n-1)\theta\cos\theta\sec\theta\, d\theta$$

$$= 2\int \sin(n-1)\theta\, d\theta$$

$$= -\frac{2}{n-1}\cos(n-1)\theta + C\text{, for some constant } C.$$

So $I_n + I_{n-2} = \dfrac{2\cos(n-1)\theta}{1-n} + C$ as required. $\boxed{2}$

(iii) $\dfrac{\cos 5\theta\sin\theta}{\cos\theta} = \sec\theta(\frac{1}{2}\sin 6\theta - \frac{1}{2}\sin 4\theta)$.

Now $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5\theta\sin\theta}{\cos\theta}\, dh = \tfrac{1}{2}\Big[I_6 - I_4\Big]_0^{\frac{\pi}{2}}$

$$= \tfrac{1}{2}\Big[I_6 + I_4 - 2I_4\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\Big[I_6 + I_4\Big]_0^{\frac{\pi}{2}} - \Big[I_4\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left[\frac{\cos 5\theta}{-5}\right]_0^{\frac{\pi}{2}} - \Big[I_4 + I_2 - I_2\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \left[\frac{2\cos 3\theta}{-3}\right]_0^{\frac{\pi}{2}} + \Big[I_2\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + \Big[I_2 + I_0 - I_0\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + \left[\frac{2\cos\theta}{-1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 0\, d\theta$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + 2 - 0$$

$= \frac{23}{15}$ as required. $\boxed{4}$

(b) (i) $\psi(x) = a_1 + a_2 + \cdots + a_k + x - (k+1)\,(a_1 a_2 \cdots a_k x)^{\frac{1}{k+1}}$.

So $\psi'(x) = 1 - (a_1 a_2 \cdots a_k x)^{\frac{1}{(k+1)}-1}\,(a_1 a_2 \cdots a_k)$

$\psi'(x) = 1 - (a_1 a_2 \cdots a_k)^{\frac{1}{(k+1)}}\, x^{\frac{1}{k+1}-1}$

$\psi'(x) = 1 - (a_1 a_2 \cdots a_k)^{\frac{1}{(k+1)}}\, x^{-\frac{k}{k+1}}$

When $\psi'(x) = 0$ then

$(a_1 a_2 \cdots a_k)^{\frac{1}{(k+1)}}\, x^{-\frac{k}{k+1}} = 1$

So $x^{-\frac{k}{k+1}} = (a_1 a_2 \cdots a_k)^{-\frac{1}{k+1}}$

So $x^k = (a_1 a_2 \cdots a_k)$

Hence $\psi'(x) = 0$, when $x = (a_1 a_2 \cdots a_k)^{\frac{1}{k}} = x_0$.

Now $\psi''(x) = \left(\dfrac{k}{k+1}\right)(a_1 \cdots a_k)^{\frac{1}{k+1}}\, x^{-\frac{2k+1}{k+1}}$

So $\psi''(x_0) = \dfrac{k}{k+1}(a_1 \cdots a_k)^{\frac{1}{k+1}}\left((a_1 \cdots a_k)^{\frac{1}{k}}\right)^{-\frac{2k+1}{k+1}}$

So $\psi''(x_0) = \dfrac{k}{k+1}(a_1 \cdots a_k)^{\frac{1}{k+1}-\frac{2k+1}{k(k+1)}}$

That is $\psi''(x_0) = \dfrac{k}{k+1}(a_1 \cdots a_k)^{-\frac{1}{k}} \overset{\text{or}}{=} \dfrac{k}{(k+1)G_k} > 0$, as $k, G_k > 0$.

Hence the minimum value of $\psi(x)$ occurs at $x = x_0$. [3]

(ii) Consider the proposition that

"if $A_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and $G_n = \sqrt[n]{a_1 a_2 \cdots a_n}$ then $A_n \geq G_n$".

Now $A_1 = a_1$ and $G_1 = \sqrt[1]{a_1} = a_1$ hence $A_1 \geq G_1$.

Hence the proposition is true for $n = 1$.

Let k be some positive integer such that the proposition is true.

That is $A_k \geq G_k$.

From (i) $\psi(a_{k+1}) \geq \psi(x_0)$.

That is $a_1 + a_2 + \cdots + a_k + a_{k+1} - (k+1)(a_1 a_2 \cdots a_{k+1})^{\frac{1}{k+1}}$

$\geq a_1 + a_2 + \cdots + a_k + G_k - (k+1)\,(a_1 a_2 \cdots a_k G_k)^{\frac{1}{k+1}}$.

$\left((a_1 a_2 \cdots a_k G_k)^{\frac{1}{k+1}} = \left((a_1 \cdots a_k)^{1+\frac{1}{k}}\right)^{\frac{1}{k+1}} = \left((a_1 \cdots a_k)^{\frac{k+1}{k}}\right)^{\frac{1}{k+1}} = G_k\right)$

That is $(k+1)\,(A_{k+1} - G_{k+1}) \geq kA_k + G_k - (k+1)G_k$

So $(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k) \geq 0$

Hence $A_{k+1} \geq G_{k+1}$.

As $A_1 \geq G_1$ and $A_k \geq G_k$ implies $A_{k+1} \geq G_{k+1}$ for some positive integer k then by the principle of mathematical induction $A_n \geq G_n$ for all positive integers n. [5]