
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2018 Trial Examination

FORM VI

MATHEMATICS EXTENSION 2

Friday 10th August 2018

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Reference Sheet
- Candidature — 77 boys

Examiner

RCF

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Given $z = i$ is a zero of $P(z) = z^4 + z^3 + az^2 + z - 6$, what is the value of a?

(A) -5

(B) 5

(C) -7

(D) 7

QUESTION TWO

A hyperbola centred on the origin with eccentricity 2 has foci at $(6, 0)$ and $(-6, 0)$. What is its Cartesian equation?

(A) $\dfrac{x^2}{27} - \dfrac{y^2}{9} = 1$

(B) $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

(C) $\dfrac{x^2}{9} - \dfrac{y^2}{27} = 1$

(D) $\dfrac{x^2}{4} - \dfrac{y^2}{36} = 1$

QUESTION THREE

Consider the curve with implicit equation $x^3 - y^3 + 3xy + 1 = 0$. Which of the following is the correct expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{y^2 + x}{x^2 - y}$

(B) $\dfrac{x^2 + y}{y^2 - x}$

(C) $\dfrac{y^2 - x}{x^2 + y}$

(D) $\dfrac{x^2 - y}{y^2 + x}$

QUESTION FOUR

Suppose w is one of the complex roots of $z^3 - 1 = 0$. What is the value of $\dfrac{1}{1+w} + \dfrac{1}{1+w^2}$?

(A) -1

(B) 0

(C) 1

(D) 2

QUESTION FIVE

The polynomial $P(z)$ with real co-efficients has a zero $z = 1 + 3i$. Which of the following quadratic polynomials must be a factor of $P(z)$?

(A) $z^2 - 2z - 8$

(B) $z^2 + 2z + 8$

(C) $z^2 + 2z - 10$

(D) $z^2 - 2z + 10$

QUESTION SIX

An object of mass m falling under gravity experiences a resistive force R proportional to the square of its velocity, that is $R = mkv^2$. Which expression best describes the object's terminal velocity?

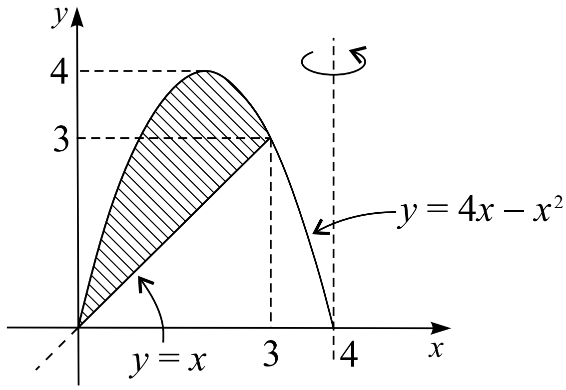
(A) $\sqrt{g+k}$

(B) $\sqrt{gk}$

(C) $\sqrt{g-k}$

(D) $\sqrt{\dfrac{g}{k}}$

QUESTION SEVEN



The graph above shows the region enclosed between the curves $y = 4x - x^2$ and $y = x$. This region is rotated about the line $x = 4$ to create a volume which can be evaluated using cylindrical shells. Which of the following integrals will give the correct value for this volume?

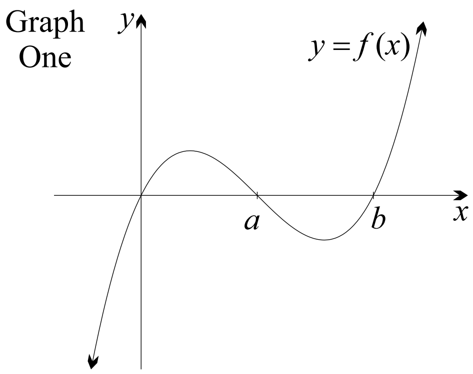
(A) $\pi \int_0^4 (4 - y)^2 \, dy$

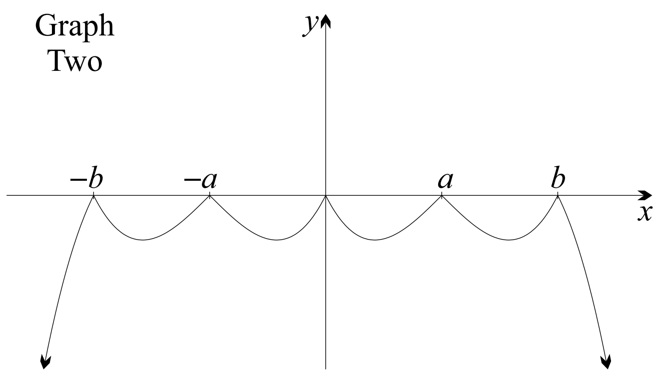
(B) $2\pi \int_0^3 x(3 - x)(4 - x) \, dx$

(C) $2\pi \int_0^3 x^2(3 - x) \, dx$

(D) $2\pi \int_0^3 x^2 - x^2(4 - x)^2 \, dx$

QUESTION EIGHT





Graph One drawn above shows $y = f(x)$. Graph Two shows a related graph. Which one of the following equations would give Graph Two?

(A) $y = -f(|x|)$

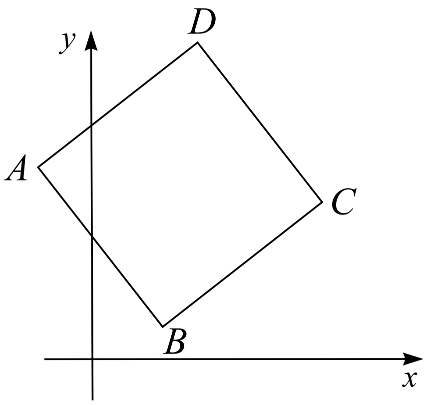
(B) $|y| = f(-x)$

(C) $-y = |f(x)|$

(D) $y = -|f(|x|)|$

Examination continues next page ...

QUESTION NINE



In the Argand diagram, $ABCD$ is a square and the vertices A and B correspond to the complex numbers u and v respectively. Which complex number corresponds to the diagonal BD?

(A) $(u - v)(1 - i)$

(B) $(u - v)(1 + i)$

(C) $(v - u)(1 + i)$

(D) $(u + v)(1 - i)$

QUESTION TEN

Which of the following integrals is equal to $\displaystyle\int_0^{2a} f(x)\,dx$?

(A) $\displaystyle\int_{-a}^{a} f(x - a)\,dx$

(B) $\displaystyle\int_{-a}^{a} f(a - x)\,dx$

(C) $\displaystyle\int_{0}^{a} f(x + a) + f(x - a)\,dx$

(D) $\displaystyle\int_{-a}^{a} f(2a - x)\,dx$

End of Section I

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Use integration by parts to find $\int x \sin x \, dx$. **2**

(b) Let $z = 1 - i$ and $w = i\sqrt{3} - 1$.

(i) Express zw in the form $a + ib$, where a and b are real. **1**

(ii) By expressing both z and w in modulus–argument form, write zw in modulus–argument form. **2**

(iii) Hence find the exact value of $\sin \dfrac{5\pi}{12}$. **1**

(iv) By using the result of part (ii), or otherwise, calculate $\left[(1-i)(i\sqrt{3}-1)\right]^6$. **1**

(c) Find:

(i) $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\, dx$ **1**

(ii) $\displaystyle\int \frac{x^2 - x + 3}{x - 1}\, dx$ **2**

(iii) $\displaystyle\int \frac{1}{x^2 - 6x + 13}\, dx$ **2**

(d) The quartic polynomial $P(x) = x^4 + 3x^3 - 6x^2 - 28x - 24$ is known to have a zero of multiplicity 3. Factorise $P(x)$ completely. **3**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

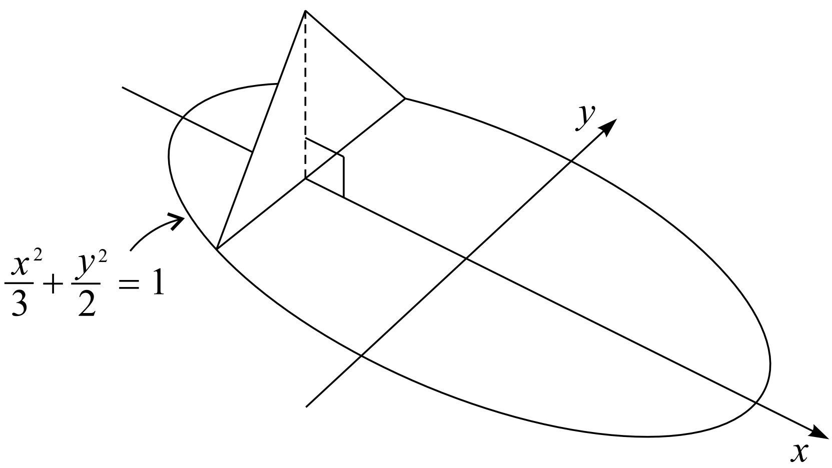
(a) (i) Sketch the region in the complex plane which simultaneously satisfies **3**

$$0 \le \arg(z-2) \le \frac{\pi}{3} \quad \text{and} \quad |z-2| \le 1.$$

Clearly label the coordinates of any corners of the region, indicating if they are included in the region.

(ii) Hence find the greatest value of $\arg z$. **1**

(b)



A camping tent is shown has an elliptical base $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$. Cross sections perpendicular to the base are equilateral triangles. The diagram above shows the base and one such triangle. All units are in metres.

(i) Show that a typical cross-sectional area is given by $2\sqrt{3}\left(1-\dfrac{x^2}{3}\right)$ square metres. **2**

(ii) Hence find the volume of the tent in cubic metres. **1**

(c) (i) Show that the normal to the hyperbola $xy=c^2$, $c \ne 0$, at $P\left(cp, \dfrac{c}{p}\right)$ is given by **2**

$$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right).$$

(ii) The normal at P meets the hyperbola again at $Q\left(cq, \dfrac{c}{q}\right)$. Find q in terms of p. **2**

(d) The polynomial equation $x^4-8x^3+16x^2-1=0$ has roots α, β, γ and δ.

(i) Find the monic polynomial equation with roots $(\alpha-2)$, $(\beta-2)$, $(\gamma-2)$ and $(\delta-2)$. **2**

(ii) Hence, or otherwise, solve $x^4-8x^3+16x^2-1=0$. **2**

Examination continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Sketch each of the following loci in the complex plane.

(i) $z^2 - (\overline{z})^2 = -8i$ **2**

(ii) $2|z| = z + \overline{z} + 4$ **2**

(b) Use the substitution $t = \tan \frac{x}{2}$ to show that **4**

$$\int_0^{\frac{\pi}{3}} \sec x \, dx = \ln(2 + \sqrt{3}\,).$$

(c) The polynomial $P(z) = z^3 + az + b$ has zeroes α, β and $3(\alpha - \beta)$.

