Sydney Grammar School
Mathematics Department
Trial Examinations 2007

FORM VI

MATHEMATICS EXTENSION 2

Examination date

Wednesday 1st August 2007

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the eight questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 8 per boy. A total of 750 booklets should be sufficient.
Candidature: 71 boys.

Examiner

DS

<u>QUESTION ONE</u> (15 marks) Use a separate writing booklet. **Marks**

(a) Show that $\int_0^{\frac{\pi}{6}} x\cos x\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$. **3**

(b) Find $\int \frac{1}{2+\sqrt{x}}\,dx$ by using the substitution $\sqrt{x} = u$. **3**

(c) Find $\int \tan^4 x\,dx$. **2**

(d) (i) Show that $\int_0^1 \frac{1}{(5x+3)(x+1)}\,dx = \frac{1}{2}\ln\frac{4}{3}$. **3**

(ii) Hence find $\int_0^{\frac{\pi}{2}} \frac{1}{4\sin x - \cos x + 4}\,dx$ using the substitution $t = \tan\frac{x}{2}$. **4**

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Given that $z = \dfrac{2+i}{1-i}$, find $z + \dfrac{1}{z}$ in the form $a + bi$, where a and b are real. $\boxed{3}$

(b) Find the two square roots of $8i$ in the form $a + bi$, where a and b are real. $\boxed{3}$

(c) Let $z = 1 + i\tan\theta$, where $0 < \theta < \frac{\pi}{2}$.
Find, in simplest form, expressions for:

(i) $|z|$ $\boxed{2}$

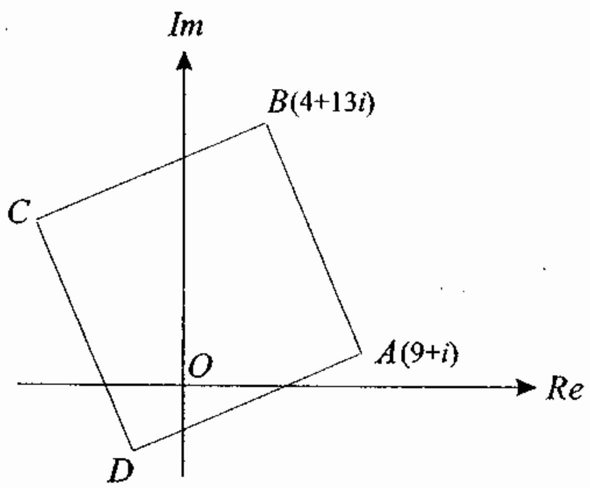
(ii) $\arg z$ $\boxed{1}$

(d) The locus of the complex number z is defined by the equation $\arg(z+1) = \frac{\pi}{4}$.

(i) Sketch the locus of z. $\boxed{1}$

(ii) Find the least value of $|z|$. $\boxed{2}$

(e)



The diagram above shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $9 + i$ and $4 + 13i$ respectively. Find the complex numbers represented by:

(i) the vector AB, $\boxed{1}$

(ii) the vertex D. $\boxed{2}$

Exam continues overleaf ...

QUESTION THREE (15 marks) Use a separate writing booklet.

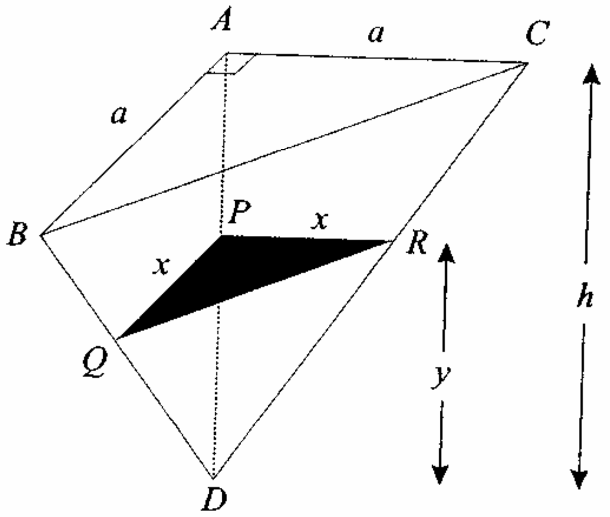
Marks

(a) (i) Use the formulae for $\cos(A+B)$ and $\cos(A-B)$ to prove that **2**

$$\cos P + \cos Q = 2\cos\tfrac{P+Q}{2}\cos\tfrac{P-Q}{2}.$$

(ii) Hence, or otherwise, solve the equation $\cos 7x + \cos 3x = 0$, for $0 \le x \le \frac{\pi}{2}$. **3**

(b)



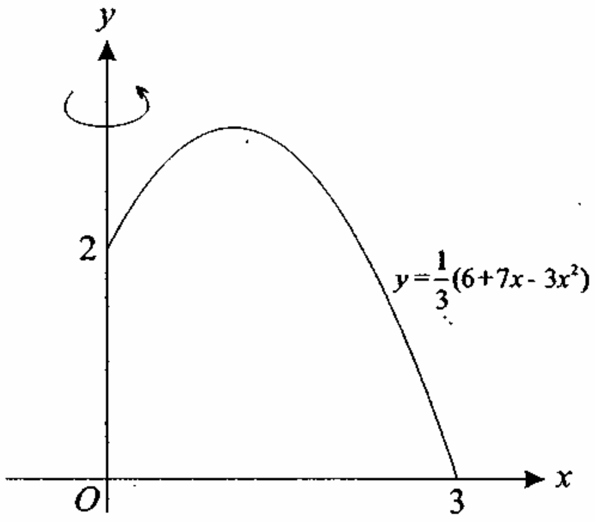
In the diagram above, $ABCD$ is a triangular pyramid. Its base ABC is a right-angled isosceles triangle with equal sides AB and AC of length a units, and its perpendicular height AD is h units. The typical triangular cross-section PQR shown is parallel to the base and y units above D. Let $PQ = PR = x$ units.

(i) Find x in terms of a, h and y. **2**

(ii) Use integration to find the volume of the pyramid. **4**

Exam continues next page ...

(c) **4**



The diagram above shows the region in the first quadrant bounded by the parabola $y = \frac{1}{3}(6 + 7x - 3x^2)$ and the x and y axes. This region is rotated through 360° about the y-axis to form a solid. Use the method of cylindrical shells to find the exact volume of the solid.

