SYDNEY GRAMMAR SCHOOL

2013 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Thursday 1st August 2013

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks
- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 68 boys

Examiner
DS/REP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which function is a primitive of $\dfrac{2x}{2x-1}$?

(A) $x + \ln(2x-1)$

(B) $\ln(2x-1)$

(C) $x + \frac{1}{2}\ln(2x-1)$

(D) $\frac{1}{2}\ln(2x-1)$

QUESTION TWO

Which expression is a correct factorisation of $z^3 - i$?

(A) $(z-i)(z^2+iz+1)$

(B) $(z+i)(z^2-iz-1)$

(C) $(z+i)(z-i)^2$

(D) $(z+i)^3$

QUESTION THREE

If $f(x)$ is an odd function, which statement is true?

(A) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(B) $\displaystyle\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(C) $\displaystyle\int_{-2a}^{a} f(x)\,dx = \int_{-a}^{2a} f(x)\,dx$

(D) $\displaystyle\int_{-a}^{2a} f(x)\,dx = \int_{a}^{2a} f(x)\,dx$

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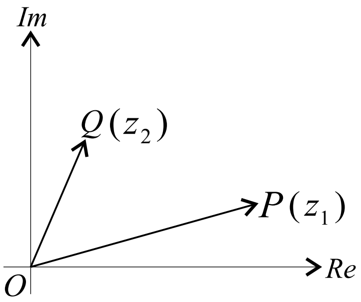
QUESTION FOUR



The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram above. Which statement about the complex number $z_2 - z_1$ is true?

(A) It is represented by the vector QP.

(B) Its principal argument lies between $\frac{\pi}{2}$ and π.

(C) Its real part is positive.

(D) Its modulus is greater than $|z_1 + z_2|$.

QUESTION FIVE

An ellipse has foci at $(-6, 0)$ and $(6, 0)$, and its directrices have equations $x = -8$ and $x = 8$. What is the eccentricity of the ellipse?

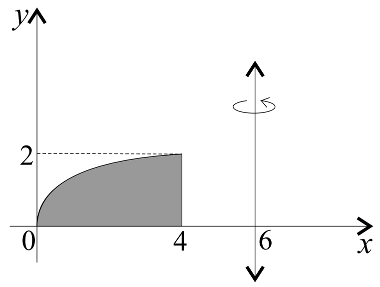
(A) $\frac{1}{2}\sqrt{3}$ (B) $\frac{1}{3}\sqrt{3}$ (C) $\frac{2}{3}\sqrt{3}$ (D) $3\sqrt{3}$

QUESTION SIX

The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero. What is the value of the double zero?

(A) -7 (B) -4 (C) 4 (D) 2

QUESTION SEVEN



The diagram above shows the region bounded by the curve $y = \sqrt{x}$ and the x-axis, from $x = 0$ to $x = 4$. The region is rotated about the line $x = 6$ to form a solid of revolution. Which integral gives the volume of the solid?

(A) $\displaystyle\int_0^2 \pi(4 - y^2)(8 - y^2)\, dy$

(B) $\displaystyle\int_0^2 4\pi(5 - y^2)\, dy$

(C) $\displaystyle\int_0^2 \pi(4 - y^2)(6 - y^2)\, dy$

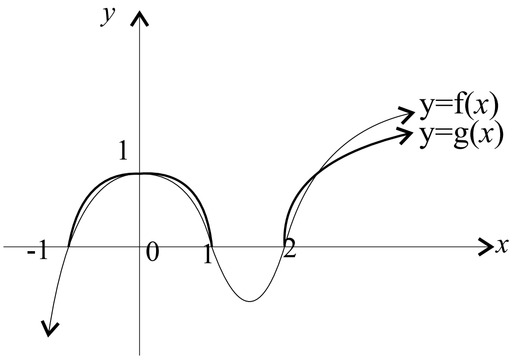
(D) $\displaystyle\int_0^2 \pi(2 - y^2)(6 - y^2)\, dy$

QUESTION EIGHT

The curve defined by the equation $x^2 - xy + 2y^2 = 4$ passes through the point $P(1, -1)$. What is the gradient of the tangent to the curve at P?

(A) $\frac{1}{2}$ (B) $-\frac{1}{5}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$

QUESTION NINE



In the diagram above, the relationship between the functions $f(x)$ and $g(x)$ could be represented by:

(A) $g(x) = \big(f(x)\big)^2$

(B) $g(x) = \log_e f(x)$

(C) $g(x) = \sqrt{f(x)}$

(D) $g(x) = |f(x)|$

QUESTION TEN

Without attempting to evaluate the integrals, determine which of the following inequalities is FALSE:

(A) $\displaystyle\int_1^2 \frac{1}{1+x}\,dx < \int_1^2 \frac{1}{x}\,dx$

(B) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x}\,dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

(D) $\displaystyle\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the quadratic equation $4z^2 + 4z + 5 = 0$. **2**

(b) Find the real values of x and y for which $\dfrac{x}{i} - \dfrac{y}{1+i} = -1 - 3i$. **2**

(c) Find $\displaystyle\int \frac{3x}{2x^2 - 5x + 2}\, dx$. **3**

(d) Find $\displaystyle\int \frac{1}{\sqrt{x^2 + 6x + 34}}\, dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4 x\, dx$. **3**

(f) (i) Use the substitution $u = a - x$ to prove that $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **1**

(ii) Hence find the value of $\displaystyle\int_0^1 x(1-x)^7\, dx$. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Shade the region in the Argand diagram where $|z + 3i| > 2|z|$. **3**

(b) (i) Express $-2 + 2i$ in modulus-argument form. **1**

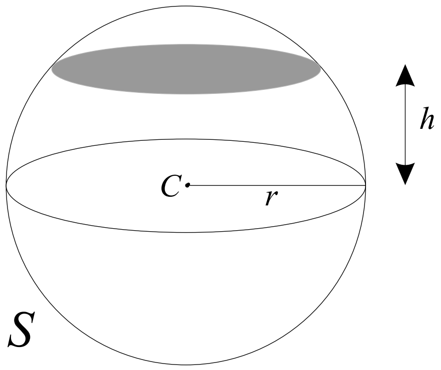
(ii) Simplify $(-2 + 2i)^{8k}$, where k is an integer. **2**

(c) A complex number z satisfies $\arg(z - 1) = \frac{\pi}{6}$.

(i) Sketch the locus of z. **1**

(ii) Show that $|z - 5| \geq 2$. **1**

(d)



In the diagram above, $\mathcal{S}$ is a sphere of radius r. The point C is the centre of the sphere. A typical horizontal cross-section h units above C is shown.

(i) Find the area of this cross-section as a function of h. **1**

(ii) Hence prove that the volume of $\mathcal{S}$ is $\frac{4}{3}\pi r^3$. **2**

(e) The polynomial $P(x) = x^4 - 4x^3 + 10x^2 - 12x - 40$ has zeroes α, β, γ and δ.

(i) Find a polynomial with zeroes $\alpha - 1$, $\beta - 1$, $\gamma - 1$ and $\delta - 1$. **2**

(ii) Hence find the zeroes of $P(x)$. **2**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down the equation of a line with gradient m that passes through the point $T(-4c, 2c)$. **1**

(ii) Solve the equation in part (i) simultaneously with the equation $xy = c^2$ to obtain a quadratic equation in x. **2**

(iii) Hence find the gradients of the two tangents to the rectangular hyperbola $xy = c^2$ that pass through the point $T(-4c, 2c)$. **2**

(b) Consider the polynomial $P(z) = z^4 + (1 - 2i)z^2 - 2i$.