(i) Show that $a = -7\alpha^2$. **2**

(ii) Show that $b = 6\alpha^3$. **1**

(iii) Deduce that the zeroes of $P(z)$ are $\dfrac{-7b}{6a}, \dfrac{-7b}{3a}$ and $\dfrac{7b}{2a}$. **1**

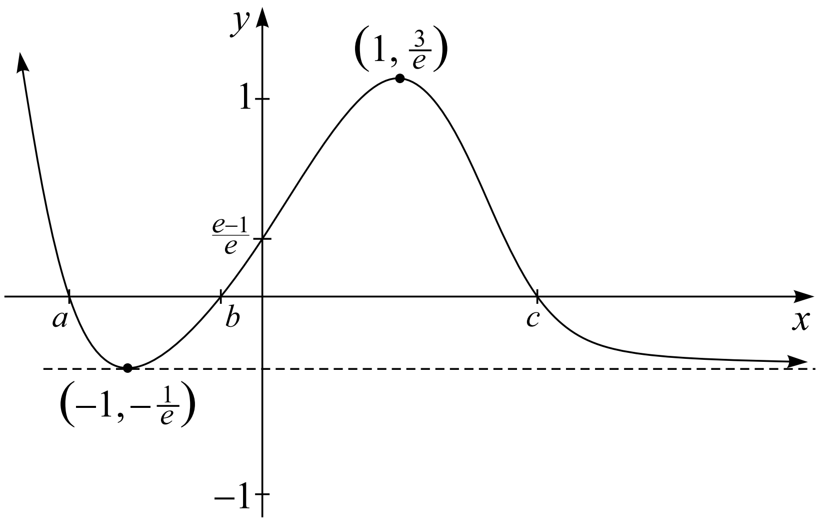
(d) Find two numbers whose sum is 4 and whose product is 29. **3**

Examination continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The circle $x^2 + y^2 = 4$ is rotated about the line $x = 3$ to form a torus. Use the method of cylindrical shells to prove that the volume of the torus is $24\pi^2$ cubic units. **4**

(b)



The function $f(x)$ graphed above has a minimum turning point at $(-1, -\frac{1}{e})$, a maximum turning point at $(1, \frac{3}{e})$ and a horizontal asymptote at $y = -\frac{1}{e}$. The x-intercepts are marked a, b and c respectively and the y-intercept is $(0, \frac{e-1}{e})$.

Without using calculus, sketch the following curves showing their essential features, taking at least one-third of a page for each graph.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \big[f(x)\big]^2$ **3**

(c) Let $I_n = \displaystyle\int_0^a (a^2 - x^2)^n \, dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_n = \dfrac{2a^2 n}{2n+1} I_{n-1}$, for $n \geq 1$. **3**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^2 (4 - x^2)^3 \, dx$. **1**

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) By considering the Binomial expansion of $(1+i)^n$ for the case when n is even, show that: **2**

$$1-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\ldots+(-1)^{\frac{n}{2}}\binom{n}{n}=2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

(b) A car of mass m kg is being driven along a straight road. The motor of the car provides a constant propelling force mP, while the car experiences a resistive force of mkv^2, where $v\,\mathrm{ms}^{-1}$ is the velocity of the car and k is a positive constant. The car is initially at rest.

(i) Show that $\dfrac{dx}{dv}=\dfrac{v}{P-kv^2}$, where x is the displacement of the car from its initial position in metres. **1**

(ii) Show that $v^2=\dfrac{P}{k}\left(1-e^{-2kx}\right)$. Hence, or otherwise, explain why the maximum velocity of the car is $V_M=\sqrt{\dfrac{P}{k}}\ \mathrm{ms}^{-1}$. **3**

(iii) Show that the distance required for the car to reach a velocity of $\frac{1}{3}V_M$ is approximately 41% of the distance required for the car to reach a velocity of $\frac{1}{2}V_M$. **1**

(c) (i) Use De Moivre's theorem to show that $\sin 5\theta = 16\sin^5\theta-20\sin^3\theta+5\sin\theta$. **1**

(ii) Hence find the four roots of the equation $16x^4-20x^2+5=0$. **1**

(iii) Show that $\sin^2\left(\dfrac{\pi}{5}\right)+\sin^2\left(\dfrac{2\pi}{5}\right)=\dfrac{5}{4}$. **2**

(iv) Hence show that $\cos\left(\dfrac{2\pi}{5}\right)=\dfrac{\sqrt{5}-1}{4}$. **2**

(d) Let $P(x)$ be a monic polynomial of degree n with rational coefficients and zeroes $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$. Prove that: **2**

$$P'(x)=P(x)\left(\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\frac{1}{x-\alpha_3}+\ldots+\frac{1}{x-\alpha_n}\right)$$

for all values of x, excluding the zeroes.

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ are distinct points on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with eccentricity e. The equation of the chord PQ is

$$y - b\sin\phi = \frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}(x - a\cos\phi)\,. \qquad \text{(Do NOT prove this)}$$

(i) If PQ is a focal chord of this ellipse, show that: **1**

$$e = \frac{\sin(\phi-\theta)}{\sin\phi - \sin\theta}.$$

(ii) If PQ subtends a right angle at the positive x-intercept of the ellipse at $R(a,0)$, show that: **3**

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}.$$

(b) (i) Sketch $y = e^x \sin x$ for the domain $0 \le x \le 2\pi$, clearly indicating the x-intercepts. **1**

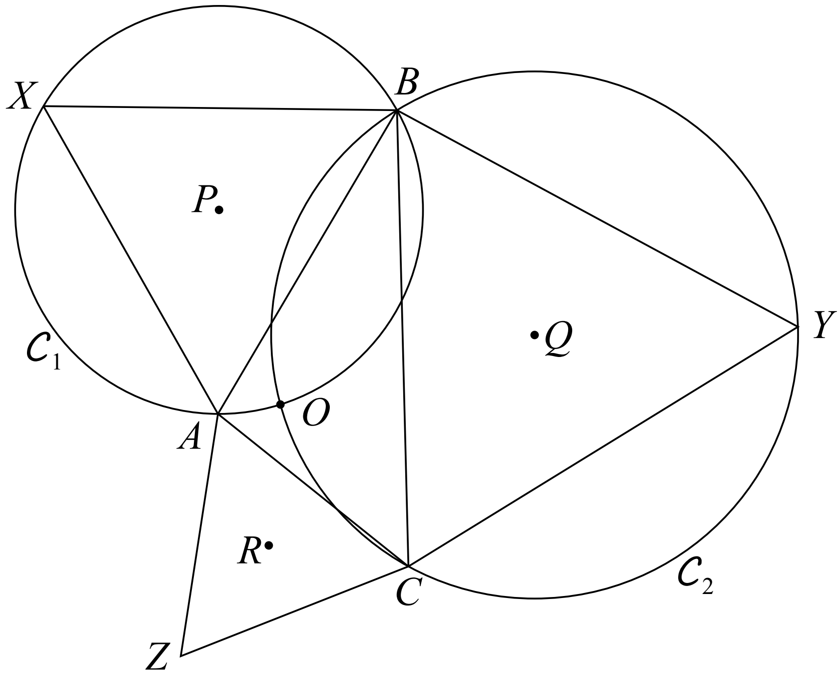
(ii) Let $I = \displaystyle\int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx$, where k is an integer. Show that: **3**

$$2I = (-1)^{k-1}e^{k\pi}(1+e^{-\pi})\,.$$

(iii) Hence find the area bounded by $y = e^x \sin x$, the x-axis and the lines $x = 0$ and $x = 2\pi$. **2**

Examination continues overleaf ...

(c) NOTE: The diagram below has been reproduced on a loose sheet so that working may be done on the diagram. Write your candidate number on the top of the sheet and place the sheet inside your answer booklet for Question Sixteen.



A scalene triangle ABC has equilateral triangles ABX, BCY and ACZ constructed on its three sides. Circle $\mathcal{C}_1$ is drawn to pass through the vertices of $\triangle ABX$ and has centre P. Similarly circle $\mathcal{C}_2$ is drawn through the vertices of $\triangle BCY$ and has centre Q. These two circles meet at B and O as shown in the diagram above.

(i) Find the sizes of $\angle AOB$, $\angle BOC$ and $\angle AOC$, giving clear geometric reasons. **1**

(ii) Prove that $AOCZ$ is a cyclic quadrilateral. **1**

(iii) Taking R as the centre of circle $AOCZ$, prove that $\triangle PQR$ is equilateral. **3**