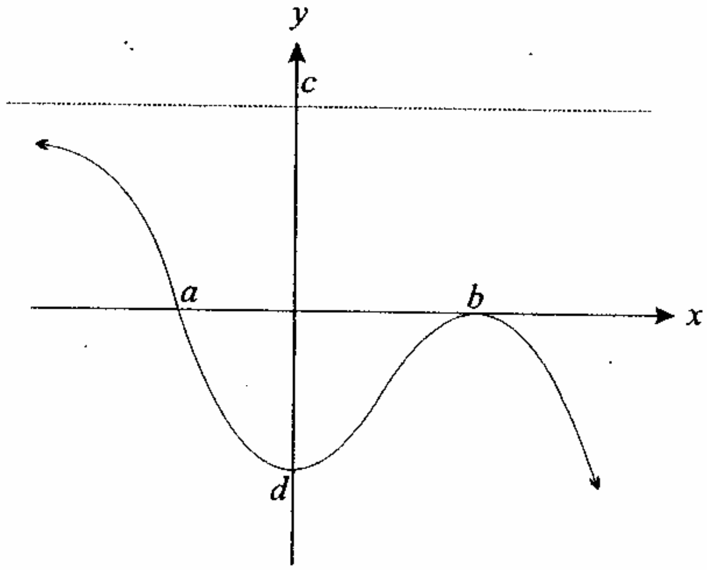
QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Expand $(\sqrt{3}+1)^2$. [1]

(ii) The polynomial equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ has roots α, β, γ and δ. Find the polynomial equation whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$ and $\delta + 1$. [3]

(iii) Hence, or otherwise, solve the equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$. [3]

(b)



The diagram above shows the graph of the function $y = f(x)$. Note that $c > |d| > 1$. On separate diagrams of roughly one-third of a page, sketch the graphs of:

(i) $y = \left(f(x)\right)^2$ [2]

(ii) $y = \dfrac{1}{f(x)}$ [2]

(c) (i) Sketch the graphs of $y = x^3$ and $y = e^{-x}$ on a number plane. [1]

(ii) Hence, on the same diagram as part (i), carefully sketch the graph of $y = x^3e^{-x}$ without any use of calculus. [3]

Exam continues next page ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

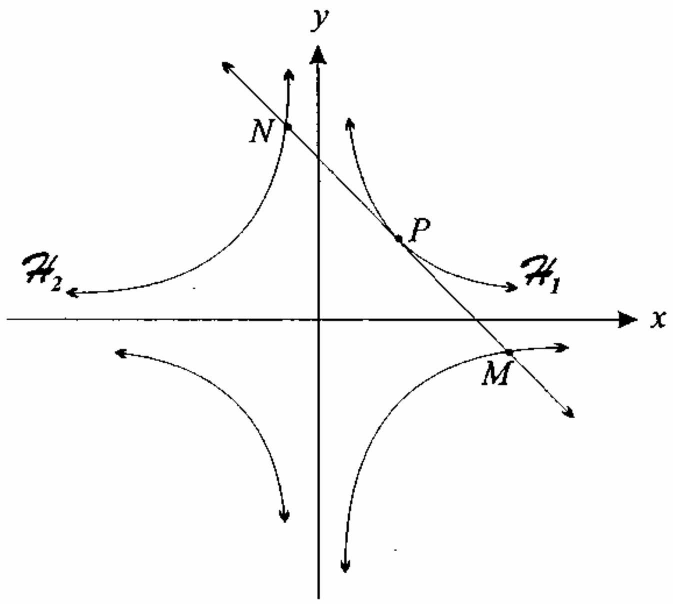
(a) The polynomial $P(x) = x^3 + ax + b$ has zeroes α, β and $2(\alpha - \beta)$.

(i) Show that $a = -13\alpha^2$. **2**

(ii) Show that $b = 12\alpha^3$. **1**

(iii) Deduce that the zeroes of $P(x)$ are $-\dfrac{13b}{12a}$, $-\dfrac{13b}{4a}$ and $\dfrac{13b}{3a}$. **2**

(b)



In the diagram above, $\mathcal{H}_1$ is the rectangular hyperbola $xy = c^2$, while $\mathcal{H}_2$ is the rectangular hyperbola $xy = -c^2$. The tangent to $\mathcal{H}_1$ at the variable point $P\left(ct, \dfrac{c}{t}\right)$ intersects $\mathcal{H}_2$ at M and N, as shown in the diagram. Let M and N be the points $\left(cp, -\dfrac{c}{p}\right)$ and $\left(cq, -\dfrac{c}{q}\right)$ respectively, and let T be the point of intersection of the tangents to $\mathcal{H}_2$ at M and N.

(i) Show that the tangent to $\mathcal{H}_1$ at P has equation $x + t^2y = 2ct$. **2**

(ii) Use the fact that M and N lie on the tangent at P to show that $p^2 + 6pq + q^2 = 0$. **3**

(iii) Find the equations of the tangents to $\mathcal{H}_2$ at M and N, and hence show that T has coordinates $\left(\dfrac{2cpq}{p+q}, \dfrac{-2c}{p+q}\right)$. **3**

(iv) Deduce that T lies on $\mathcal{H}_1$. **2**

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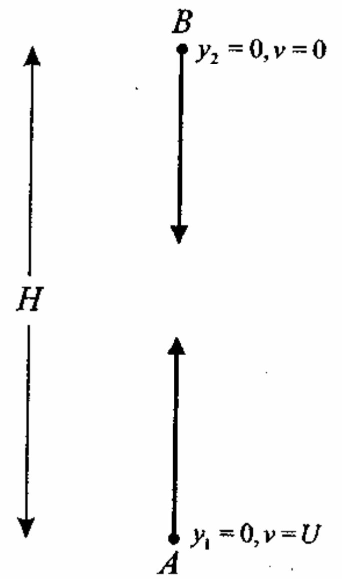
QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



A particle P_1 of mass m is projected vertically upwards from a point A with initial velocity U. At the same instant, a second particle P_2, also of mass m, is dropped from a point B directly above A. The distance H between A and B is equal to the maximum height that P_1 would reach were it not to collide with P_2. As the particles P_1 and P_2 move, they each experience air resistance of magnitude mkv^2, where k is a positive constant and v is velocity. At the instant the particles collide, P_2 has reached 50% of its terminal velocity V. Let y_1 be the distance of P_1 above A, and y_2 the distance of P_2 below B.

(i) Show that $V = \sqrt{\dfrac{g}{k}}$. **2**

(ii) Show that $y_1 = \dfrac{1}{2k} \ln \left(\dfrac{g + kU^2}{g + kv^2} \right)$, where v is the velocity of P_1. **3**

(iii) Hence show that $H = \dfrac{1}{2k} \ln \left(1 + \dfrac{U^2}{V^2} \right)$. **2**

(iv) Assuming that $y_2 = \dfrac{1}{2k} \ln \left| \dfrac{g}{g - kv^2} \right|$, show that at the instant the particles collide, $y_2 = \dfrac{1}{2k} \ln \dfrac{4}{3}$. **2**

(v) Deduce that the speed of P_1 at the instant the particles collide is $\dfrac{V}{\sqrt{3}}$. **2**

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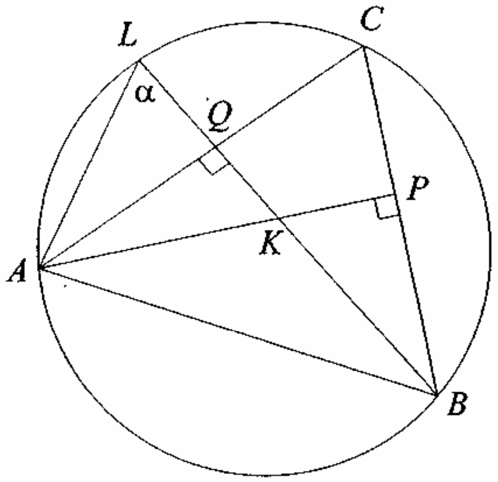
(b) $\boxed{4}$



The points A, B and C lie on a circle, as shown in the diagram above. The altitudes AP and BQ of $\triangle ABC$ intersect at K. The interval BQ produced meets the circle at L. Let $\angle ALQ = \alpha$.