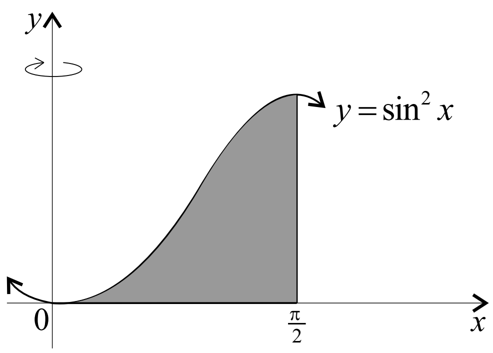
(i) Show that $P(i) = 0$. **1**

(ii) Explain why $P(-i)$ must also be zero. **1**

(iii) Suppose that the other two zeroes of $P(z)$ are w and $-w$.
Use the product of the zeroes to find w. **3**

(c) (i) Find $\displaystyle\int x \cos 2x \, dx$. **2**

(ii) **3**



The diagram above shows the region bounded by the curve $y = \sin^2 x$, the x-axis and the line $x = \frac{\pi}{2}$. The region is rotated about the y-axis through $360°$. Use the method of cylindrical shells to find the exact volume of the solid formed.

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) An object of mass 2 kg is projected vertically upwards at 20 m/s and experiences air resistance of magnitude $\frac{1}{10}v^2$ Newtons, where v is the speed of the object after t seconds. Take $g = 10\,\text{m/s}^2$.

(i) Show that the maximum height reached by the object is $10\ln 3$ metres. **3**

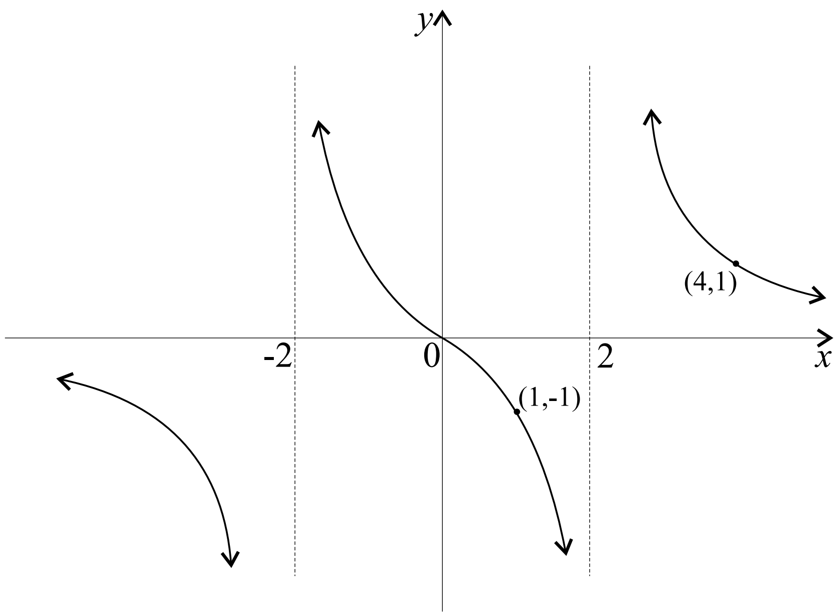
(ii) Find the speed of the object, correct to three significant figures, at the instant it reaches half its maximum height. **2**

(b) Let $I_n = \displaystyle\int_1^e (1 + \ln x)^n \, dx$, where $n \geq 0$.

(i) Use integration by parts to show that $I_n = (2^n)\, e - 1 - n\, I_{n-1}$. **2**

(ii) Hence find the exact value of $\displaystyle\int_1^e (2 + \ln x)(1 + \ln x)^2 \, dx$. **2**

(c)



The diagram above shows the graph of the odd function $y = f(x)$, where

$$f(x) = \frac{3x}{x^2 - 4}.$$

Sketch the graphs of each of the following functions on large separate diagrams, showing the x-intercepts and asymptotes. You are NOT expected to find any stationary points.

(i) $y = \dfrac{1}{f(x)}$ **2**

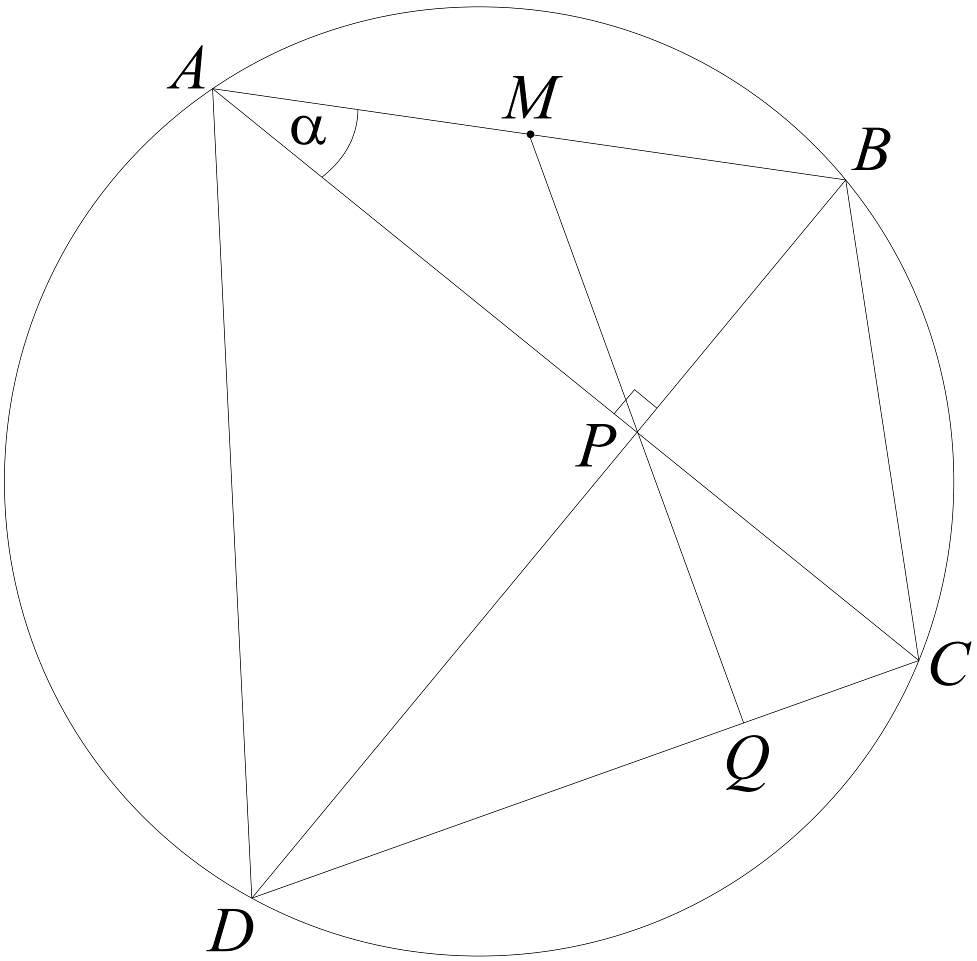
(ii) $y = \log_e f(x)$ **2**

(iii) $y = x + f(x)$ **2**

Exam continues overleaf ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



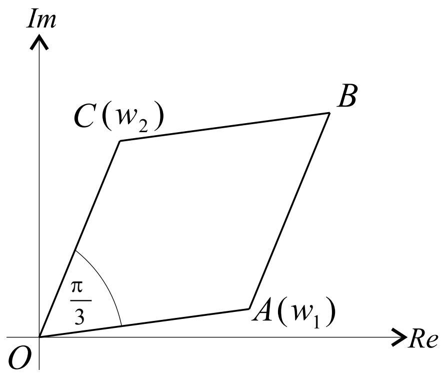
In the diagram above $ABCD$ is a cyclic quadrilateral whose diagonals are perpendicular and intersect at P. Let M be the midpoint of AB, and suppose that MP produced meets DC at Q. Let $\angle PAM = \alpha$.