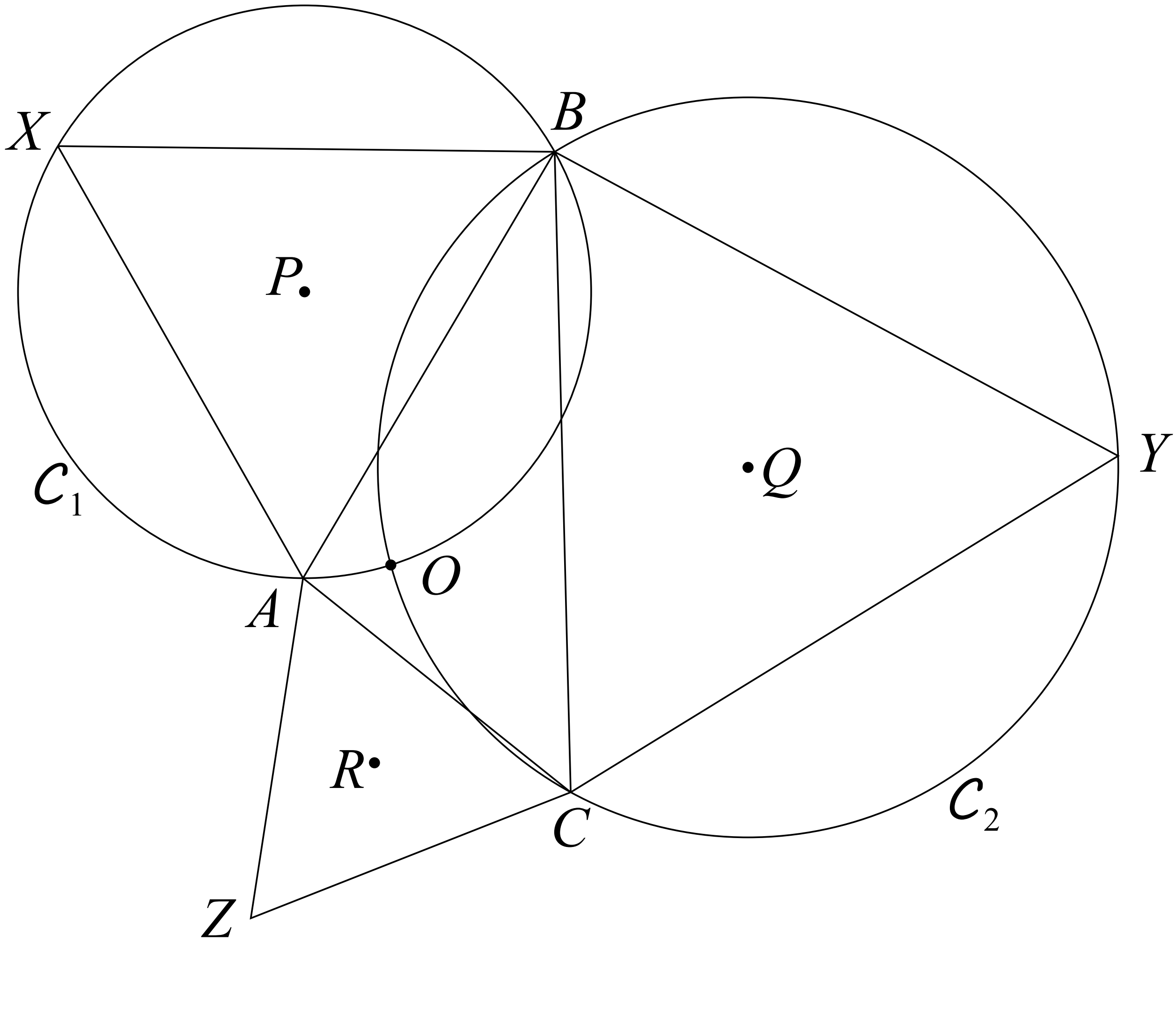
End of Section II

END OF EXAMINATION

CANDIDATE NUMBER

Bundle this sheet with the rest of question sixteen.

QUESTION SIXTEEN



SGS 2018 4U TRIAL

MULTIPLE CHOICE SUMMARY

1. A
2. C
3. B
4. C
5. D
6. D
7. B
8. D
9. A
10. B

(1)
$P(z) = z^4 + z^3 + az^2 + z - 6$

sub $z = i$

$P(i) = i^4 + i^3 + ai^2 + i - 6$

$= 1 - i - a + i - 6$

$= -5 - a$

If $P(i) = 0 \quad a = (-5)$ (A)

(2)
$e = 2 \quad (6,0) \Rightarrow a = 3$

$b^2 = a^2(e^2 - 1)$

$b^2 = a^2(4 - 1)$

$b^2 = 27$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \therefore \frac{x^2}{9} - \frac{y^2}{27} = 1$ (C)

(3)
$x^3 - y^3 + 3xy + 1 = 0$

Diff wrt x

$3x^2 - 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$

$3x^2 + 3y = (3y^2 - 3x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x^2 + y}{y^2 - x}$ (B)

(4)
$z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$

ω is complex root $\therefore \omega^2 + \omega + 1 = 0$

$\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = \frac{(1+\omega^2) + (1+\omega)}{(1+\omega)(1+\omega^2)}$

$= \frac{1 + \omega + \omega^2 + 1}{1 + \omega + \omega^2 + \omega^3}$ (C)

$= \frac{0+1}{0+1} = 1$

⑤

One zero $1+3i$

Other zero $1-3i$ since Real Coefficients

Quadratic Factor $z^2 - 2\,\mathrm{Re}(\alpha)z + \alpha\bar{\alpha}$

$z^2 - 2z + 10$ (D)

⑥

At Terminal velocity, no net force

$$mkv^2 = mg$$

$$v^2 = \frac{g}{k}$$

$$v = \sqrt{\frac{g}{k}} \quad (v>0) \quad (D)$$

⑦ $$V = \int_{r_{IN}}^{r_{OUT}} 2\pi r h\, dr$$

$r = 4 - x$

$dr = -dx$

$r_{IN} \Rightarrow x = 3$

$r_{OUT} \Rightarrow x = 0$

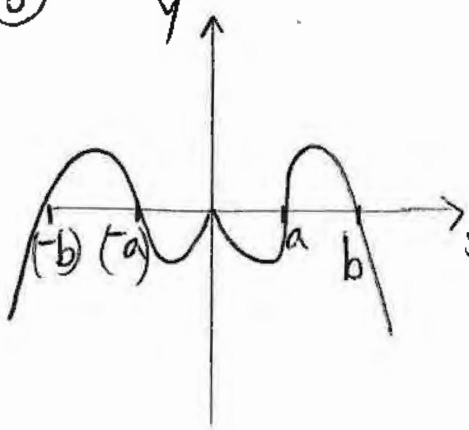
$h = (4x - x^2) - x$

$= 3x - x^2$

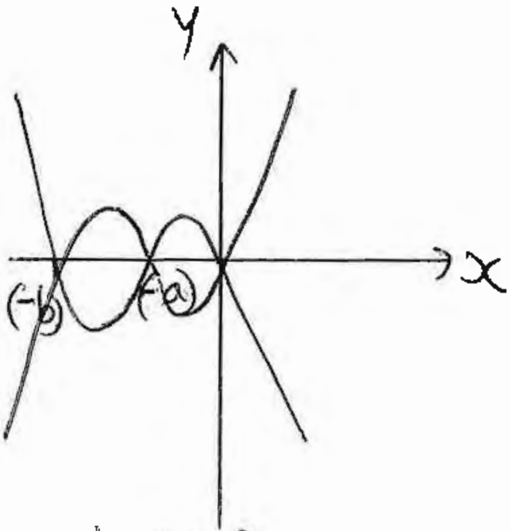
$$V = \int_{x=3}^{0} 2\pi(4-x)(3x-x^2)(-dx)$$

$$= 2\pi \int_0^3 x(3-x)(4-x)\,dx \quad (B)$$

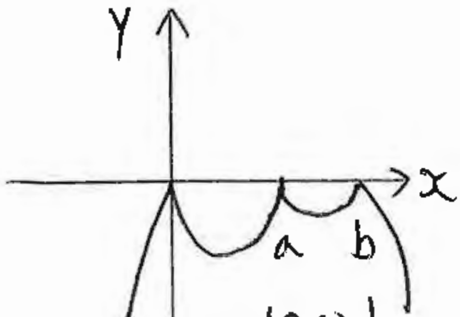
⑧



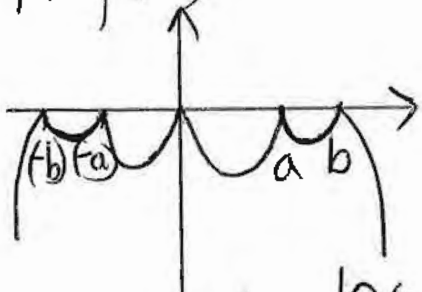
$y = -f(|x|)$



$|y| = f(-x)$



$-y = |f(x)|$



$y = -|f(|x|)|$

(D)

⑨ $\vec{OA}$ represents u
$\vec{OB}$ represents v
$\vec{BA} = \vec{OA} - \vec{OB}$
represents $(u - v)$
$\vec{BC}$ is $\vec{AB}$ rotated $90°$ clockwise
represents $\dfrac{u-v}{i} = -i(u-v)$

$\vec{BD} = \vec{BC} + \vec{CD}$
$= \vec{BC} + \vec{BA}$
$= -i(u-v) + (u-v)$
$= (1-i)(u-v)$.

(A)

<u>Alternatively:</u>
$\vec{BA}$ rotated by $\left(-\frac{\pi}{4}\right)$
and enlarged by $\sqrt{2}$
$\times \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$
$= (1 - i)$

(10) A translates function to the right by a
hence requires limits $x = a$ & $x = 3a$
B reflects in vertical line $x = \frac{a}{2}$
hence correct limits $x = a$ and $x = -a$ (B)
C has first term translating function to the left by a, but second section translates function to the right hence not part of original area
D reflects function in vertical line $x = a$
hence requires limits $x = 0$ & $x = 2a$

(11)

a) $\int x\sin x\,dx$ $\quad u = x \quad \frac{dv}{dx} = \sin x$

$= -x\cos x - \int -\cos x\,dx \quad \frac{du}{dx} = 1 \quad v = -\cos x$ ✓

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C$ ✓ (Do not penalise +c)

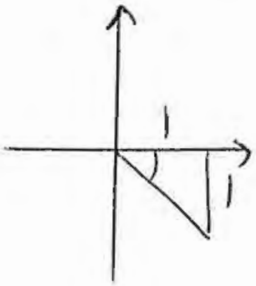
b) $z = 1 - i \qquad \omega = i\sqrt{3} - 1$

(i) $z\omega = (1-i)(i\sqrt{3}-1)$

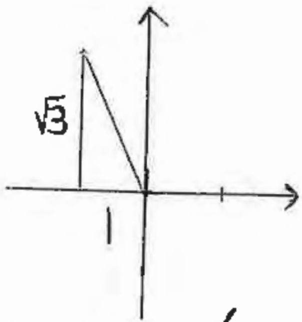
$= i\sqrt{3} - 1 + \sqrt{3} + i$

$= (\sqrt{3}-1) + i(\sqrt{3}+1)$ ✓

(ii)



$z = \sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$



$\omega = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$ ✓

(Both ω & z correct)

$z\omega = 2\sqrt{2}\,\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$ ✓

$= 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$

(iii) Equating imaginary parts

$\sqrt{3} + 1 = 2\sqrt{2}\sin\frac{5\pi}{12}$

$\therefore \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \begin{matrix}(\times\sqrt{2})\\(\times\sqrt{2})\end{matrix}$ ✓ (or equivalent)

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

(iv) $(z\omega)^6 = \left(2\sqrt{2}\,\text{cis}\frac{5\pi}{12}\right)^6$ $\qquad$ (also: $8^3 i$)

$= (2\sqrt{2})^6\,\text{cis}\left(\frac{30\pi}{12}\right) = 512\,\text{cis}\frac{5\pi}{2} = 512i$ ✓ (or)

11 c)

(i) $\int \frac{x}{\sqrt{4+x^2}}\,dx$ Reverse Chain Rule or substitute $u=4+x^2$

$= (4+x^2)^{\frac{1}{2}} + C$ ✓

(ii) $\int \frac{x^2-x+3}{x-1}\,dx = \int \frac{x(x-1)+3}{x-1}\,dx$ ✓

$= \int x + \frac{3}{x-1}\,dx$ ✓

$= \frac{x^2}{2} + 3\ln|x-1| + C$ ✓

(iii) $\int \frac{1}{x^2-6x+13}\,dx = \int \frac{1}{(x-3)^2+4}\,dx$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$ ✓

d) $P(x) = x^4 + 3x^3 - 6x^2 - 28x - 24$

$P'(x) = 4x^3 + 9x^2 - 12x - 28$

$P''(x) = 12x^2 + 18x - 12$

Zero of multiplicity 3 $\Rightarrow P''(x) = 0$

$\therefore 2x^2 + 3x - 2 = 0$

$(2x-1)(x+2) = 0$

$x = \frac{1}{2}$ OR $x = -2$ ✓

either $x = \frac{1}{2}$ or (-2) could be multiple zero

$P(-2) = 16 - 24 - 24 + 56 - 24$
$= 0$

$P'(-2) = -32 + 36 + 24 - 28$
$= 0$ ✓

$\therefore$ Triple zero is $x = -2$ ✓

(Given question stated it did have root of multiplicity 3 showing $P(\frac{1}{2}) \neq 0$ was sufficient here but would not have been if worded differently)