Prove that $AK = AL$.

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ and that $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$. $\boxed{3}$

(ii) Hence prove that $\boxed{4}$

$$\cos^3\theta\sin^4\theta = \tfrac{1}{64}\left(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\right).$$

(b) (i) Use the substitution $u = \pi - x$ to show that, for any function $f(x)$, $\boxed{3}$

$$\int_0^{\pi} x\, f(\sin x)\, dx = \tfrac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$$

(ii) Hence show that $\boxed{5}$

$$\int_0^{\pi} \frac{x\sin^3 x}{1 + \cos^2 x}\, dx = \tfrac{\pi}{2}(\pi - 2).$$

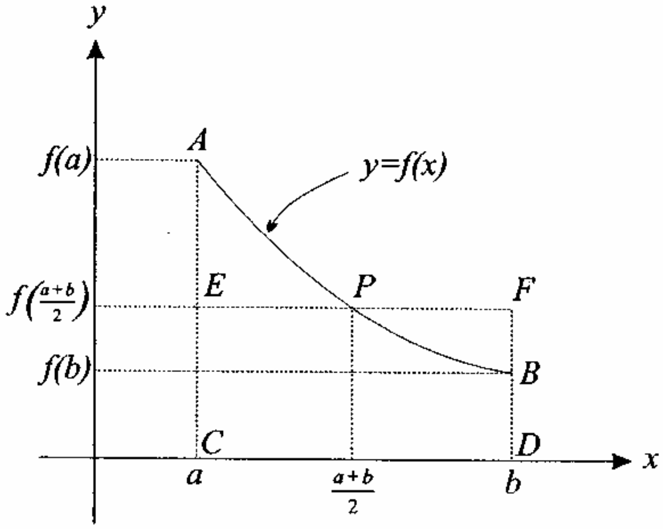
QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) The complex numbers ω_1 and ω_2 have modulus 1, and arguments α_1 and α_2 respectively, where $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$.

(i) Draw a diagram showing all the given information. **2**

(ii) Show that $\arg(\omega_1 - \omega_2) = \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$. **3**

(b)



The diagram above shows the curve $y = f(x)$ for $a \le x \le b$. Note that $f''(x)$ is positive for $a \le x \le b$.

(i) Copy the diagram, and then use areas to explain briefly why **3**

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}.$$

(ii) Use the result in part (i) with $f(x) = \dfrac{1}{x^2}$, $a = n-1$ and $b = n$, where n is an integer greater than 1, to show that **2**

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right).$$

(iii) Deduce that **2**

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

(iv) Show that **1**

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

(v) Hence show that $\dfrac{3}{2} < \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$. **2**

END OF EXAMINATION

Extension 2 Trial Solutions, 2007

TOTAL is 8×15

(1) (a) $\int_0^{\frac{\pi}{6}} x\cos x\,dx$

$= \left[x\sin x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x\,dx$ ✓

$= \frac{\pi}{6}\cdot\frac{1}{2} - 0 + \left[\cos x\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ ✓

Let $u = x$

$\therefore u' = 1$

Let $v' = \cos x$

$\therefore v = \sin x$ ✓

(b) $\int \frac{1}{2+\sqrt{x}}\,dx$

$= \int \frac{2u}{2+u}\,du$ ✓

$= 2\int \frac{(2+u)-2}{2+u}\,du$

$= 2\int 1\,du - 4\int \frac{1}{2+u}\,du$ ✓

$= 2u - 4\ln|2+u| + c$

$= 2\sqrt{x} - 4\ln|2+\sqrt{x}| + c$ ✓

Let $\sqrt{x} = u$

$\therefore x = u^2$

$\therefore dx = 2u\,du$

(c) $\int \tan^4 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$ ✓

$= \int \tan^2 x\sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + c$ ✓

(d)(i) Let $\frac{1}{(5x+3)(x+1)} = \frac{A}{5x+3} + \frac{B}{x+1}$

$\therefore \quad 1 = A(x+1) + B(5x+3)$

Let $x = -1$.

$\therefore \quad 1 = -2B$

$\therefore \quad B = -\frac{1}{2}$

Let $x = -\frac{3}{5}$.

$\therefore \quad 1 = \frac{2}{5}A$

$\therefore \quad A = \frac{5}{2}$ ✓

$$\therefore \int_0^1 \frac{1}{(5x+3)(x+1)}\,dx = \frac{1}{2}\int_0^1 \frac{5}{5x+3} - \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx \quad ✓$$

$$= \frac{1}{2}\Big[\ln|5x+3|\Big]_0^1 - \frac{1}{2}\Big[\ln|x+1|\Big]_0^1$$

$$\left.\begin{aligned} &= \frac{1}{2}\left[\ln\left|\frac{5x+3}{x+1}\right|\right]_0^1 \\ &= \frac{1}{2}(\ln 4 - \ln 3) \\ &= \frac{1}{2}\ln\frac{4}{3} \end{aligned}\right\} ✓$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{1}{4\sin x - \cos x + 4}\,dx$

Let $t = \tan\frac{x}{2}$
$\therefore \quad x = 2\tan^{-1}t$
$\therefore dx = \frac{2}{1+t^2}\,dt$ ✓

x	0	$\frac{\pi}{2}$
t	0	1

$$= \int_0^1 \frac{1}{4\cdot\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 4} \cdot \frac{2}{1+t^2}\,dt \quad ✓$$

$$= \int_0^1 \frac{2}{8t - 1 + t^2 + 4 + 4t^2}\,dt$$

$$= \int_0^1 \frac{2}{5t^2 + 8t + 3}\,dt \quad ✓$$

$$\left.\begin{aligned} &= 2\int_0^1 \frac{1}{(5t+3)(t+1)}\,dt \\ &= 2\cdot\frac{1}{2}\ln\frac{4}{3} \quad (\text{using (i)}) \\ &= \ln\frac{4}{3} \end{aligned}\right\} ✓$$

(2) (a) $z + \dfrac{1}{z} = \dfrac{2+i}{1-i} + \dfrac{1-i}{2+i}$

$$= \frac{(2+i)(1+i)}{2} + \frac{(1-i)(2-i)}{5}$$

$$= \frac{1+3i}{2} + \frac{1-3i}{5}$$

$$\left.\begin{aligned} &= \frac{5+15i+2-6i}{10} \\ &= \frac{7}{10} + \frac{9}{10}i \end{aligned}\right\}$$

(b) Let $(a+bi)^2 = 8i$.