(i) Explain why $AM = PM$. **1**

(ii) Prove that $MQ \perp DC$. **3**

QUESTION FIFTEEN (Continued)

(b)



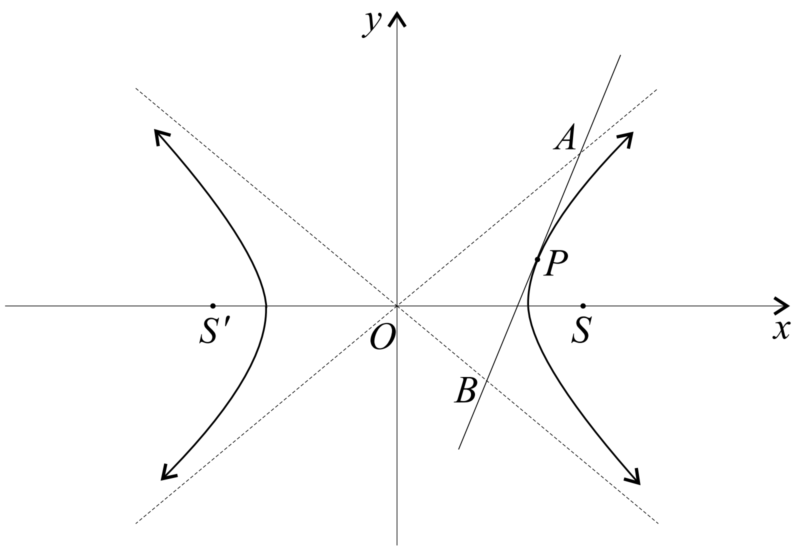
In the Argand diagram above, $OABC$ is a rhombus with $\angle COA = \frac{\pi}{3}$. The points A and C represent the complex numbers w_1 and w_2 respectively.

(i) Explain why $w_2 = w_1 \operatorname{cis} \frac{\pi}{3}$. **1**

(ii) Write down, in terms of w_1 only, the complex numbers represented by the vectors OB and AC. **2**

(iii) By considering $i(w_1 + w_2)$, show that the diagonals OB and AC of the rhombus are perpendicular. **2**

(c)



The diagram above shows the tangent at a point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meeting the asymptotes of the hyperbola at A and B. The points S and S' are the foci of the hyperbola.

(i) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$. **1**

(ii) Show that $OA \times OB = a^2 + b^2$, where O is the origin. **2**

(iii) Extend AO to A' so that $OA' = OB$ and extend BO to B' so that $OB' = OA$. Explain why the points A, B, A' and B' are concyclic. **1**

(iv) Hence show that the points A, S, B and S' are concyclic. **2**

Exam continues overleaf ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence use the identity $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ to show that **2**

$$\left(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta.$$

(iii) Use part (ii) and the identity $\cos 3A = 4\cos^3 A - 3\cos A$ to deduce that **1**

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}.$$

(iv) Hence show that $\cos\frac{2\pi}{7}$ is a root of the equation $8x^3 + 4x^2 - 4x - 1 = 0$. **1**

(b) (i) Use a suitable double angle formula to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. **1**

(ii) Find $\cos 4\theta$ in terms of $\cos\theta$. **1**

(iii) Let $I = \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\, dx$.

(α) Show, by using the substitution $x = \sin 4\theta$, that **2**

$$I = \int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{\cos^2\theta}\, d\theta.$$

(β) Hence find the exact value of I. **2**

(c) (i) Explain why **1**

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}.$$

(ii) Prove by mathematical induction, or otherwise, that for all $n \geq 2$, **3**

$$n\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\,\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\,,\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\,,\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE: } \ln x = \log_e x,\ x > 0$$

SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER: ....

2013
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FORM VI
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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Extension 2 TRIAL HSC 2013

Multiple Choice

(1) $\int \frac{2x}{2x-1}\,dx = \int \left(1 + \frac{1}{2x-1}\right) dx$

$= x + \frac{1}{2}\ln(2x-1) + c$

C

(2) $z^3 - i = z^3 + i^3$

$= (z+i)(z^2 - iz - 1)$

B

(3) $\int_{-a}^{2a} f(x)\,dx = \int_{-a}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx$

$= 0 + \int_{a}^{2a} f(x)\,dx$

D

(4) $\frac{\pi}{2} < \arg(z_2 - z_1) < \pi$

B

(5) $ae = 6$ and $\frac{a}{e} = 8$

So $8e^2 = 6$

$e = \frac{\sqrt{3}}{2}$

A

(6) $P'(x) = 3(x^2 + 2x - 8)$

$= 3(x+4)(x-2)$

$P(-4) \neq 0$

$P(2) = 0$

D

(7) $A(y) = \pi(R-r)(R+r)$, where $\begin{cases} R = 6-x \\ r = 2 \end{cases}$

$= \pi(4-x)(8-x)$

$= \pi(4-y^2)(8-y^2)$

A

(8) $2x - y - x\cdot\frac{dy}{dx} + 4y\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{y-2x}{4y-x}$, $m = \frac{-1-2}{-4-1}$

C

(9) C

(10) C

Written Response

(11)(a) $4z^2+4z+5=0$, $\Delta = 16-80$
$= -64$

$$z = \frac{-4 \pm 8i}{8}$$
$$= -\frac{1}{2} \pm i$$

(b) $$\frac{x}{i} - \frac{y}{1+i} = -1-3i$$
$$-ix - \frac{y(1-i)}{2} = -1-3i$$
$$-y + i(y-2x) = -2-6i$$
$$y=2 \text{ and } x=4$$

(c) $$\int \frac{3x}{2x^2-5x+2}\,dx = \int \left(\frac{2}{x-2} - \frac{1}{2x-1}\right)dx$$
$$= 2\ln|x-2| - \frac{1}{2}\ln|2x-1| + c$$

(d) $$\int \frac{1}{\sqrt{x^2+6x+34}}\,dx = \int \frac{1}{\sqrt{(x+3)^2+25}}\,dx$$
$$= \ln\left|(x+3) + \sqrt{x^2+6x+34}\right| + c$$

(e) $$\int_0^{\frac{\pi}{3}} \tan^4 x\,dx = \int_0^{\frac{\pi}{3}} \tan^2 x(\sec^2 x - 1)\,dx$$
$$= \int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x\,dx - \int_0^{\frac{\pi}{3}} (\sec^2 x - 1)\,dx$$
$$= \left[\frac{1}{3}\tan^3 x\right]_0^{\frac{\pi}{3}} - \left[\tan x - x\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{3}\cdot 3\sqrt{3} - \left(\sqrt{3} - \frac{\pi}{3}\right)$$
$$= \frac{\pi}{3}$$

(f)(i) $$\int_0^a f(x)\,dx$$
$$= \int_a^0 f(a-u)\cdot -du$$
$$= \int_0^a f(a-u)\,du$$
$$= \int_0^a f(a-x)\,dx$$
(dummy variable)