$P(x) = (x+2)^3(x-\alpha)$

Consider constant term $-8\alpha = -24$

$\therefore \alpha = 3$

$\underline{P(x) = (x+2)^3(x-3)}$ ✓

Alternatively: (Product of roots) $\sum \alpha\beta\gamma\delta = \frac{e}{a} = -24$

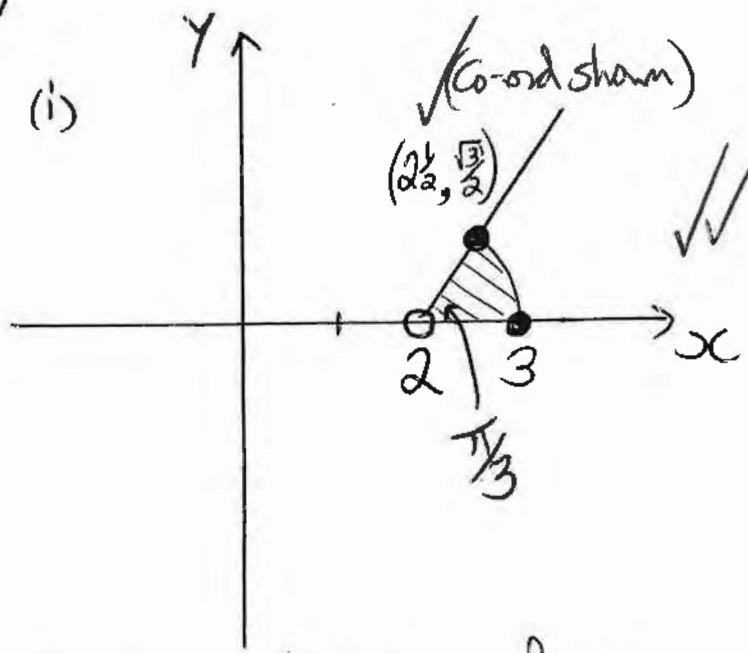
$(-2)^3 \times \alpha = -24$

$-8\alpha = -24$

$\alpha = 3$

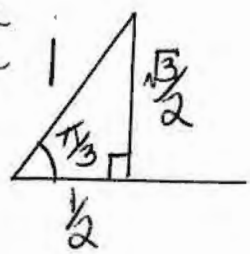
12a

(i)



✓ (Co-ord shown)

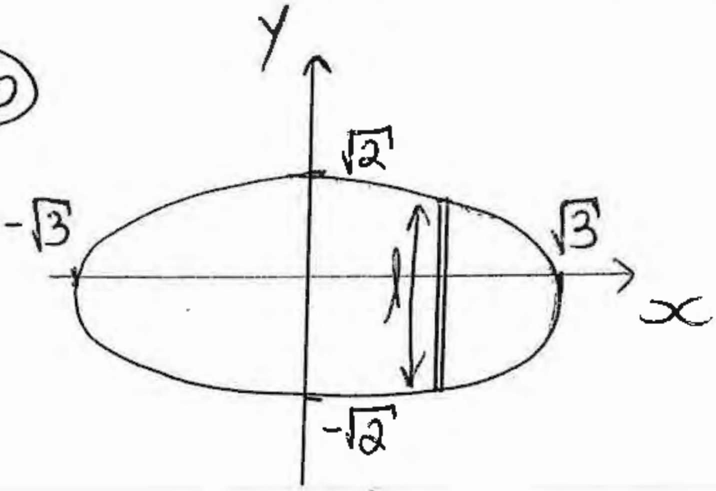
✓✓ (Must have open circle at (2,0))

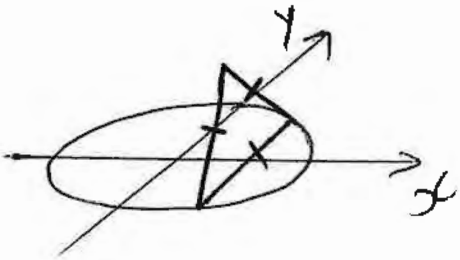


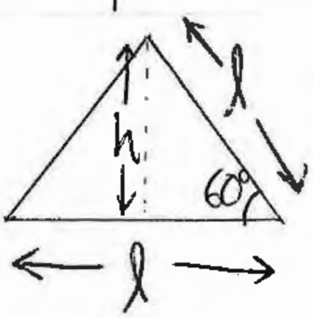
(ii) Greatest value of $\arg z$ occurs at $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$

hence max $\arg z = \underline{\tan^{-1}\left(\sqrt{3}/5\right)}$ ✓

12b







$$h = l\sin 60° = \frac{\sqrt{3}\,l}{2}$$

$$A = \frac{1}{2}lh = \frac{1}{2}l \times \frac{\sqrt{3}\,l}{2} = \frac{\sqrt{3}\,l^2}{4}$$

$$\frac{x^2}{3}+\frac{y^2}{2}=1$$

$$y^2=2\left(1-\frac{x^2}{3}\right)$$

$$y=\pm\sqrt{\tfrac{2}{3}(3-x^2)}$$ ✓

hence $l=2\sqrt{\tfrac{2}{3}(3-x^2)}$

$$l^2=4\left(\tfrac{2}{3}[3-x^2]\right)$$

$$=\tfrac{8}{3}(3-x^2)$$

$$\text{Area}=\frac{\sqrt{3}}{4}\times\frac{8}{3}(3-x^2)$$

$$=\frac{2}{\sqrt{3}}(3-x^2)$$

$$=2\sqrt{3}\left(1-\tfrac{x^2}{3}\right)\ m^2$$ ✓SHOW

(ii) $$V=\int_{x=-\sqrt{3}}^{\sqrt{3}}2\sqrt{3}\left(1-\tfrac{x^2}{3}\right)dx$$

$$=\int_0^{\sqrt{3}}4\sqrt{3}\left(1-\tfrac{x^2}{3}\right)dx \quad \text{(by even symmetry)}$$

$$=4\sqrt{3}\left[x-\tfrac{x^3}{9}\right]_0^{\sqrt{3}}$$

$$=4\sqrt{3}\left(\sqrt{3}-\tfrac{3\sqrt{3}}{9}\right)$$

$$=4\sqrt{3}\left(\frac{2\sqrt{3}}{3}\right)$$

$$=8\ m^3$$ ✓

c) $xy = c^2$ $\quad P\left(cp, \frac{c}{p}\right)$

Differentiate parametrically

$$\frac{dx}{dp} = c \quad \frac{dy}{dp} = -\frac{c}{p^2}$$

$$\frac{dy}{dx} = \frac{dy/dp}{dx/dp} = -\frac{1}{p^2} \checkmark$$

or explicitly

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\left(\frac{dy}{dx}\right)_p = \frac{-c^2}{(cp)^2} = -\frac{1}{p^2}$$

or implicitly

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\left(\frac{dy}{dx}\right)_p = \frac{-c/p}{cp} = -\frac{1}{p^2}$$

(i) Gradient of normal at P $= p^2$

$$\left.\begin{aligned} y - \frac{c}{p} &= p^2(x - cp) \\ cp^3 - \frac{c}{p} &= p^3x - y \end{aligned}\right\} \text{"SHOW"} \checkmark$$

$$c\left(p^2 - \frac{1}{p^2}\right) = px - \frac{y}{p} \quad \text{as required}$$

Method 1

(ii) Gradient of PQ $= \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \dfrac{\frac{p-q}{pq}}{q-p} = -\dfrac{1}{pq} \checkmark$

Equating gradient of normal and chord PQ

$$p^2 = -\frac{1}{pq}$$

$$\therefore q = -\frac{1}{p^3} \checkmark$$

or Method 2

Find point of intersection of normal and hyperbola

$$xy = c^2 \quad ①$$

$$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right) \quad ②$$

from ① $y = c^2/x$

sub in ② $\quad px - \dfrac{c^2}{px} = c\left(p^2 - \dfrac{1}{p^2}\right)$

$$p^2x^2 - cpx\left(p^2 - \frac{1}{p^2}\right) - c^2 = 0 \checkmark$$

roots of this eqn are x values of points P & Q.

$\sum \alpha\beta$ (Product of roots) $\therefore cp \times cq = \dfrac{-c^2}{p^2} \checkmark$

$$p = -\frac{1}{q^3}$$

or $\sum \alpha$ (Sum of roots) $\quad cp + cq = \dfrac{cp\left(p^2 - \frac{1}{p^2}\right)}{p^2}$

Method 3

The normal meets the hyperbola again at Q hence $Q(cq, \frac{c}{q})$ satisfies eqn of normal.

$$pcq - \frac{c}{pq} = c(p^2 - \frac{1}{p^2})$$

$$pq - p^2 = \frac{1}{pq} - \frac{1}{p^2} \qquad c \neq 0$$

$$p(q-p) = \frac{1}{p^2 q}(p-q) \qquad p \neq q$$

$$p = -\frac{1}{p^2 q}$$

$$q = -\frac{1}{p^3}$$

12d

(i) $P(x) = x^4 - 8x^3 + 16x^2 - 1$

For related eqn replace x with $(x+2)$

$$P(x+2) = (x+2)^4 - 8(x+2)^3 + 16(x+2)^2 - 1$$

$$= x^4 + 4x^3 \times 2 + 6x^2 \times 4 + 4x \times 8 + 16$$
$$- 8(x^3 + 6x^2 + 12x + 8) + 16x^2 + 64x + 64 - 1$$

$$= x^4 + 8x^3 - 8x^3 + 24x^2 - 48x^2 + 16x^2$$
$$+ 32x - 96x + 64x + 16 - 64 + 64 - 1$$

$$= x^4 - 8x^2 + 15$$

hence $\underline{x^4 - 8x^2 + 15 = 0}$ is monic polynomial eqn with roots $(\alpha - 2), (\beta - 2) \ldots$

(ii) $(x^2 - 5)(x^2 - 3) = 0$

$$(x + \sqrt{5})(x - \sqrt{5})(x + \sqrt{3})(x - \sqrt{3}) = 0$$

$$x = \pm\sqrt{3}, \pm\sqrt{5}$$

Hence solutions to original quartic eqn

$\underline{x = 2 \pm \sqrt{3}, 2 \pm \sqrt{5}}$

13

a) (i) $z^2 - \bar{z}^2 = -8i$

$(z-\bar{z})(z+\bar{z}) = -8i$

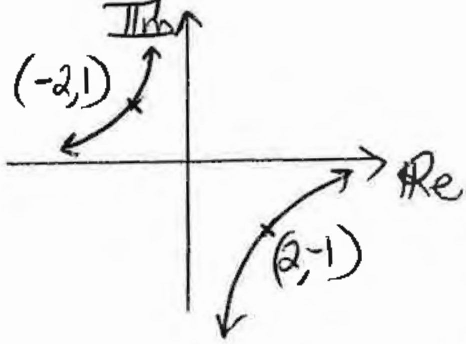
Let $z = x+iy$

$\bar{z} = x-iy$

$2iy \times 2x = -8i$

$4xyi = -8i$

$xy = -2$ ✓



✓ (Do not penalise no points)

or $(x+iy)^2 - (x-iy)^2 = -8i$

$(x^2+2xyi-y^2)-(x^2-2xyi-y^2) = -8i$

$4xyi = -8i$

(ii) $2|z| = z + \bar{z} + 4$

Let $z = x+iy$.