$\therefore (a^2-b^2) + 2abi = 0 + 8i$

$\therefore a^2 - b^2 = 0$ and $ab = 4$

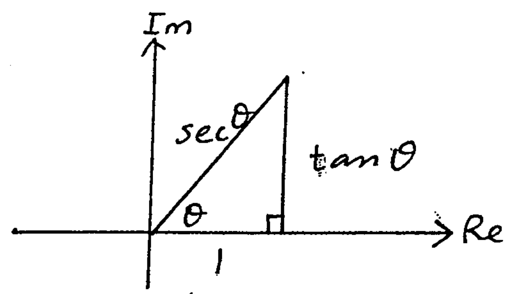
$\therefore (a,b) = (2,2)$ or $(-2,-2)$

So the square roots of $8i$ are $2+2i$ and $-2-2i$.

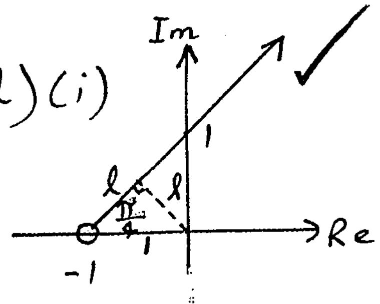
(c) (i) $|z| = \sqrt{1+\tan^2\theta}$

$= \sec\theta$

(ii) $\arg z = \theta$



(d) (i)



(ii) $l^2 + l^2 = 1$ (Pythagoras)

$\therefore l = \dfrac{1}{\sqrt{2}}$

So the least value of $|z|$ is $\dfrac{1}{\sqrt{2}}$.

(e) (i) $\vec{AB} = \vec{OB} - \vec{OA}$

$\therefore \vec{AB}$ represents $(4+13i)-(9+i)$

$= -5+12i$.

(ii) $\vec{AD}$ represents $(-5+12i)i$

$= -12-5i$.

$\vec{OD} = \vec{OA} + \vec{AD}$,

so $\vec{OD}$ represents $(9+i)+(-12-5i)$

$= -3-4i$,

so D represents $-3-4i$.

(3) (a) (i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ ①

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ ②

① + ② : $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ ③

Let $A+B = P$ and let $A-B = Q$.

$\therefore A = \frac{P+Q}{2}$ and $B = \frac{P-Q}{2}$

Substitute into ③ : $\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

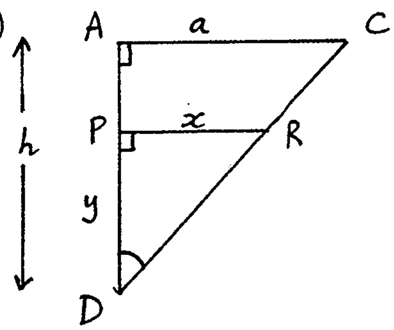
(ii) $\cos 7x + \cos 3x = 0$, $0 \le x \le \frac{\pi}{2}$

$\therefore 2\cos 5x \cos 2x = 0$, $0 \le 5x \le \frac{5\pi}{2}$ and $0 \le 2x \le \pi$

$\therefore 5x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $2x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}$ or $\frac{\pi}{4}$

(b) (i)



$\Delta PRD \parallel\!\!| \Delta ACD$ (equiangular)

$\therefore \frac{x}{a} = \frac{y}{h}$ (matching sides in equal ratios)

$\therefore x = \frac{ay}{h}$

(ii) $V = \int_{y=0}^{y=h} \frac{1}{2}x^2\,dy$

$= \int_0^h \frac{1}{2}\left(\frac{ay}{h}\right)^2 dy$

$= \frac{a^2}{2h^2}\int_0^h y^2\,dy$

$= \frac{a^2}{2h^2}\left[\frac{y^3}{3}\right]_0^h$

$= \frac{a^2}{2h^2}\cdot\frac{h^3}{3}$

$= \frac{1}{6}a^2h$

(c) $V = \pi \int_0^3 2\pi r h \, dx$, where $r = x$ and $h = y$

$= 2\pi \int_0^3 xy \, dx$ ✓

$= \frac{2\pi}{3} \int_0^3 (6x + 7x^2 - 3x^3) dx$ ✓

$= \frac{2\pi}{3} \left[3x^2 + \frac{7x^3}{3} - \frac{3x^4}{4} \right]_0^3$ ✓

$= \frac{2\pi}{3} \left(27 + 63 - \frac{243}{4} \right)$

$= \frac{39\pi}{2} \ u^3$ ✓

(4)(a)(i) $(\sqrt{3}+1)^2 = 4+2\sqrt{3}$ ✓

(ii) Replace x with $x-1$.

The required equation is

$$(x-1)^4 + 4(x-1)^3 - 2(x-1)^2 - 12(x-1) - 3 = 0 \checkmark$$

$$x^4 - 4x^3 + 6x^2 - 4x + 1 + 4x^3 - 12x^2 + 12x - 4 - 2x^2 + 4x - 2$$
$$- 12x + 12 - 3 = 0 \checkmark$$

$$x^4 - 8x^2 + 4 = 0 \checkmark$$

(iii) $$x^2 = \frac{8 \pm \sqrt{48}}{2}$$

$$= 4 \pm 2\sqrt{3} \checkmark$$

So the new equation has roots

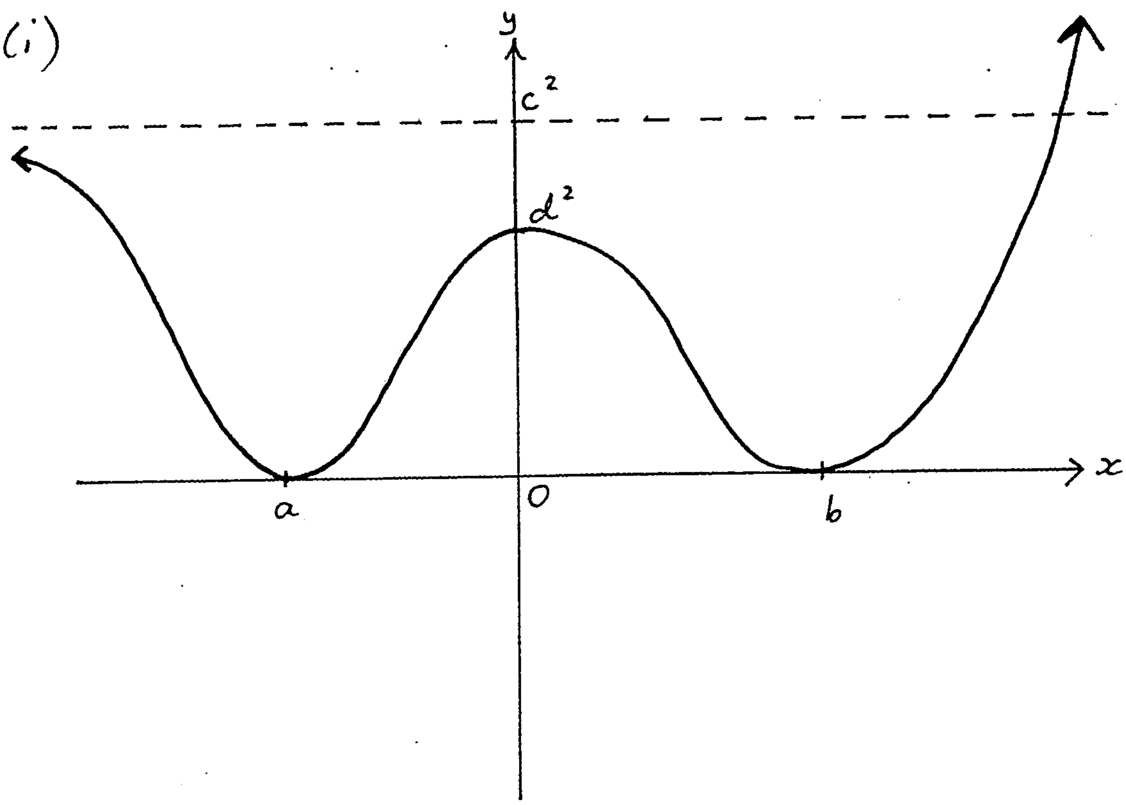
$x = \sqrt{3}+1, \ -\sqrt{3}-1, \ \sqrt{3}-1$ or $-\sqrt{3}+1$ ✓