$u = a-x$

So $dx = -du$.

x	0	a
u	a	0

(ii) $$\int_0^1 x(1-x)^7\,dx$$
$$= \int_0^1 (1-x)x^7\,dx \quad \text{(using part (i))}$$
$$= \int_0^1 (x^7 - x^8)\,dx$$
$$= \left[\frac{x^8}{8} - \frac{x^9}{9}\right]_0^1$$
$$= \frac{1}{8} - \frac{1}{9}$$
$$= \frac{1}{72}$$

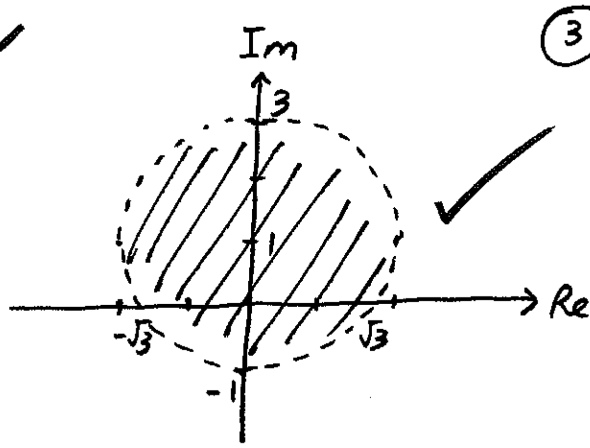
(12) (a) Let $z = x + iy$.

Then $x^2 + (y+3)^2 > 4(x^2+y^2)$

$6y + 9 > 3x^2 + 3y^2$

$x^2 + y^2 - 2y < 3$

$x^2 + (y-1)^2 < 4$



(b)(i) $-2 + 2i = 2\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$

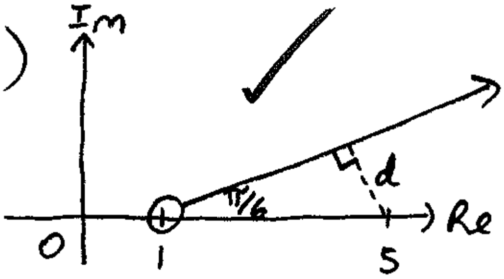
(ii) $(-2+2i)^{8k} = (2\sqrt{2})^{8k}\,\text{cis}\left(\frac{3\pi}{4}\cdot 8k\right)$

$= 2^{8k}\cdot 2^{4k}\,\text{cis}(6\pi k)$

$= 2^{12k}\,\text{cis}\,6\pi k$

$= 2^{12k}$, $k \in \mathbb{Z}$

(c)(i)



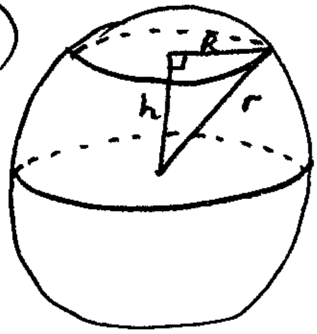
(ii) Let d be the minimum value of $|z-5|$. (shown on diagram)

$d = 4\sin\frac{\pi}{6}$

$= 2$

So $|z-5| \geq 2$.

(d)



(i) $A(h) = \pi R^2$

$= \pi(r^2 - h^2)$

(ii) $V = 2\int_0^r \pi(r^2 - h^2)\,dh$

$= 2\pi\left[r^2 h - \frac{1}{3}h^3\right]_0^r$

$= 2\pi\left(r^3 - \frac{1}{3}r^3\right)$

$= 2\pi\cdot\frac{2}{3}r^3$

$= \frac{4}{3}\pi r^3$

(e)(i) $P(x+1) = (x+1)^4 - 4(x+1)^3 + 10(x+1)^2 - 12(x+1) - 40$

$= x^4 + 4x^3 + 6x^2 + 4x + 1 - 4x^3 - 12x^2 - 12x - 4$
$+ 10x^2 + 20x + 10 - 12x - 12 - 40$

$= x^4 + 4x^2 - 45$

(ii) $P(x+1) = (x^2+9)(x^2-5)$ and so has zeroes $\pm 3i$ and $\pm\sqrt{5}$.
So $P(x)$ has zeroes $1 \pm 3i$ and $1 \pm \sqrt{5}$.

(13)(a)(i) $y - 2c = m(x + 4c)$

(ii) $y = mx + 2c(2m+1)$ ①

$xy = c^2$ ②

Subst ① into ②: $mx^2 + 2c(2m+1)x - c^2 = 0$

(iii) Line ① is a tangent to curve ②
when $\Delta = 0$.

So $4c^2(2m+1)^2 + 4c^2 m = 0$

$4m^2 + 5m + 1 = 0$

$(4m+1)(m+1) = 0$

$m = -\frac{1}{4}$ or -1

(b) $P(z) = z^4 + (1-2i)z^2 - 2i$

(i) $P(i) = i^4 + (1-2i)i^2 - 2i$
$= 1 - 1 + 2i - 2i$
$= 0$

(ii) $P(z)$ is even, since it only involves even powers of z.

(iii) Product of zeroes is $-2i$.

So $i^2 w^2 = -2i$

$w^2 = 2i$

Let $w = x + iy$.

Then $x^2 - y^2 = 0$ and $xy = 1$.

So $w = 1 + i$ (or $-1 - i$).

(13)(c)(i) $\int x\cos 2x\,dx$

$= uv - \int vu'$

$= \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

Let $u = x$.
Then $u' = 1$.

Let $v' = \cos 2x$.
Then $v = \frac{1}{2}\sin 2x$.

(ii) $V = \int_0^{\frac{\pi}{2}} 2\pi r h\,dx$, where $r = x$ and $h = y = \sin^2 x$

$= 2\pi\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$

$= 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}x(1-\cos 2x)\,dx$

$= 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}x\,dx - 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}x\cos 2x\,dx$

$= \pi\left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}} - \pi\left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \pi \cdot \frac{\pi^2}{8} - \pi\left(-\frac{1}{4} - \frac{1}{4}\right)$

$= \pi\left(\frac{\pi^2}{8} + \frac{1}{2}\right)$

$= \frac{1}{8}\pi(\pi^2 + 4)$ units^3

(14)(a)(i) $m\ddot{x} = -mg - \frac{v^2}{10}$

$$2\ddot{x} = -20 - \frac{v^2}{10}$$

$$\ddot{x} = \frac{-200 - v^2}{20}$$

$$v.\frac{dv}{dx} = \frac{-200-v^2}{20}$$

$$\frac{dx}{dv} = \frac{-20v}{200+v^2}$$

$$x = -10\int \frac{2v}{200+v^2}\,dv$$

$$= -10\ln(200+v^2) + c$$

When $t=0$, $x=0$ and $v=20$.

So $c = 10\ln 600$.

So $x = 10\ln\left(\frac{600}{200+v^2}\right)$.

So when $v=0$, $x = 10\ln 3$.

So the maximum height is $10\ln 3$ metres.

(ii) When $x = 5\ln 3$,
$$5\ln 3 = 10\ln\left(\frac{600}{200+v^2}\right)$$

$$e^{\frac{1}{2}\ln 3} = \frac{600}{200+v^2}$$

$$\frac{200+v^2}{600} = \frac{1}{\sqrt{3}}$$

$$200 + v^2 = 200\sqrt{3}$$

$$v^2 = 200(\sqrt{3}-1) = 146.41\ldots$$

So $v \doteqdot 12.1 \text{ ms}^{-1}$

(b)(i) $I_n = [uv]_1^e - \int_1^e vu'$

Let $u = (1+\ln x)^n$.
Then $u' = \frac{n}{x}(1+\ln x)^{n-1}$.