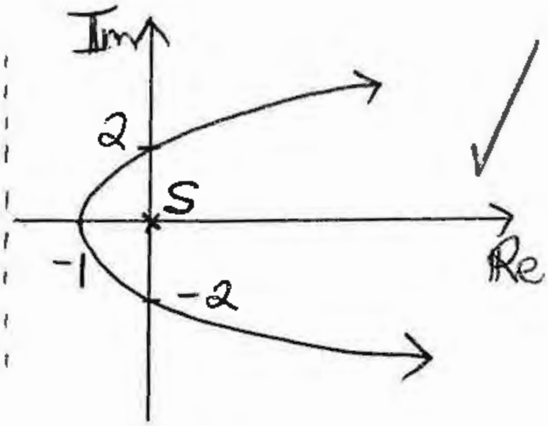
$2\sqrt{(x+iy)(x-iy)} = (x+iy)+(x-iy)+4$

$2\sqrt{x^2+y^2} = 2x+4$

$\sqrt{x^2+y^2} = x+2$

$x^2+y^2 = x^2+4x+4$

$y^2 = 4(x+1)$ ✓



✓

b) $I=\int_0^{\pi/3} \sec x \, dx$

$t = \tan \frac{x}{2}$

$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$

$dt = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) dx$

$\frac{2dt}{1+t^2} = dx$

$\sec x = \frac{1}{\cos x}$

$= \frac{1+t^2}{1-t^2}$

✓

x	0	$\frac{\pi}{3}$
t	0	$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$I = \int_0^{1/\sqrt{3}} \frac{1+t^2}{1-t^2} \times \frac{2dt}{1+t^2}$$

$$= \int_0^{1/\sqrt{3}} \frac{2\,dt}{1-t^2} \quad ✓$$

$\frac{2}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t}$

(Using cover-up rule)

$A=1, \; B=1$

$$= \int_0^{1/\sqrt{3}} \frac{1}{1-t} + \frac{1}{1+t} \, dt \quad ✓$$

$$= \Big[-\ln|1-t| + \ln|1+t|\Big]_0^{1/\sqrt{3}}$$

$$= \left[\ln\left(\frac{1+t}{1-t}\right)\right]_0^{1/\sqrt{3}}$$

$$= \ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right) - \ln 1$$

$$= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) \quad \frac{(\times\sqrt{3}+1)}{(\times\sqrt{3}+1)}$$

$$= \ln\left(\frac{3+2\sqrt{3}+1}{3-1}\right)$$

$$= \ln(2+\sqrt{3})$$

"SHOW"

✓

13c $P(z) = z^3 + az + b$ has zeroes α, β and $3(\alpha-\beta)$

(i) $\left(\Sigma\alpha\right)$ ie sum of roots

$$\alpha+\beta+3(\alpha-\beta) = 0$$
$$4\alpha - 2\beta = 0$$
$$\beta = 2\alpha \quad \checkmark$$

$\left(\Sigma\alpha\beta\right)$ sum of pairs

$$\alpha\beta + 3\alpha(\alpha-\beta) + 3\beta(\alpha-\beta) = a$$
$$3\alpha^2 - 3\beta^2 + \alpha\beta = a$$

sub $\beta = 2\alpha$

$$3\alpha^2 - 3\times 4\alpha^2 + 2\alpha^2 = a \quad \checkmark \text{"SHOW"}$$
$$a = -7\alpha^2 \quad (1) \text{ (as required)}$$

(ii) $\left(\Sigma\alpha\beta\gamma\right)$ product of roots

$$3\alpha\beta(\alpha-\beta) = -b \quad \checkmark \text{"SHOW"}$$
$$6\alpha^2(-\alpha) = -b$$
$$b = 6\alpha^3 \quad (2) \text{ (as required)}$$

(iii) (2) ÷ (1)

$$\frac{b}{a} = \frac{6\alpha^3}{-7\alpha^2}$$
$$-\frac{7b}{6a} = \alpha$$
$$\therefore \beta = 2\alpha$$
$$= -\frac{7b}{3a}$$
$$3(\alpha-\beta) = 3\left(\frac{-7b}{6a} - \frac{-7b}{3a}\right) \quad \checkmark \text{"SHOW"}$$
$$= \frac{-7b}{2a} + \frac{7b}{a}$$
$$= \frac{7b}{2a}$$

Zeroes of $P(z)$ are $-\frac{7b}{6a}$, $\frac{-7b}{3a}$ and $\frac{7b}{2a}$

(13d) Question means solve $z^2-4z+29=0$ ✓

$(z-2)^2+25=0$ ✓

$(z-2)^2=-25$ ✓

$z-2=\pm 5i$ ✓

$z=2\pm 5i$ ✓

alternatively: let roots be α & β.

$\alpha+\beta=4$ ① $\alpha\beta=29$ ②

sub ② into ① $\alpha+\frac{29}{\alpha}=4$

$\alpha^2-4\alpha+29=0$ as before ...

alternatively: let roots be $a+ib$ and $a-ib$ justifying use of conjugates since sum and product are REAL. ✓

$(a+ib)(a-ib)=29$

$a^2+b^2=29$ ① ✓ Pair of Eqns

$(a+ib)+(a-ib)=4$

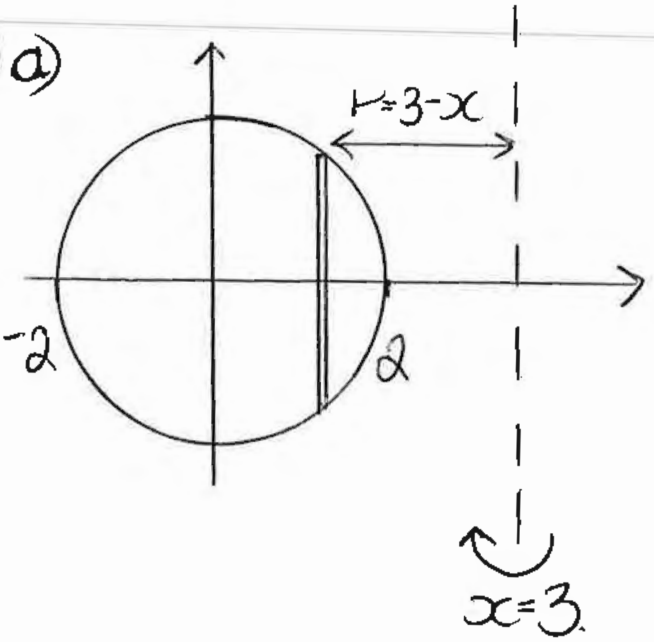
$2a=4$ ② $\Rightarrow$ $a=2$

$b^2=25$

$b=\pm 5$

$z=2+5i$ and $2-5i$ ✓

14 a)



$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

$$r = 3 - x$$
$$dr = (-dx)$$
$$r_{IN} \Rightarrow x = 2$$
$$r_{OUT} \Rightarrow x = (-2)$$
$$h = 2\sqrt{4 - x^2}$$

$$V = \int_{x=2}^{-2} 2\pi(3 - x)\,2\sqrt{4 - x^2}\,(-dx)$$

$$= \int_{-2}^{2} 4\pi(3 - x)(4 - x^2)^{\frac{1}{2}}\,dx$$

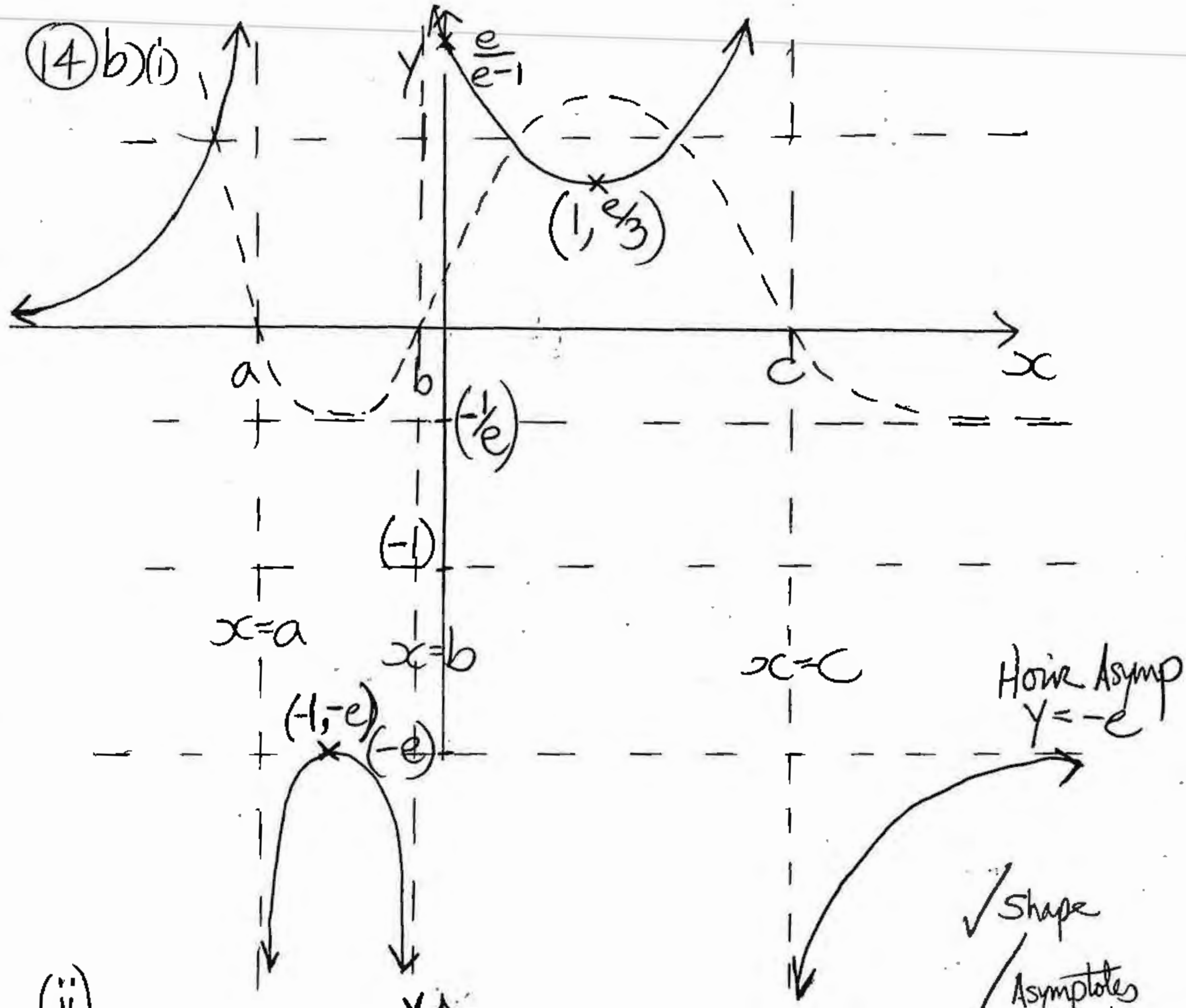
$$= 4\pi\int_{-2}^{2} 3\sqrt{4 - x^2}\,dx - 4\pi\int_{-2}^{2} x(4 - x^2)^{\frac{1}{2}}\,dx$$