(using part (i)).

So the original equation has roots

$x = \sqrt{3}, \ -\sqrt{3}-2, \ \sqrt{3}-2$ or $-\sqrt{3}$. ✓

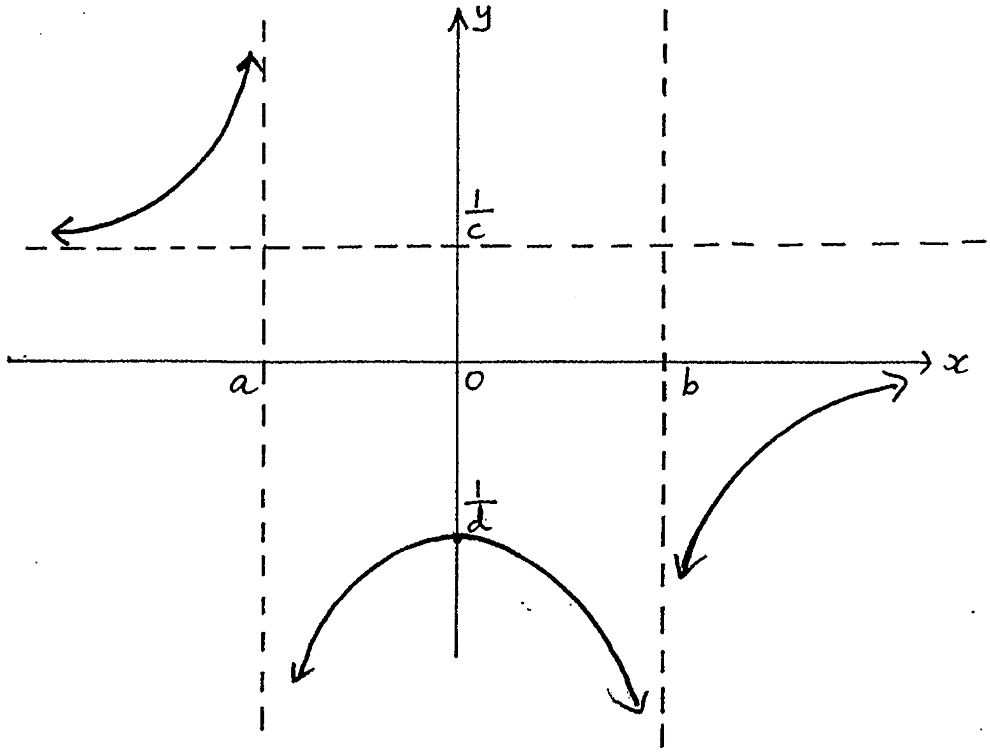
(b)(i)



✓✓

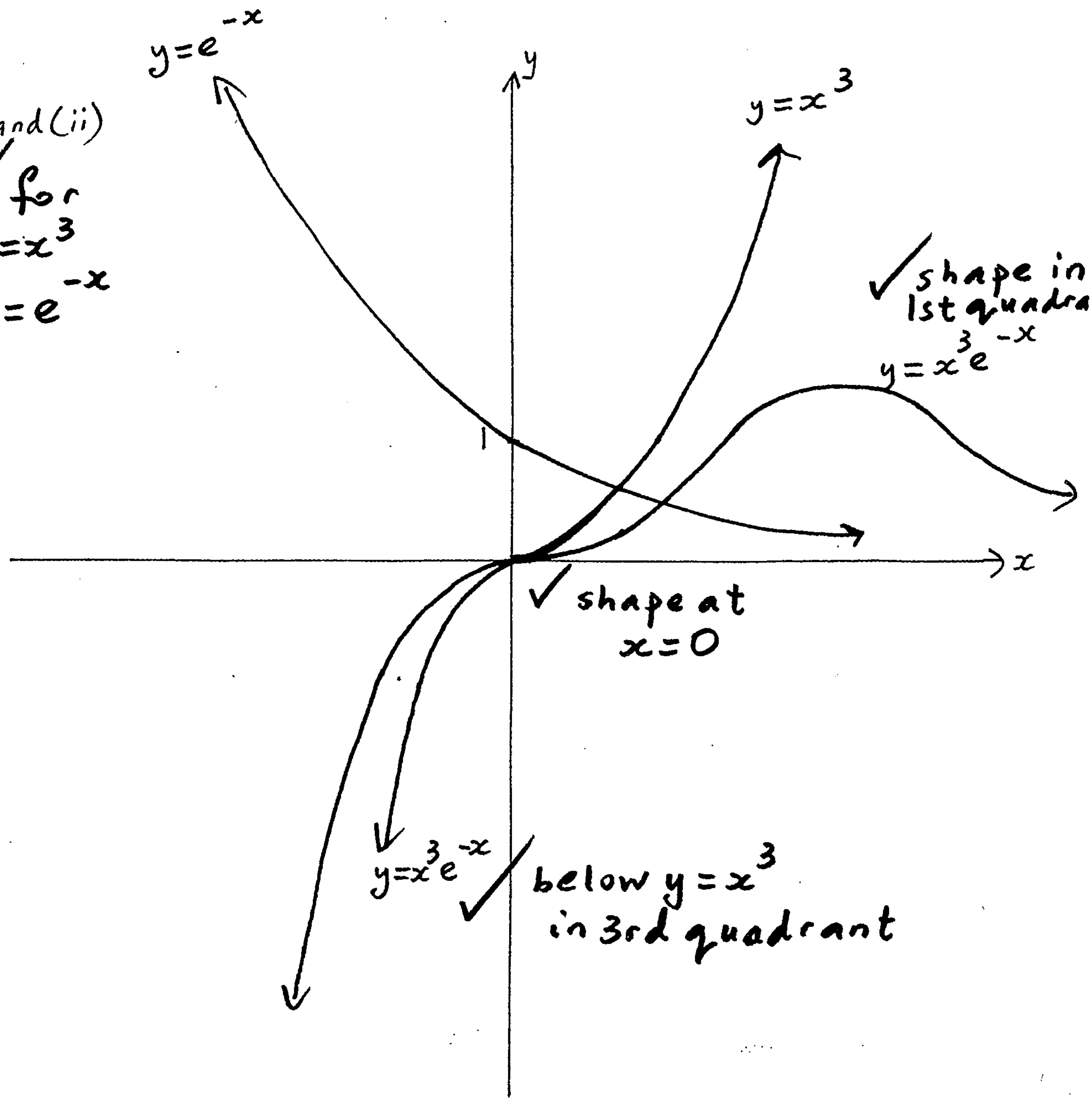
-1 per error

(ii)