Let $v' = 1$.
Then $v = x$.

$$= \left[x(1+\ln x)^n\right]_1^e - n\int_1^e (1+\ln x)^{n-1}\,dx$$

$$= e.2^n - 1.1 - n.I_{n-1}$$

$$= (2^n)e - 1 - nI_{n-1}$$

(14)(b)(ii) $\int_1^e (2+\ln x)(1+\ln x)^2\,dx$

$$= \int_1^e \big(1+(1+\ln x)\big)(1+\ln x)^2\,dx$$

$$= I_2 + I_3$$

Now, $I_0 = \int_1^e dx = e-1$

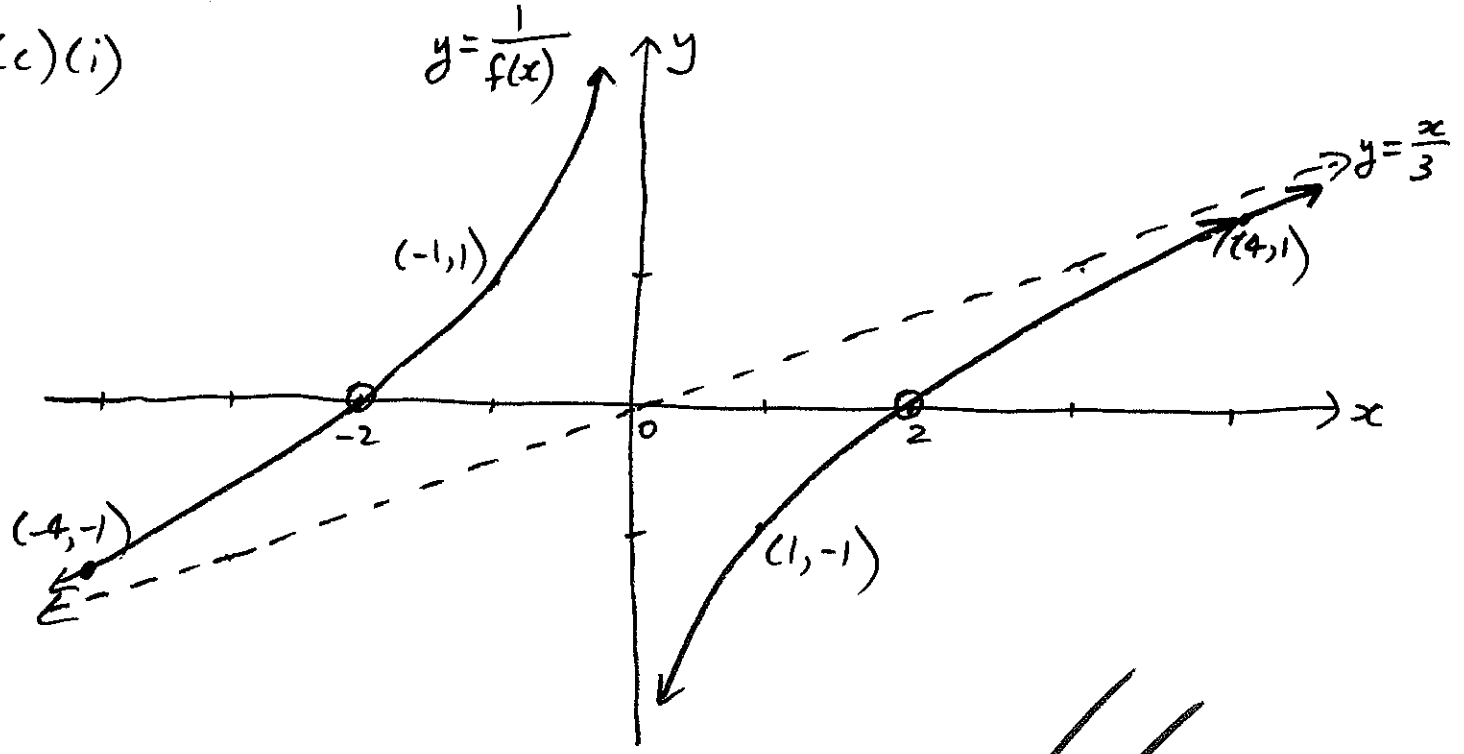
$I_1 = 2e-1-(e-1) = e$

$I_2 = 4e-1-2e = 2e-1$

$I_3 = 8e-1-3(2e-1) = 2e+2.$

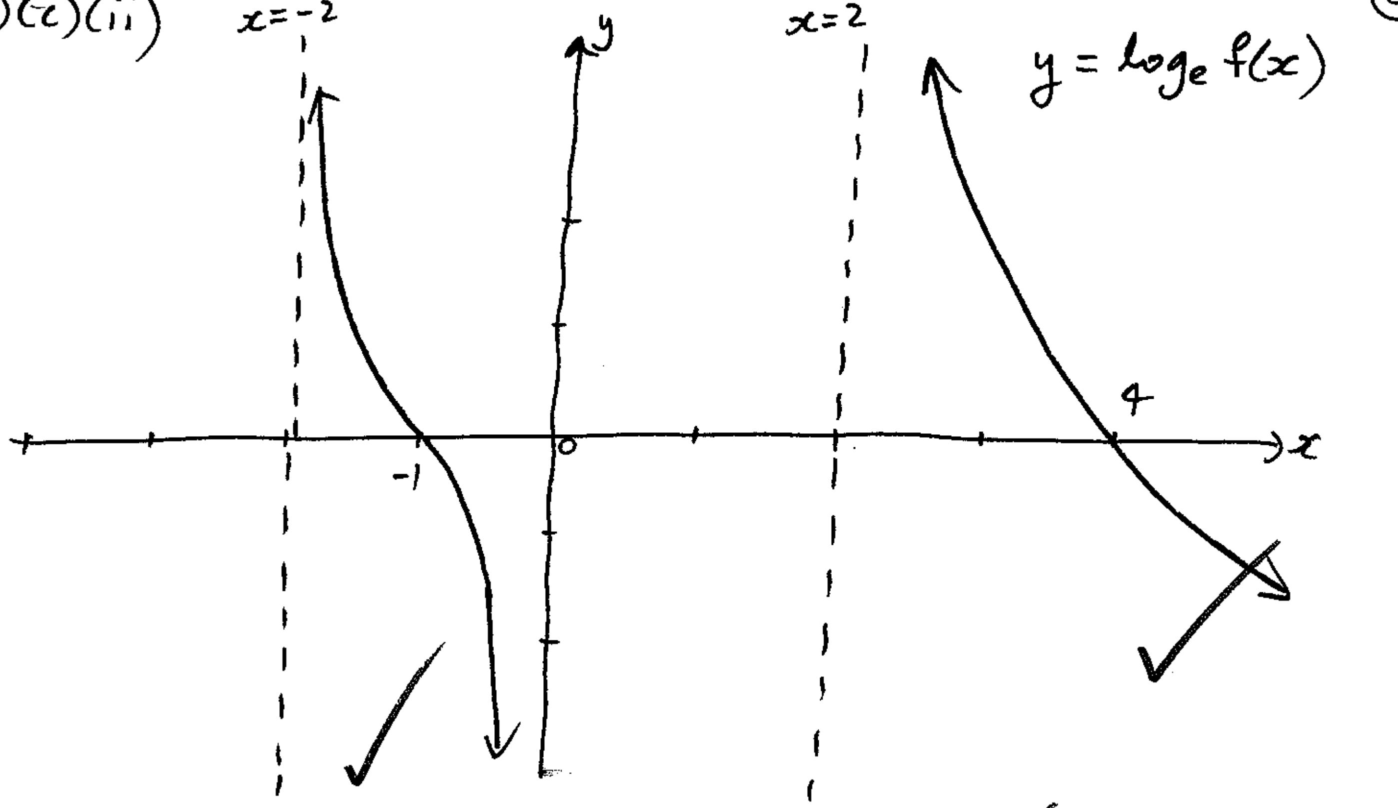
So $\int_1^e (2+\ln x)(1+\ln x)^2\,dx = 4e+1.$