Area of semicircle radius 2 — Odd symmetry over symmetric interval

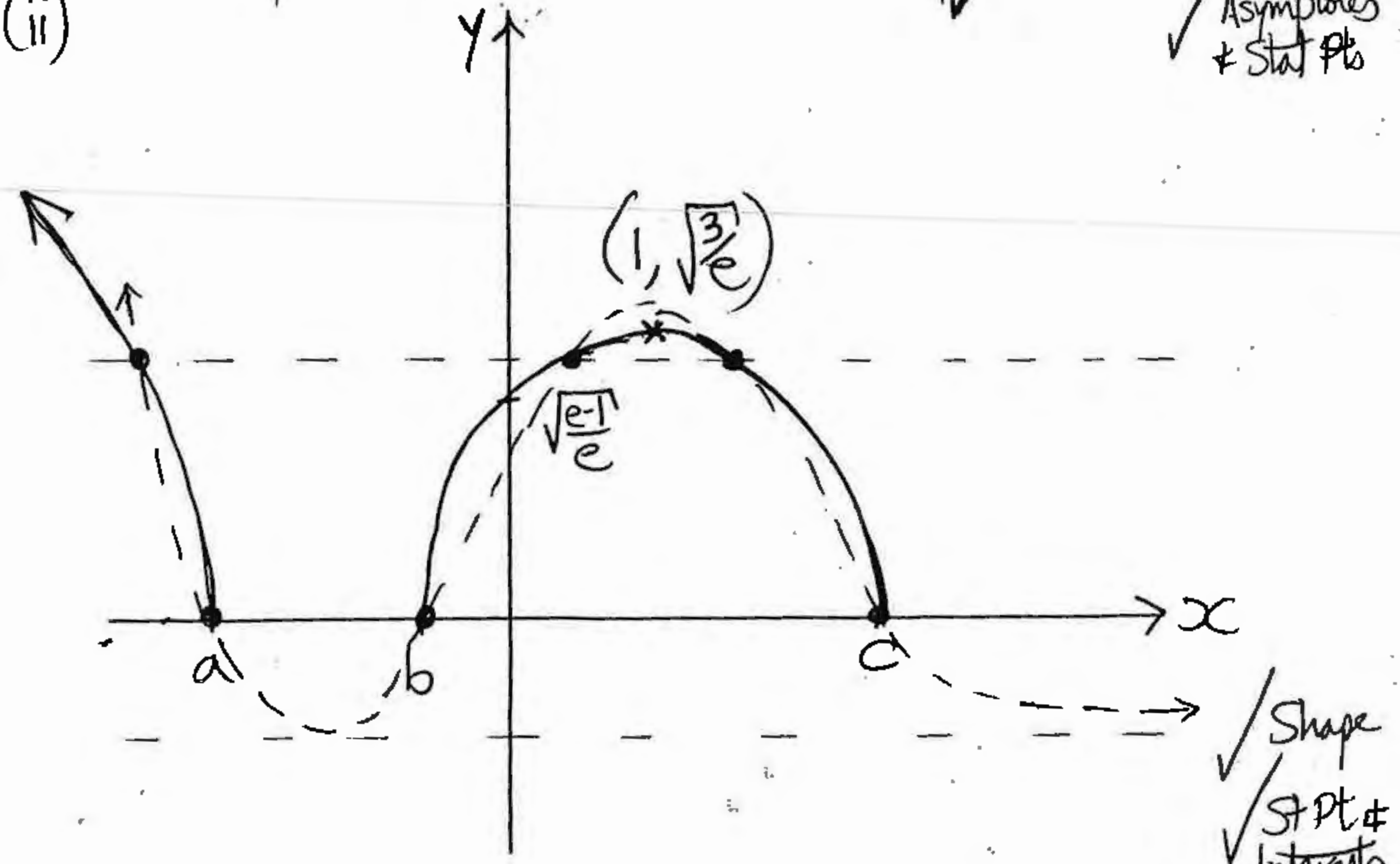
$$= 4\pi \times 3 \times \tfrac{1}{2}\pi 2^2 - 0$$

$$= 24\pi^2\ u^3$$ as required

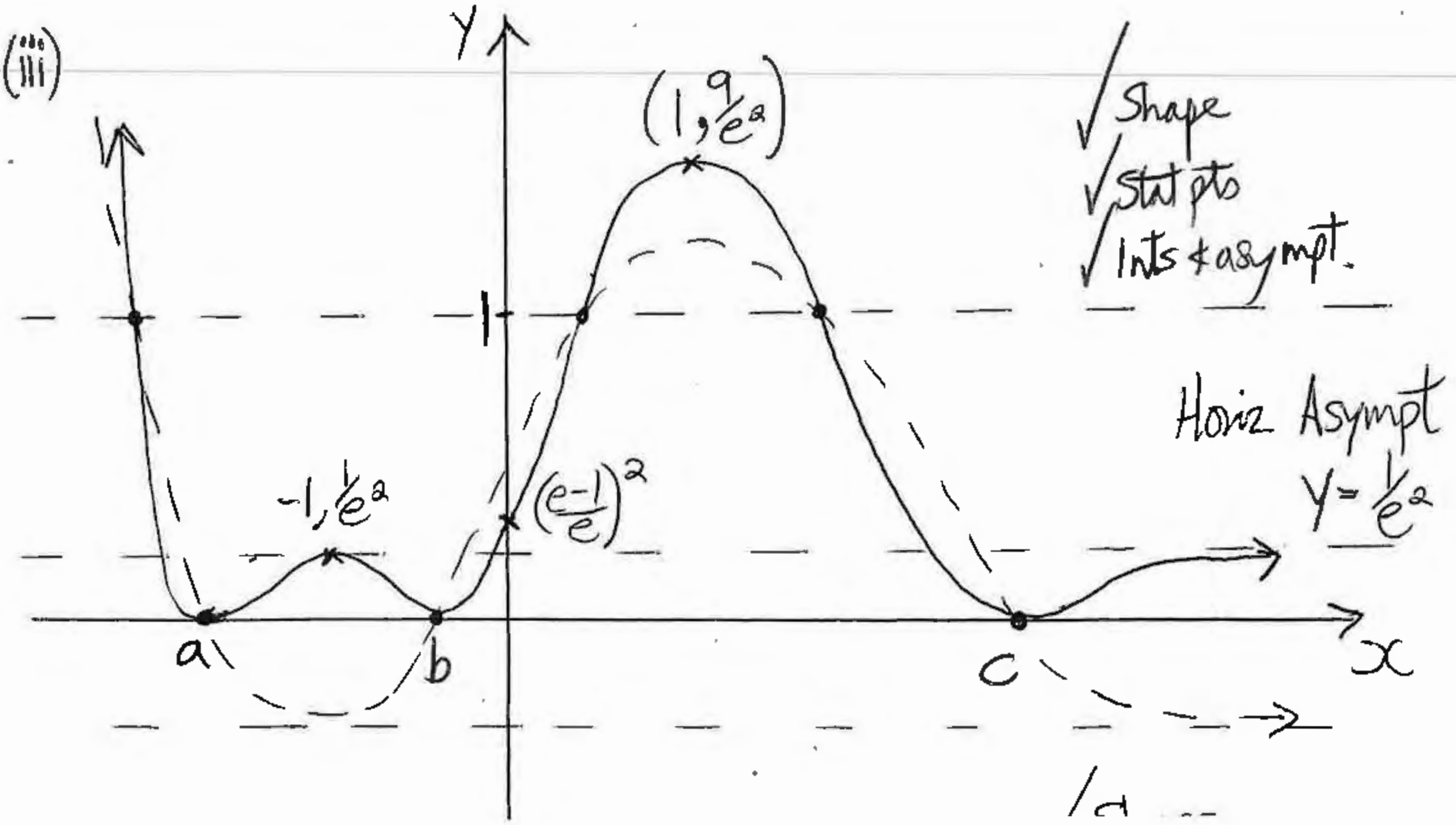
14 b) (i)


y
$\frac{e}{e-1}$
$(1, \frac{e}{3})$
x
a
b
c
$(-\frac{1}{e})$
(-1)
$x=a$
$x=b$
$x=c$
Horiz Asymp
$y=-e$
$(-1,-e)$
$(-e)$
✓ Shape
✓ Asymptotes & Stat Pts

(ii)


y
$(1, \sqrt{\frac{3}{e}})$
$\sqrt{\frac{e-1}{e}}$
x
a
b
c
✓ Shape
✓ St Pt & Intercepts

(iii)



(14c) $I_n = \int_0^a (a^2-x^2)^n \, dx$ $\qquad u = (a^2-x^2)^n \quad \frac{dv}{dx} = 1$

$\qquad \frac{du}{dx} = n(a^2-x^2)^{n-1} \times (-2x) \quad v = x$

$= \left[x(a^2-x^2)^n\right]_0^a - \int_0^a -2nx(a^2-x^2)^{n-1} \times x \, dx$ ✓

$= 0 + 2n\int_0^a x^2(a^2-x^2)^{n-1} \, dx$

$= 2n\int_0^a -(a^2-x^2-a^2)(a^2-x^2)^{n-1} \, dx$ ✓

$= -2n\int_0^a (a^2-x^2)^n dx + 2na^2\int (a^2-x^2)^{n-1} dx$

✓ "SHOW"