√√

−1 per error



(c)(i) and (ii)

√ for $y=x^3$ and $y=e^{-x}$

√ shape in 1st quadrant

√ shape at $x=0$

√ below $y=x^3$ in 3rd quadrant

(5)(a)(i) Sum of zeros $= 0$

$\therefore \quad \alpha + \beta + 2\alpha - 2\beta = 0$

$\therefore \quad 3\alpha = \beta \quad (1)$ ✓

Sum of zeros multiplied in pairs $= a$

$\therefore \quad \alpha\beta + 2\alpha(\alpha - \beta) + 2\beta(\alpha - \beta) = a$

$\alpha\beta + 2\alpha^2 - \cancel{2\alpha\beta} + \cancel{2\alpha\beta} - 2\beta^2 = a$

Substitute $\beta = 3\alpha$ from (1):

$\therefore \quad 3\alpha^2 + 2\alpha^2 - 18\alpha^2 = a$

$\therefore \quad a = -13\alpha^2$ ✓

(ii) Product of zeros $= -b$

$\therefore \quad 2\alpha\beta(\alpha - \beta) = -b$

Substitute $\beta = 3\alpha$ from (1):

$6\alpha^2 . (-2\alpha) = -b$

$\therefore \quad b = 12\alpha^3$ ✓

(iii) Dividing the results in (ii) and (i),

$$\frac{b}{a} = -\frac{12\alpha}{13}$$

$$\therefore \quad \alpha = -\frac{13b}{12a}$$

Using (1), $\beta = 3 . \left(\frac{-13b}{12a}\right)$

$$= \frac{-13b}{4a}$$ ✓

Finally, $2(\alpha - \beta) = 2\left(-\frac{13b}{12a} + \frac{13b}{4a}\right)$

$$= 26b . \frac{-1 + 3}{12a}$$

$$= \frac{26b}{6a}$$

$$= \frac{13b}{3a}$$ ✓

So the zeros of $P(x)$ are $\frac{-13b}{12a}$, $\frac{-13b}{4a}$ and $\frac{13b}{3a}$.

(b) (i) $\mathcal{H}_1$ has equation $y = c^2x^{-1}$

$$\therefore \quad y' = -c^2x^{-2}$$
$$= \frac{-c^2}{x^2}$$

$\therefore$ at P, gradient is $\frac{-c^2}{c^2t^2} = \frac{-1}{t^2}$

$\therefore$ tangent at P has equation

$$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$$
$$t^2y - ct = -x + ct$$
$$x + t^2y = 2ct$$

(ii) $M\left(cp, \frac{-c}{p}\right)$ and $N\left(cq, \frac{-c}{q}\right)$ lie on the line $x + t^2y = 2ct$.

$\therefore \quad cp - \frac{ct^2}{p} = 2ct \quad$ and $\quad cq - \frac{ct^2}{q} = 2ct$

$\therefore \quad cp^2 - ct^2 = 2cpt$ (*) and $cq^2 - ct^2 = 2cqt$

Subtracting, we get $\quad c(p^2 - q^2) = 2ct(p - q)$

$$\therefore \quad t = \frac{1}{2}(p + q)$$

Substitute into (*):

$$cp^2 - c \cdot \frac{1}{4}(p+q)^2 = 2cp \cdot \frac{1}{2}(p+q)$$
$$\frac{1}{4}(p+q)^2 + p(p+q) - p^2 = 0$$
$$\frac{1}{4}(p^2 + 2pq + q^2) + pq = 0$$
$$p^2 + 2pq + q^2 + 4pq = 0$$
$$p^2 + 6pq + q^2 = 0$$

(iii) $\mathcal{H}_2$ has equation $\quad y = -c^2x^{-1}$

$$\therefore y' = +c^2x^{-2}$$
$$= \frac{c^2}{x^2}$$

$\therefore$ at M, gradient is $\frac{c^2}{c^2p^2} = \frac{1}{p^2}$

$\therefore$ tangent at M has equation

$$y + \frac{c}{p} = \frac{1}{p^2}(x - cp)$$
$$p^2y + cp = x - cp$$
$$x - p^2y = 2cp \quad (1)$$

Similarly, the tangent at N has equation

$$x - q^2 y = 2cq \quad (2)$$

$(2) - (1)$: $(p^2 - q^2)y = -2c(p-q)$

$$\therefore y = \frac{-2c}{p+q}$$

Substitute into (1):

$$x = 2cp + p^2 \cdot \frac{-2c}{p+q}$$

$$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$$

$$= \frac{2cpq}{p+q}$$

$\therefore T$ is the point $\left(\frac{2cpq}{p+q}, \frac{-2c}{p+q}\right)$

(iv) T lies on $\mathcal{H}_1 : xy = c^2$
if its coordinates satisfy $xy = c^2$.

That is, if $\dfrac{-4c^2pq}{(p+q)^2} = c^2$

$$-4pq = (p+q)^2$$

$$p^2 + 6pq + q^2 = 0$$

We know from part (ii) that this condition is satisfied, so T lies on $\mathcal{H}_1$.

(6) (a) (i) For P_2 :

Forces acting :

$mkv^2 =$ air resistance $\uparrow$

$\downarrow$ $mg =$ gravity

$$\left.\begin{aligned} \therefore m\ddot{y}_2 &= mg - mkv^2 \\ \ddot{y}_2 &= g - kv^2 \end{aligned}\right\} \text{(taking downwards as positive)}$$

$$\left.\begin{aligned} &\text{When } \ddot{y}_2 = 0,\ v = V \\ &\therefore 0 = g - kV^2 \\ &\therefore V = \sqrt{\frac{g}{k}} \quad (V > 0) \end{aligned}\right\}$$

(ii) For P_1 :

Forces acting :

$\downarrow mg$ $\quad \downarrow mkv^2$

$$\therefore m\ddot{y}_1 = -mg - mkv^2 \text{ (taking upwards as positive)}$$

$$\ddot{y}_1 = -g - kv^2$$

$$\therefore v \cdot \frac{dv}{dy_1} = -g - kv^2$$

$$\frac{dy_1}{dv} = \frac{-v}{g + kv^2}$$

$$\therefore y_1 = -\frac{1}{2k}\int \frac{2kv}{g + kv^2}\,dv$$

$$= -\frac{1}{2k}\ln\left|g + kv^2\right| + c$$

$$\left.\begin{aligned} &\text{When } t = 0,\ y_1 = 0 \text{ and } v = U. \\ &\therefore c = \frac{1}{2k}\ln\left|g + kU^2\right| \\ &\therefore y_1 = \frac{1}{2k}\left(\ln\left|g + kU^2\right| - \ln\left|g + kv^2\right|\right) \\ &\therefore y_1 = \frac{1}{2k}\ln\left|\frac{g + kU^2}{g + kv^2}\right| \end{aligned}\right\}$$

(Or $\frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv^2}\right)$, since numerator and denominator are both positive.)