(c)(i)

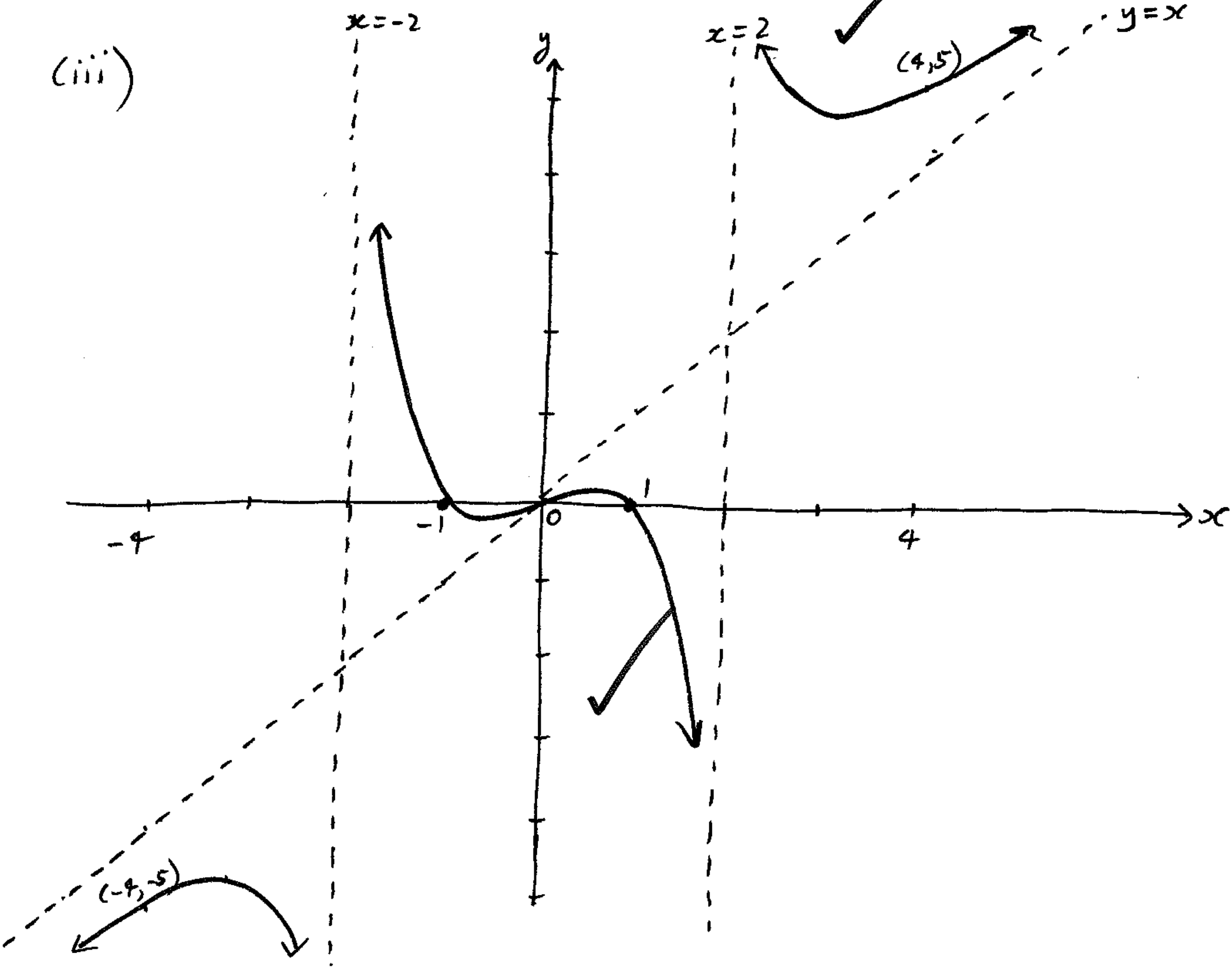


Maximum of one mark if either the open circles or the oblique asymptote are omitted.

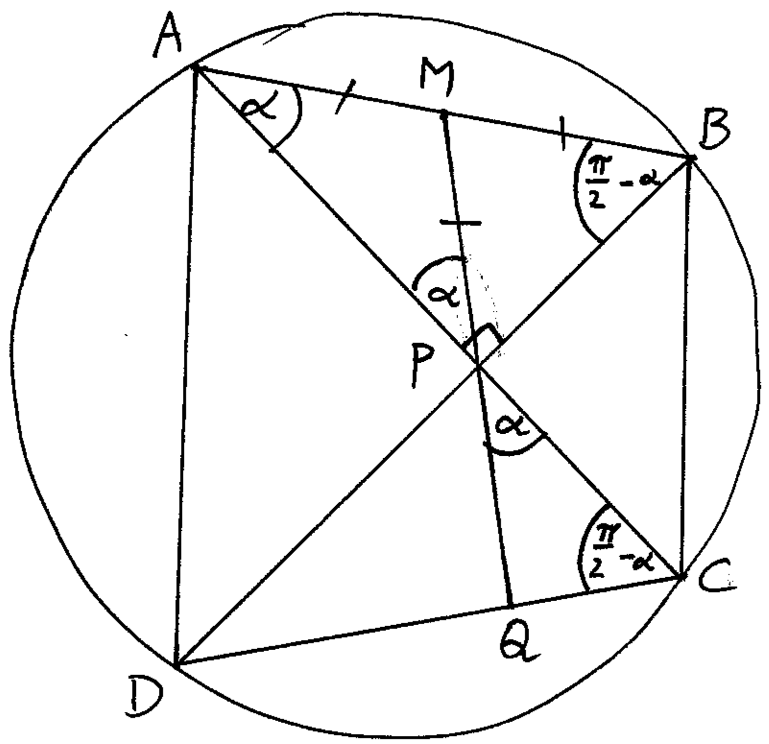
(14)(c)(ii)



(iii)



(15)(a)