$I_n = -2nI_n + 2na^2 I_{n-1}$

$(1+2n)I_n = 2na^2 I_{n-1}$

$I_n = \frac{2na^2}{1+2n} I_{n-1}$ as required

(ii) $\int_0^2 (4-x^2)^3\,dx \Rightarrow a=2 \quad n=3$

$I_0 = \int_0^2 (4-x^2)^0\,dx$

$= \int_0^2 dx$

$= [x]_0^2$

$= 2.$

or $I_1 = \int_0^2 4-x^2\,dx$

$= \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3}$

$= \frac{16}{3}$

$I_3 = \frac{2\times3\times2^2}{1+2\times3} I_2$

$= \frac{24}{7} I_2$

$= \frac{24}{7} \times \frac{16}{5} I_1$

$= \frac{24}{7} \times \frac{16}{5} \times \frac{8}{3} I_0$

$= \frac{24}{7} \times \frac{16}{5} \times \frac{8}{3} \times 2$ ✓

$= \frac{2048}{35}$

Alternatively: Binomial expansion of integrand

$(4-x^2)^3 = 4^3 + 3\times4^2\times(-x^2) + 3\times4\times(-x^2)^2 + (-x^2)^3$

$= 64 - 48x^2 + 12x^4 - x^6$

$\int_0^2 64 - 48x^2 + 12x^4 - x^6\,dx$

$= \left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right]_0^2$

$= \left(\cancel{128} - \cancel{128} + \frac{12\times32}{5} - \frac{128}{7}\right) - 0$

$= \frac{2048}{35}$

(15a) $(1+i)^n = 1^n + \binom{n}{1}1^{n-1}i + \binom{n}{2}1^{n-2}i^2 + \binom{n}{3}i^3 + \ldots + \binom{n}{n}i^n$

$\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^n = 1 + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i + \binom{n}{4} + \ldots + i^n\binom{n}{n}$ ✓

By DeMoivres $2^{\frac{n}{2}}\operatorname{cis}\frac{n\pi}{4} = \left[1 - \binom{n}{2} + \binom{n}{4} - \ldots \binom{n}{n}i^4\right] + i\left[\binom{n}{1} - \binom{n}{3} + \ldots\right]$

since n even $i^n = \pm 1$

if n a multiple of four $i^n = +1$

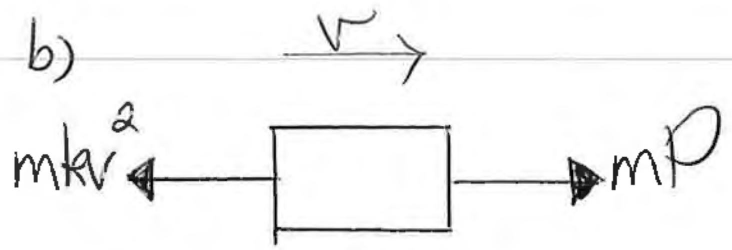
otherwise $i^n = (-1)$

✓"SHOW"

Equating Real Parts

$$2^{\frac{n}{2}}\cos\frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \ldots + (-1)^{\frac{n}{2}}\binom{n}{n}$$

b)



(i) Eqn of motion

$$mP - mkv^2 = ma$$
$$P - kv^2 = v\frac{dv}{dx}$$
$$\frac{P - kv^2}{v} = \frac{dv}{dx} \quad \checkmark \text{ "SHOW"}$$
$$\therefore \frac{dx}{dv} = \frac{v}{P - kv^2}$$

(ii)
$$\int_0^x dx = \int_0^v \frac{v}{P - kv^2}\,dv$$

NB Car initially at rest at origin

$P > kv^2$

$$x = \left[-\frac{1}{2k}\ln\left(P - kv^2\right)\right]_0^v$$
$$= \frac{1}{2k}\ln P - \frac{1}{2k}\ln\left(P - kv^2\right) \checkmark$$
$$2kx = \ln\left(\frac{P}{P - kv^2}\right)$$
$$e^{2kx} = \frac{P}{P - kv^2} \quad \checkmark \text{ "SHOW"}$$
$$P - kv^2 = Pe^{-2kx}$$
$$kv^2 = P\left(1 - e^{-2kx}\right)$$
$$v^2 = \frac{P}{k}\left(1 - e^{-2kx}\right) \quad \text{as required}$$

Max velocity of car as $x \to \infty$ $\quad e^{-2kx} \to 0$

$$v^2 \to \frac{P}{k} \qquad v \to \sqrt{\frac{P}{k}}\ \text{m/s}^{-1} \checkmark$$

Alternatively Terminal Velocity

No Net Force $\quad mP = mkv^2 \qquad v^2 = \frac{P}{k} \quad v = \sqrt{\frac{P}{k}} \quad (v \geq 0)$

(iii) $x = \frac{1}{2k} \ln\left(\frac{P}{P-kV^2}\right)$ $\qquad V_M = \sqrt{\frac{P}{k}}$

Distance to reach $\frac{V_M}{2}$ $\qquad x_{\frac{1}{2}V_M} = \frac{1}{2k} \ln\left(\frac{P}{P - k \times \frac{1}{4}\frac{P}{k}}\right)$

$$= \frac{1}{2k} \ln\left(\frac{P}{3P/4}\right)$$

$$= \frac{1}{2k} \ln \frac{4}{3}$$

$\frac{V_M}{3}$ $\qquad x_{\frac{V_M}{3}} = \frac{1}{2k} \ln\left(\frac{P}{P - k \times \frac{1}{9}\frac{P}{k}}\right)$

$$= \frac{1}{2k} \ln\left(\frac{P}{8P/9}\right)$$

$$= \frac{1}{2k} \ln \frac{9}{8}$$

$\therefore \frac{\frac{1}{3}V_M}{\frac{1}{2}V_M} = \frac{\ln \frac{9}{8}}{\ln \frac{4}{3}}$ ✓ "SHOW"

$\doteqdot 0.409\ldots$

Hence distance required to reach $\frac{1}{3}V_M$ is 41% of distance required to reach $\frac{1}{2}V_M$.

$-\left[\sin^2\frac{2\pi}{5}+\sin^2\frac{\pi}{5}\right]=-\frac{5}{4}$ ✓ "SHOW"

$\sin^2\frac{\pi}{5}+\sin^2\frac{2\pi}{5}=\frac{5}{4}$ as required

(iv) Using double angle formula $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$
and Pythagorean identity $\sin^2\theta=(1-\cos^2\theta)$ ✓

$\frac{1}{2}\left(1-\cos\frac{2\pi}{5}\right)+\left(1-\cos^2\frac{2\pi}{5}\right)=\frac{5}{4}$

$\cos^2\frac{2\pi}{5}+\frac{1}{2}\cos\frac{2\pi}{5}=\frac{3}{2}-\frac{5}{4}$

$=\frac{1}{4}$ ✓

$(\times 4)$ $4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$ ✓ Let $u=\cos\frac{2\pi}{5}$

ie $4u^2+2u-1=0$

$\Delta=4-4\times4\times(-1)$

$=20$

$u=\frac{-2\pm\sqrt{20}}{8}$

$u=\frac{-1\pm\sqrt{5}}{4}$

✓ "SHOW"

$\therefore \cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$ (since $\frac{2\pi}{5}$ is first quadrant $\therefore \cos\frac{2\pi}{5}>0$)

c)(i) $z^5 = (\cos\theta + i\sin\theta)^5$

by De Moivres theorem

$$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

✓ "SHOW"

$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta \quad \text{as required}$$

(ii) $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin\theta(16\sin^4\theta - 20\sin^2\theta + 5)$

sub $x = \sin\theta$.

consider ①, $\sin 5\theta = 0$ means $x = 0$

or $16x^4 - 20x^2 + 5 = 0$

distinct solutions of eqn ①

$5\theta = 0, \pm\pi, \pm 2\pi$

$\theta = 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$

$\therefore x = \sin\frac{\pi}{5}, \sin\left(\frac{-\pi}{5}\right), \sin\frac{2\pi}{5} \;\&\; \sin\left(\frac{-2\pi}{5}\right)$ ✓

are four roots of given quartic

(iii) Taking sum of paired roots $= c/a$ and using odd symmetry of sine fn $\sin\left(\frac{-\pi}{5}\right) = -\sin\frac{\pi}{5}$ gives ✓

$$-\sin^2\left(\frac{2\pi}{5}\right) - \sin^2\left(\frac{\pi}{5}\right) + \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right) - \sin\frac{\pi}{5}\sin\frac{2\pi}{5} - \sin\frac{\pi}{5}\sin\frac{2\pi}{5} + \sin\frac{\pi}{5}\sin\frac{2\pi}{5}$$

$$= -20/16$$

d) $P(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)$

take logarithms

$\ln P(x) = \ln(x-\alpha_1) + \ln(x-\alpha_2) + \dots + \ln(x-\alpha_n)$ ✓

differentiate wrt x

$$\frac{1}{P(x)} \times P'(x) = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \dots + \frac{1}{x-\alpha_n}$$ ✓

$$P'(x) = P(x)\left[\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \dots + \frac{1}{x-\alpha_n}\right]$$

Alternative Method: (Extended Product Rule)

$$P(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

$$\begin{aligned}P'(x) = {} & 1\times(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) + (x-\alpha_1)\times 1\times(x-\alpha_3)\dots(x-\alpha_n) \\ & + (x-\alpha_1)(x-\alpha_2)\times 1\times(x-\alpha_3)\dots(x-\alpha_n) + \dots \\ & + (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{n-1})\times 1\end{aligned}$$ ✓

$$= \frac{P(x)}{x-\alpha_1} + \frac{P(x)}{x-\alpha_2} + \frac{P(x)}{x-\alpha_3} + \dots + \frac{P(x)}{x-\alpha_n}$$ ✓

$$= P(x)\left[\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \dots + \frac{1}{x-\alpha_n}\right]$$ ✓