(iii) When $v = 0$, $y_1 = H$ (if no collision occurs).

$$\therefore \quad H = \frac{1}{2k} \ln\left(\frac{g + kU^2}{g}\right)$$

$$= \frac{1}{2k} \ln\left(1 + \frac{kU^2}{g}\right)$$

$$= \frac{1}{2k} \ln\left(1 + \frac{U^2}{g/k}\right)$$

$$= \frac{1}{2k} \ln\left(1 + \frac{U^2}{V^2}\right)$$

(iv) At the instant the particles collide, $y_1 + y_2 = H$, and the speed of P_2 is 50% of $V = \frac{V}{2}$.

So $$y_2 = \frac{1}{2k} \ln\left|\frac{g}{g - k.\frac{V^2}{4}}\right|$$

$$= \frac{1}{2k} \ln\left|\frac{4g}{4g - kV^2}\right|$$

$$= \frac{1}{2k} \ln\left|\frac{4g}{4g - k.\frac{g}{k}}\right| \quad \text{(using (i))}$$

$$= \frac{1}{2k} \ln \frac{4}{3}$$

(v) Find v (the speed of P_1) when $y_1 + y_2 = H$.

$$\frac{1}{2k} \ln\left(\frac{g + kU^2}{g + kv^2}\right) + \frac{1}{2k} \ln\frac{4}{3} = \frac{1}{2k} \ln\left(1 + \frac{U^2}{V^2}\right)$$

(using (iii))

$$\therefore \quad \frac{4}{3} \cdot \frac{g/k + U^2}{g/k + v^2} = 1 + \frac{U^2}{V^2}$$

$$\frac{4}{3} \cdot \frac{V^2 + U^2}{V^2 + v^2} = \frac{V^2 + U^2}{V^2}$$

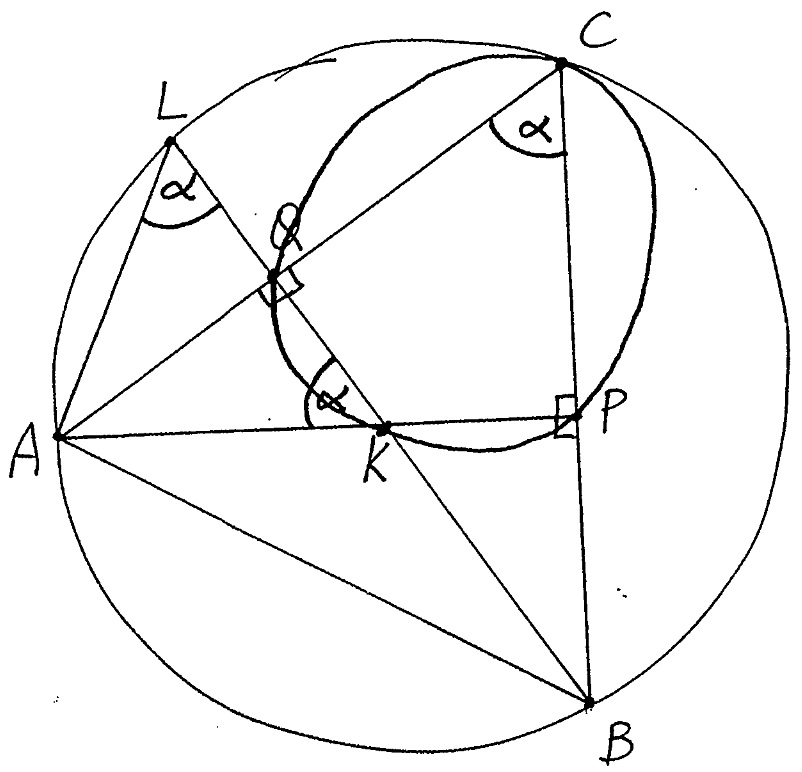
$$\frac{4}{3V^2 + 3v^2} = \frac{1}{V^2}$$

$$3v^2 = V^2$$

$$\therefore \quad v^2 = \frac{V^2}{3}$$

$$\therefore \quad v = \frac{V}{\sqrt{3}} \quad (v > 0)$$

(b)



✓ $\angle ACB = \angle ALB = \alpha$ (angles at the circumference standing on arc AB).

✓ But quadrilateral QCPK is cyclic (opposite angles CQK and CPK are supplementary).

✓ $\therefore \angle AKQ = \alpha$ (exterior angle of cyclic quadrilateral QCPK).

✓ $\therefore \Delta AKL$ is isosceles (since two of its angles ALK and AKL are equal).

$\therefore AK = AL$ (sides opposite equal angles).

(Don't be too strict with the reasons.)

(7) (a) (i) $z + \frac{1}{z} = \cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^{-1}$

$= \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$

(de Moivre's theorem)

$$\left.\begin{aligned} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ &= 2\cos\theta \end{aligned}\right\}$$

$\therefore \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Similarly, $z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

$\therefore \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

(ii) $\cos^3\theta \sin^4\theta = \frac{1}{8}\left(z + \frac{1}{z}\right)^3 \cdot \frac{1}{16}\left(z - \frac{1}{z}\right)^4$

$= \frac{1}{128}\left(z^2 - \frac{1}{z^2}\right)^3 \left(z - \frac{1}{z}\right)$

$= \frac{1}{128}\left(z - \frac{1}{z}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$

$= \frac{1}{128}\left(z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} - z^5 + 3z - \frac{3}{z^3} + \frac{1}{z^7}\right)$

$= \frac{1}{128}\left(\left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)\right)$

$= \frac{1}{128}\left(2\cos 7\theta - 2\cos 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)\right)$

$= \frac{1}{64}\left(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\right)$

(b)(i) $\int_0^\pi x \cdot f(\sin x)\,dx$

$= -\int_\pi^0 (\pi-u)\cdot f(\sin(\pi-u))\,du$ ✓

✓ $\begin{cases} = \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u \cdot f(\sin u)\,du \\ = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x \cdot f(\sin x)\,dx \end{cases}$

Let $u = \pi - x$

$\therefore dx = -du$

x	0	π
u	π	0

$\left.\begin{aligned} &\therefore 2\int_0^\pi x \cdot f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx \\ &\therefore \int_0^\pi x \cdot f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx \end{aligned}\right\}$ ✓