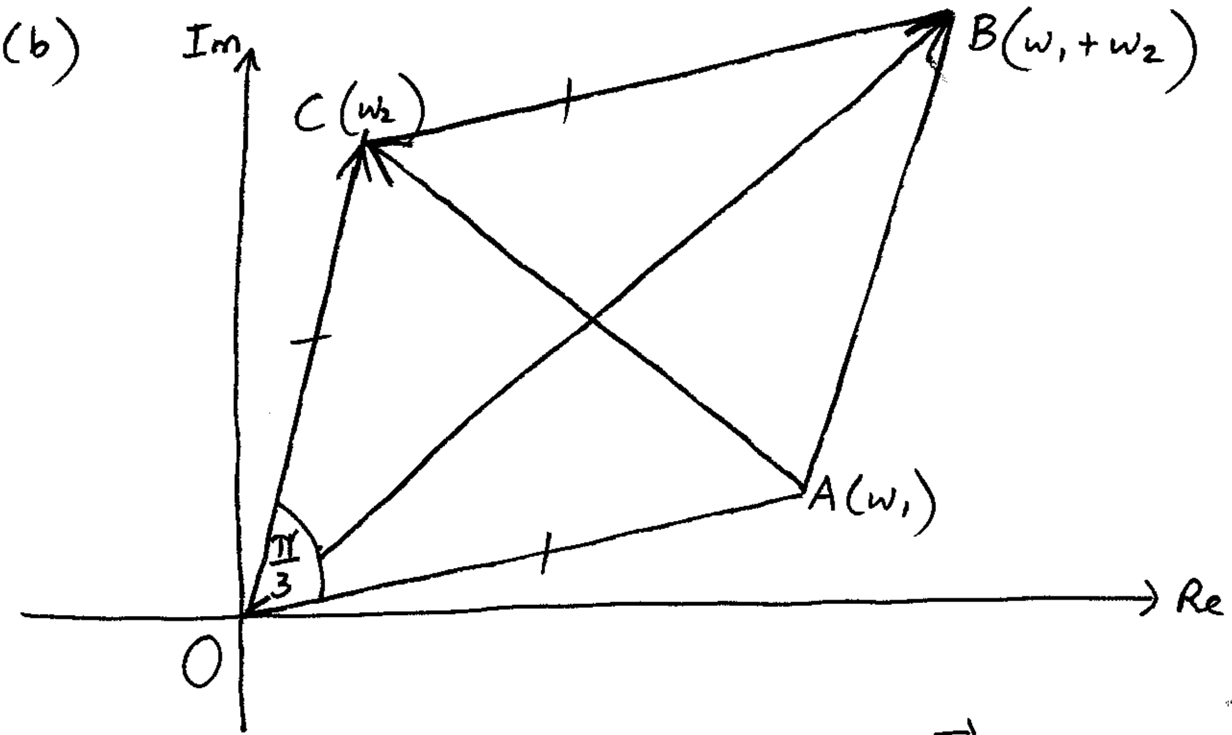


(i) $\angle APB = \frac{\pi}{2}$, so it is an angle in a semicircle with diameter AB.
So M is the centre of the semicircle.
So AM and PM are equal radii. ✓

(ii) $AM = PM$ (from (i)),
so $\angle APM = \alpha$ (base angles of isosceles triangle),
so $\angle CPQ = \alpha$ (vertically opposite). ✓

Now, $\angle ABD = \frac{\pi}{2} - \alpha$ (angle sum of ΔAPB),
so $\angle ACD = \frac{\pi}{2} - \alpha$ (angles at circumference on same arc) ✓
so $\angle MQC = \frac{\pi}{2}$ (angle sum of ΔPQC).
So $MQ \perp DC$. ✓

(15)(b)



(i) $\overrightarrow{OC}$ is the anticlockwise rotation of $\overrightarrow{OA}$ through $\frac{\pi}{3}$ about O. So $w_2 = w_1 \cdot 1\,\text{cis}\frac{\pi}{3}$

$= w_1\,\text{cis}\frac{\pi}{3}$.

(ii) $\overrightarrow{OB}$ represents $w_1 + w_2 = w_1 + w_1\,\text{cis}\frac{\pi}{3}$

$= w_1\left(1 + \text{cis}\frac{\pi}{3}\right)$

and $\overrightarrow{AC}$ represents $w_2 - w_1 = w_1\,\text{cis}\frac{\pi}{3} - w_1$

$= w_1\left(\text{cis}\frac{\pi}{3} - 1\right)$.

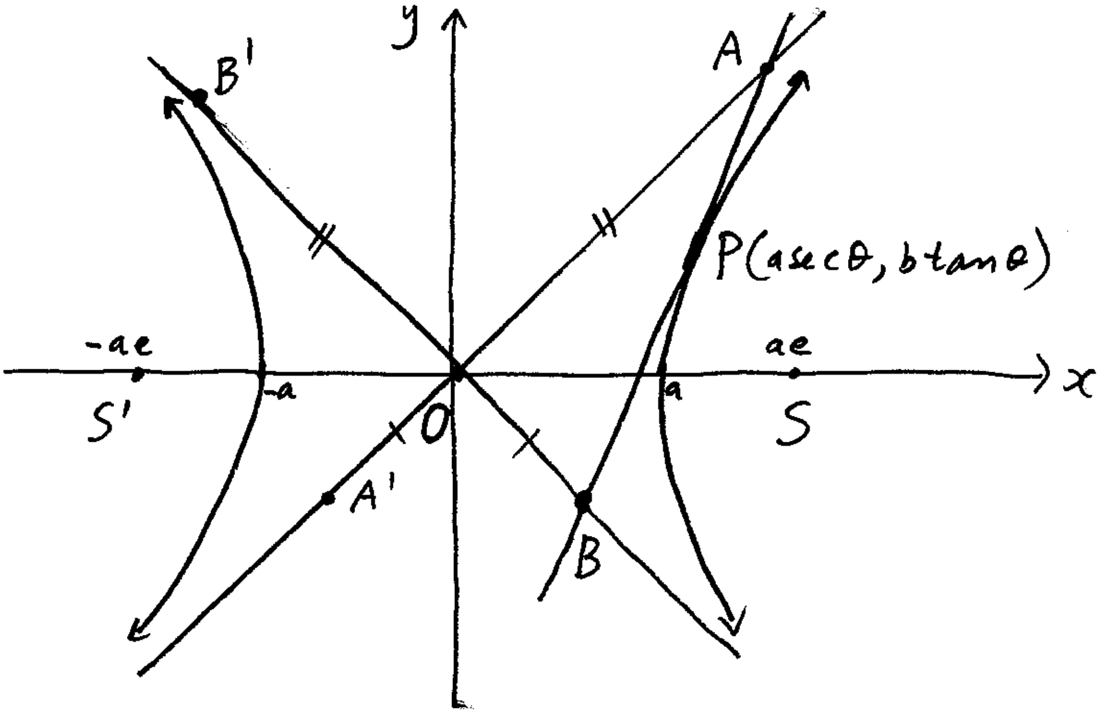
(iii)
$$\begin{aligned} i(w_1+w_2) &= iw_1\left(1+\text{cis}\frac{\pi}{3}\right) \\ &= iw_1\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= w_1\left(-\frac{\sqrt{3}}{2} + \frac{3}{2}i\right) \\ &= \sqrt{3}w_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= \sqrt{3}w_1\left(\text{cis}\frac{\pi}{3} - 1\right) \\ &= \sqrt{3}\left(w_2 - w_1\right) \end{aligned}$$

So $w_2 - w_1 = \frac{1}{\sqrt{3}}i(w_1+w_2)$.

So $w_2 - w_1 = (w_1+w_2) \cdot r\,\text{cis}\frac{\pi}{2}$, where $r = \frac{1}{\sqrt{3}}$.

So AC and OB are perpendicular.

(15)(c)



(i) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$

$= \dfrac{b\sec\theta}{a\tan\theta}$

So the tangent at P has equation

$y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$ ✓

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab$

(ii) Solve the tangent equation simultaneously with $y = \dfrac{bx}{a}$:

$bx\sec\theta - bx\tan\theta = ab$

$x = \dfrac{a}{\sec\theta - \tan\theta} \cdot \dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$

$= a(\sec\theta + \tan\theta)$

So $A = (a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$.