16 a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $P(a\cos\theta, b\sin\theta)$ $Q(a\cos\phi, b\sin\phi)$

$$y-b\sin\phi=\frac{b\sin\phi-b\sin\theta}{a\cos\phi-a\cos\theta}(x-a\cos\phi)$$

Focal chord passes through $(ae,0)$

$$-b\sin\phi(a\cos\phi-a\cos\theta)=(b\sin\phi-b\sin\theta)(ae-a\cos\phi)$$

$$\sin\phi(\cos\theta-\cos\phi)=(\sin\phi-\sin\theta)(e-\cos\phi)$$

$$\frac{\sin\phi(\cos\theta-\cos\phi)}{\sin\phi-\sin\theta}=e-\cos\phi$$

$$\frac{\sin\phi\cos\theta-\cancel{\sin\phi\cos\phi}+\cos\phi(\cancel{\sin\phi}-\sin\theta)}{\sin\phi-\sin\theta}=e \quad \checkmark \text{"SHOW"}$$

$$e=\frac{\sin(\phi-\theta)}{\sin\phi-\sin\theta}$$

(ii) $\angle PRQ=90^\circ \Rightarrow m_{PR}\times m_{RQ}=-1$ $R(a,0)$

$m_{PR}=\frac{b\sin\theta}{a(\cos\theta-1)}$ $m_{RQ}=\frac{b\sin\phi}{a(\cos\phi-1)}$

$$\frac{b\sin\theta}{a(\cos\theta-1)}\times\frac{b\sin\phi}{a(\cos\phi-1)}=-1 \quad \checkmark$$

$$-\frac{b^2}{a^2}=\frac{(\cos\theta-1)(\cos\phi-1)}{\sin\theta\sin\phi}$$

Method 1: Double Angle Formulae

$$\text{RHS} = \frac{\left(1-2\sin^2\left(\frac{\theta}{2}\right)-1\right)\left(1-2\sin^2\left(\frac{\phi}{2}\right)-1\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$ ✓

using
$\sin 2A = 2\sin A\cos A$
$\cos 2A = 1-2\sin^2 A$

$$= \frac{4\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}}{4\sin\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}}$$ ✓ "show"

$$= \tan\frac{\theta}{2}\tan\frac{\phi}{2}$$ (as required)

Method 2: Using t-formulae

Consider $\left(\frac{\cos\theta - 1}{\sin\theta}\right)$

Let $t = \tan\frac{\theta}{2}$
$\sin\theta = \frac{2t}{1+t^2}$
$\cos\theta = \frac{1-t^2}{1+t^2}$

$$= \frac{\frac{1-t^2}{1+t^2} - 1}{\frac{2t}{1+t^2}} \begin{matrix} \times(1+t^2) \\ \times(1+t^2) \end{matrix}$$ ✓

$$= \frac{1-t^2-(1+t^2)}{2t}$$

$$= \frac{-2t^2}{2t}$$

$$= -t$$

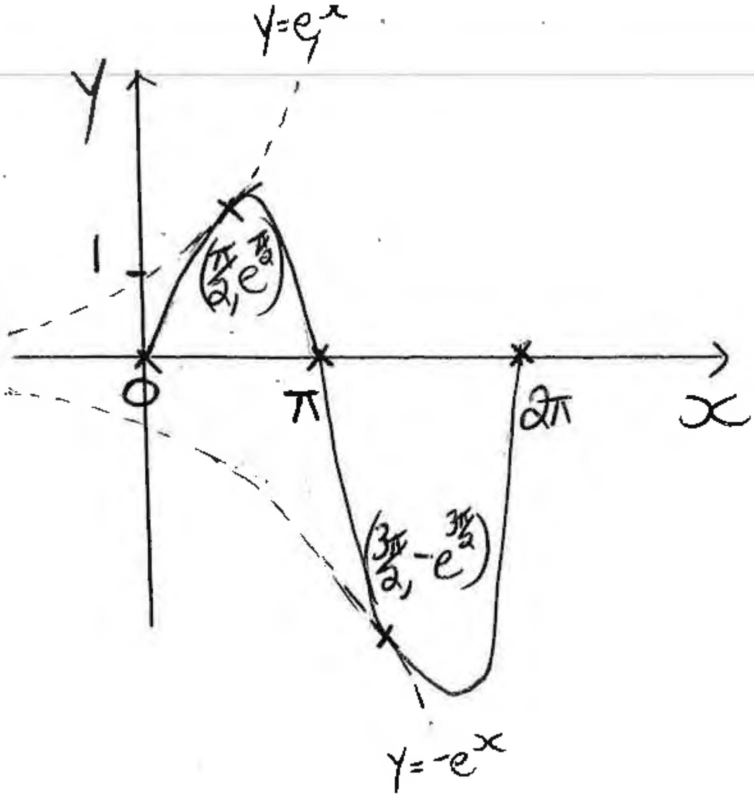
$$= -\tan\frac{\theta}{2}$$

similarly $\frac{\cos\phi - 1}{\sin\phi} = -\tan\frac{\phi}{2}$ ✓

$$\therefore -\frac{b^2}{a^2} = \left(-\tan\frac{\theta}{2}\right)\times\left(-\tan\frac{\phi}{2}\right)$$

$$= \tan\frac{\theta}{2}\tan\frac{\phi}{2}$$ (as required)

b)



✓ Shape with x-ints clearly. Must indicate second arch significantly larger ~~than first~~

Note: Pts $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$ were good points to mark BUT are not in fact the stationary pts of this function.

(ii) $I = \displaystyle\int_{(k-1)\pi}^{k\pi} e^x \sin x \, dx$

By parts $u = e^x$, $\frac{dv}{dx} = \sin x$, $\frac{du}{dx} = e^x$, $v = -\cos x$

$$= \left[-e^x \cos x\right]_{(k-1)\pi}^{k\pi} - \int_{(k-1)\pi}^{k\pi} -e^x \cos x \, dx$$

$U = e^x$, $\frac{dV}{dx} = \cos x$, $\frac{dU}{dx} = e^x$, $V = \sin x$

$$= \left[-e^x \cos x\right]_{(k-1)\pi}^{k\pi} + \left[e^x \sin x\right]_{(k-1)\pi}^{k\pi} - \int_{(k-1)\pi}^{k\pi} e^x \sin x \, dx \quad ✓$$

$$I = \left[e^x(\sin x - \cos x)\right]_{(k-1)\pi}^{k\pi} - I$$

$$2I = \left[e^x(\sin x - \cos x)\right]_{(k-1)\pi}^{k\pi}$$

$$= e^{k\pi}(\sin k\pi - \cos k\pi) - e^{(k-1)\pi}(\sin(k-1)\pi - \cos(k-1)\pi)$$

for integer k.

$\sin k\pi = 0$

$\sin (k-1)\pi = 0$

$$2I = -e^{k\pi} \cos k\pi + e^{(k-1)\pi} \cos (k-1)\pi \quad ✓$$

$\cos k\pi = 1$ if k even
or -1 if k odd

$\therefore \cos k\pi = (-1)^k$ similarly $\cos(k-1)\pi = (-1)^{k-1}$

$\therefore 2I = (-1)^k(-e^{k\pi}) + (-1)^{k-1} e^{(k-1)\pi}$

$= (-1)^{k-1} e^{k\pi} + (-1)^{k-1} e^{k\pi} \times e^{-\pi}$

$= (-1)^{k-1} e^{k\pi}(1+e^{-\pi})$ (as required) ✓

Alternatively: $u = \sin x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = \cos x$ $\quad v = e^x$

$I = \left[e^x \sin x\right]_{(k-1)\pi}^{k\pi} - \int_{(k-1)\pi}^{k\pi} e^x \cos x \, dx$

$U = \cos x$ $\quad \frac{dV}{dx} = e^x$

$\frac{dU}{dx} = -\sin x$ $\quad V = e^x$

$= \left[e^x \sin x\right] - \left\{\left[e^x \cos x\right] - \int -\sin x \, e^x \, dx\right\}$

$I = \left[e^x \sin x - e^x \cos x\right] - I$ as before

(iii) Required area

$= \int_0^{\pi} e^x \sin dx + \left|\int_{\pi}^{2\pi} e^x \sin x \, dx\right|$

$\uparrow$ $k=1$ $\quad \uparrow$ $k=2$

Note: 2^{nd} integral gives a negative since below x-axis ✓

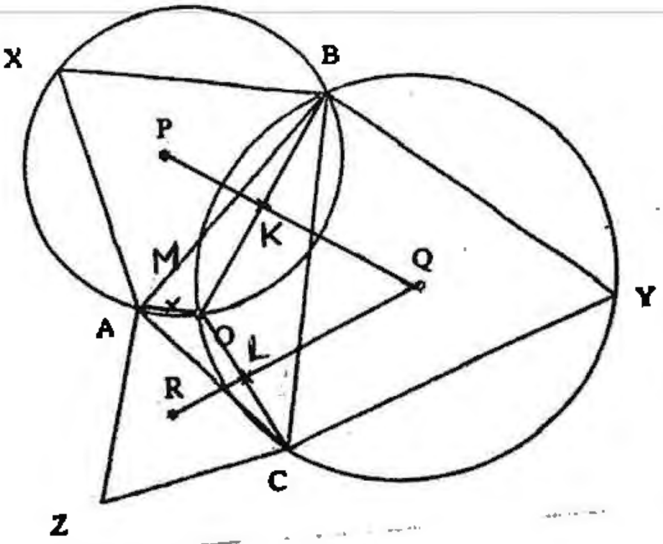
$= \frac{1}{2}e^{\pi}(1+e^{-\pi}) + \left|-\frac{1}{2}e^{2\pi}(1+e^{-\pi})\right|$

$= \frac{e^{\pi}(1+e^{-\pi})(1+e^{\pi})}{2}$

$= \frac{(e^{\pi}+1)(e^{\pi}+1)}{2}$

$= \frac{(e^{\pi}+1)^2}{2} \; u^2$ ✓

16c) c)



(i) $\angle AXB = 60^\circ$ (Equilateral $\triangle XAB$)
$\therefore \angle AOB = 120^\circ$ (Opposite $\angle$ in cyclic quadrilateral XAOB)
similarly $\angle BOC = 120^\circ$ (Equilateral $\triangle$ BCY & Cyclic quad BOCY) ✓
$\angle AOC = 120^\circ$ (Angles at point O OR Full Revolution)

(ii) $\angle AZC = 60^\circ$ (Equilateral $\triangle AZC$)
$\therefore \angle AOC + \angle AZC = 180^\circ$
$\therefore$ AOCZ is a cyclic quadrilateral ✓"SHOW" (Opposite angles are supplementary)

(iii) Construct common chords BO, OC and OA.
Centre of circle P lies on perpendicular bisector of OB. Similarly centre of second circle Q lies on same perp. bisector. Let midpoint of chord be point K. P, K & Q are colinear and $\angle OKQ$ is 90°
Similarly define L as midpt of common chord OC
O, L, C collinear and $\angle OLQ = 90^\circ$ ✓
Hence KQLO a cyclic quadrilateral since opposite angles supplementary $\therefore \angle KOL + \angle KQL = 180^\circ$

but $\angle KOL = \angle BOC$
$= 120°$ from (1)
$\therefore \angle KQL = \angle PQR$
$= 60°$ ✓

similar argument using other pairs of common chords
OA & OB with midpoint of OA defined as M
gives PKOM as cyclic quadrilateral
hence $\angle RPQ = 60°$
and/or OA & OC, OLRM a cyclic quadrilateral
hence $\angle PRQ = 60°$
showing any pairs of two angles in $\triangle PQR = 60°$
proves triangle is equilateral. ✓