(ii) $\int_0^\pi \frac{x\sin^3 x}{1+\cos^2 x}\,dx$

$= \int_0^\pi x \cdot \frac{\sin^3 x}{2-\sin^2 x}\,dx$

$= \int_0^\pi x \cdot f(\sin x)\,dx$, where $f(\sin x) = \frac{\sin^3 x}{2-\sin^2 x}$

$= \frac{\pi}{2}\int_0^\pi \frac{\sin^3 x}{2-\sin^2 x}\,dx$ (using (i)) ✓

$= \frac{\pi}{2}\int_0^\pi \frac{1-\cos^2 x}{1+\cos^2 x}\cdot \sin x\,dx$ ✓

Let $u = \cos x$

$\therefore \sin x\,dx = -du$

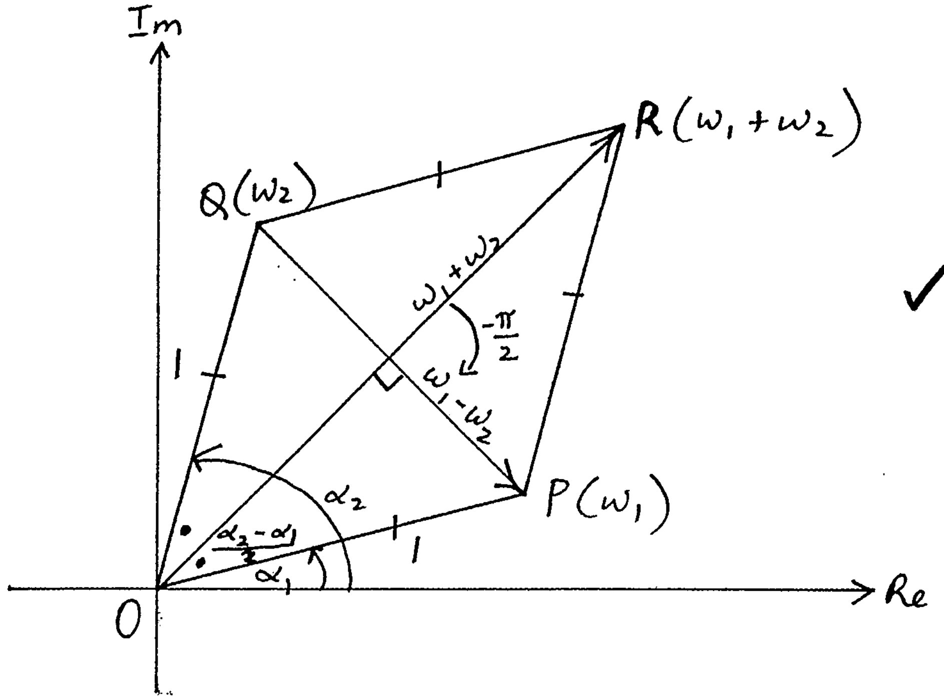
x	0	π
u	1	-1

$= -\frac{\pi}{2}\int_1^{-1} \frac{1-u^2}{1+u^2}\,du$ ✓

$= \frac{\pi}{2}\int_{-1}^{+1} \frac{-(1+u^2)+2}{1+u^2}\,du$ ✓

$\left.\begin{aligned} &= \frac{\pi}{2}\left[-u + 2\tan^{-1}u\right]_{-1}^{1} \\ &= \frac{\pi}{2}\left[-1+\frac{\pi}{2} - \left(1-\frac{\pi}{2}\right)\right] \\ &= \frac{\pi}{2}(\pi-2) \end{aligned}\right\}$ ✓

(8)(a)(i)



(ii) OPRQ is a rhombus, since $|w_1| = |w_2|$.

$\therefore \angle QOR = \angle POR = \frac{\alpha_2 - \alpha_1}{2}$ (diagonal OR of rhombus bisects $\angle QOP$)

$\therefore \arg(w_1+w_2) = \alpha_1 + \angle POR$

$= \alpha_1 + \frac{\alpha_2-\alpha_1}{2}$

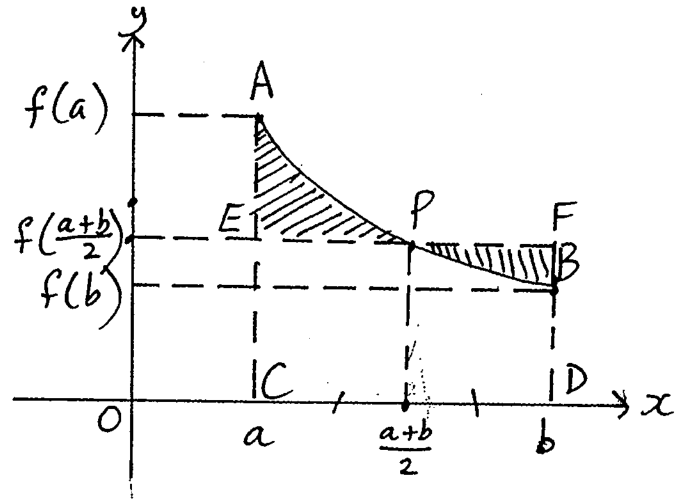
$= \frac{\alpha_1+\alpha_2}{2}$

$\therefore \arg(w_1 - w_2) = \arg(w_1+w_2) - \frac{\pi}{2}$ (the diagonals OR and QP are perpendicular)

$= \frac{\alpha_1+\alpha_2}{2} - \frac{\pi}{2}$

$= \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$

(b)



(i) Exact area between curve and x-axis < area of trapezium ACDB

$$\therefore \int_a^b f(x)\,dx < \frac{b-a}{2}\left(f(a)+f(b)\right)$$

Also, area of portion PFB < area of portion PEA, since the arc AP is steeper than the arc PB.

$\therefore$ area of rectangle EFDC < exact area between curve and x-axis

$$\therefore (b-a).\, f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx$$

(ii) Let $f(x) = \frac{1}{x^2}$, $a = n-1$, $b = n$ in (i),

so that $\frac{a+b}{2} = \frac{2n-1}{2}$.

$$\therefore \frac{4}{(2n-1)^2} < \int_{n-1}^{n} x^{-2}\,dx < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$$

$$\therefore \frac{4}{(2n-1)^2} < \left[-\frac{1}{x}\right]_{n-1}^{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$$

$$\therefore \frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$$

(iii) Put $n = 2, 3, 4, \dots$ into the result in (ii) and add: ✓(for the idea)

$$\frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{7^2} + \dots < \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$$

$$< \frac{1}{2}\left(\left(\frac{1}{1^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \dots\right) \checkmark$$

$$\therefore\ 4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)$$

(iv) $\text{LHS} = \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\right)$

$$< \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \dots\right) \checkmark$$

$$= \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$= \text{RHS}$$

(v) ✓ From (iv), $2\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\right) < 4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right)$

So using (iii),

$$2\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)$$

$$\therefore\ \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots < \frac{1}{2} \text{ and } \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots > \frac{1}{2}$$

$$\therefore\ 1 + \frac{1}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2}$$

$$\text{i.e. } \frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4} \checkmark$$