Similarly, $B = (a(\sec\theta - \tan\theta), b(\sec\theta - \tan\theta))$ ✓

So $OA^2 . OB^2 = (a^2(\sec\theta + \tan\theta)^2 + b^2(\sec\theta + \tan\theta)^2)$

$\cdot (a^2(\sec\theta - \tan\theta)^2 + b^2(\sec\theta - \tan\theta)^2)$

$= (a^2+b^2)(\sec\theta + \tan\theta)^2 . (a^2+b^2)(\sec\theta - \tan\theta)^2$ ✓

$= (a^2+b^2)^2(\sec^2\theta - \tan^2\theta)^2$.

So $OA . OB = a^2 + b^2$.

(15)(c)(iii) $OA \cdot OA' = OB \cdot OB'$, by the definitions of A' and B'.

So, by the converse of the intersecting chords theorem, the points A, B, A' and B' are concyclic.

(iv) $OA \cdot OA' = OB \cdot OB'$

$= OA \cdot OB$

$= a^2 + b^2$ (from (ii))

But $OS \cdot OS' = a^2 e^2$

$= a^2 + b^2$ (since $b^2 = a^2(e^2 - 1)$).

So, by the converse of the intersecting chords theorem, S and S' lie on the same circle as A, B, A' and B'.

So A, B, S and S' are concylic.

(16)(a)(i) $z^n + z^{-n} = (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}$

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$ (de Moivre)

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) $\text{LHS} = \left((z^{2n} + z^{-2n}) + (z^{2n-2} + z^{-(2n-2)}) + \dots + (z^2 + z^{-2}) + z^0\right)\sin\theta$

$= \left(2\cos 2n\theta + 2\cos(2n-2)\theta + \dots + 2\cos 2\theta + 1\right)\sin\theta$

$= \left(\sin(2n+1)\theta - \sin(2n-1)\theta\right) + \left(\sin(2n-1)\theta - \sin(2n-3)\theta\right)$

$+ \dots + \left(\sin 3\theta - \sin\theta\right) + \sin\theta$

$= \sin(2n+1)\theta = \text{RHS}$

(iii) Let $n = 3$.

Then $(2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta + 1)\sin\theta = \sin 7\theta$

$2(4\cos^3 2\theta - 3\cos 2\theta) + 2(2\cos^2 2\theta - 1) + 2\cos 2\theta + 1 = \dfrac{\sin 7\theta}{\sin\theta}$

$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \dfrac{\sin 7\theta}{\sin\theta}$.

(iv) Let $\theta = \dfrac{\pi}{7}$.

Then $8\cos^3\dfrac{2\pi}{7} + 4\cos^2\dfrac{2\pi}{7} - 4\cos\dfrac{2\pi}{7} - 1 = \dfrac{\sin\pi}{\sin\frac{\pi}{7}} = 0.$

So it follows that $\cos\dfrac{2\pi}{7}$ is a root of the equation

$8x^3 + 4x^2 - 4x - 1 = 0.$

(16)(b)(i) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Let $\theta = \frac{\pi}{8}$.

Then $\tan\frac{\pi}{4} = \dfrac{2t}{1-t^2}$, where $t = \tan\frac{\pi}{8}$

$$t^2+2t-1=0$$
$$(t+1)^2 = 2$$
$$t = \pm\sqrt{2}-1$$

But $t>0$, so $t = \tan\frac{\pi}{8} = \sqrt{2}-1$.

(ii)
$$\begin{aligned}\cos 4\theta &= 2\cos^2 2\theta - 1\\ &= 2(2\cos^2\theta-1)^2-1\\ &= 2(4\cos^4\theta - 4\cos^2\theta+1)-1\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

(iii)(α)

$x = \sin 4\theta$

$dx = 4\cos 4\theta\, d\theta$

x	-1	1
θ	$-\frac{\pi}{8}$	$\frac{\pi}{8}$

$$I = \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{4\cos 4\theta}{\sqrt{1+\sin 4\theta}+\sqrt{1-\sin 4\theta}+2}\,d\theta$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{4\cos 4\theta}{\sqrt{\cos^2 2\theta+\sin^2 2\theta+\underset{2\sin 2\theta\cos 2\theta}{\cancel{+}}}+\sqrt{\cos^2 2\theta+\sin^2 2\theta\underset{-2\sin 2\theta\cos 2\theta}{\cancel{+}}}+2}\,d\theta$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{4\cos 4\theta}{(\cos 2\theta+\sin 2\theta)+(\cos 2\theta-\sin 2\theta)+2}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{8}} \frac{4\cos 4\theta}{2(\cos 2\theta+1)}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{2\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{8}} \frac{2\cos 4\theta}{\cos^2\theta}\,d\theta$$

(β)
$$\begin{aligned}I &= 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4\theta - 8\cos^2\theta+1}{\cos^2\theta}\,d\theta\\ &= 2\int_0^{\frac{\pi}{8}} (8\cos^2\theta - 8 + \sec^2\theta)\,d\theta\\ &= 2\int_0^{\frac{\pi}{8}} (4+4\cos 2\theta - 8+\sec^2\theta)\,d\theta\\ &= 2\left[2\sin 2\theta - 4\theta + \tan\theta\right]_0^{\frac{\pi}{8}}\\ &= 2\left(2\sqrt{2}-\frac{\pi}{2}+(\sqrt{2}-1)\right)\\ &= 4\sqrt{2}-\pi-2\end{aligned}$$

(16)(c)(i) $LHS = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \ldots + \frac{1}{(2k-1)(2k)}$

$> \frac{1}{(2k+1)(2k+2)} + \frac{1}{(2k+1)(2k+2)} + \ldots + \frac{1}{(2k+1)(2k+2)}$ (k terms)

$= \frac{k}{(2k+1)(2k+2)}$

$= RHS$

(ii) Proof by induction:

When $n=2$, $LHS = 2\times\frac{4}{3} = 2\frac{2}{3}$ and $RHS = 3\times\frac{3}{4} = 2\frac{1}{4}$.

So the result is true when $n=2$.

Assume that the result is true for the positive integer $n=k$.

That is, assume that $k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)$. $\circledast$

Prove that the result is true for $n=k+1$.

That is, prove that

$$(k+1)\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}+\frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{1}{2k+2}\right).$$

$$LHS = (k+1)\left(\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{1}{2k+1}\right)$$

$$= k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right) + 1\left(1+\frac{1}{3}+\ldots\frac{1}{2k-1}\right) + \frac{k+1}{2k+1}$$

$$> (k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right) + \left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{k}{(2k+1)(2k+2)}\right) + \frac{k+1}{2k+1}$$

(using $\circledast$) (using part (i))

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right) + \frac{k+(k+1)(2k+2)}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right) + \frac{\cancel{(2k+1)}(k+2)}{\cancel{(2k+1)}(2k+2)}$$

$= RHS$

So, by mathematical induction, the result is true for all $n \geq 2$.

(16)(c)(ii) Otherwise:

$$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} > 1 + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}$$

$$= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$$

$$> \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right),$$

since $\frac{n}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$ (n terms)

$> \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}$

so that $\frac{1}{2} > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$.

So $$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} > \left(\frac{1}{n} + 1\right)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$$

$$= \frac{n+1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right),$$

so $$n\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right).$$