CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2019 Trial Examination

FORM VI

MATHEMATICS EXTENSION 2

Monday 12th August 2019

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Question 16 Working Sheet
- Reference sheet
- Candidature — 85 boys

Examiner

DWH/GMC

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which expression is equal to $\displaystyle\int xe^x\,dx$?

(A) $xe^x - e^x + C$

(B) $xe^x + e^x + C$

(C) $xe^x - e^{-x} + C$

(D) $xe^x + e^{-x} + C$

QUESTION TWO

If z is any complex number, which of the following is a correct simplification of $z\overline{z} - |z|^2$?

(A) 1

(B) 0

(C) $(z + \overline{z})(z - \overline{z})$

(D) $(z + iz)(z - iz)$

QUESTION THREE

Let $P(x) = x^4 + x^2 - 12$. Which of the following is a factor of $P(x)$?

(A) $x + 3$

(B) $x + \sqrt{3}$

(C) $x + 3i$

(D) $x + i\sqrt{3}$

QUESTION FOUR

Which expression is always equal to $\displaystyle\int_0^{2a} f(x)\,dx$?

(A) $\displaystyle\int_0^{2a} f(2a-x)\,dx$

(B) $\displaystyle\int_0^{2a} f(a-x)\,dx$

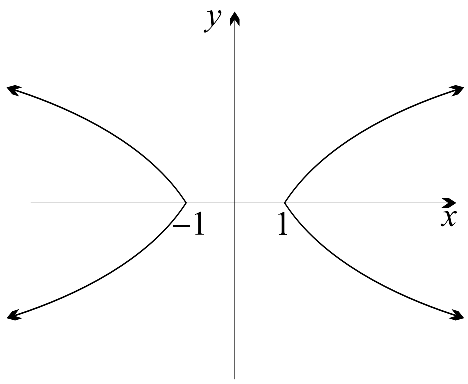
(C) $\displaystyle 2\int_0^{a} f(2a-x)\,dx$

(D) $\displaystyle 2\int_0^{a} f(a-x)\,dx$

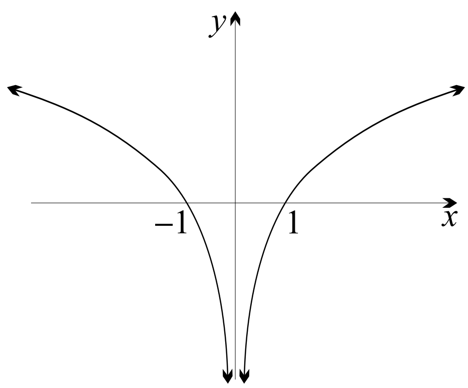
QUESTION FIVE

Which graph best represents the curve $y = \ln|x|$?

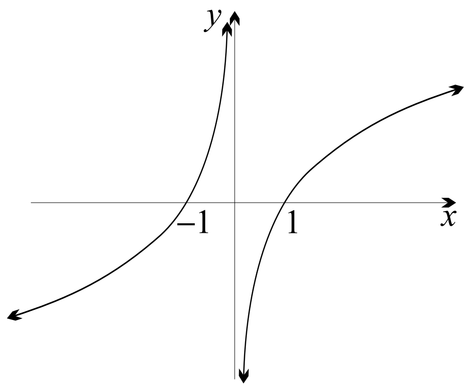
(A)



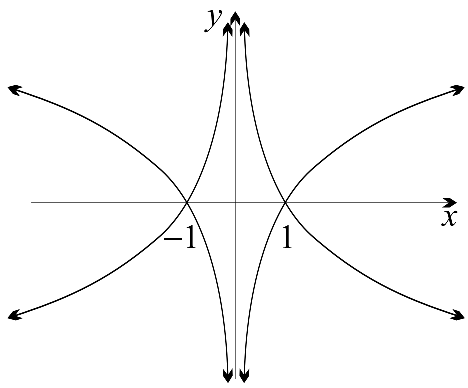
(B)



(C)



(D)



Examination continues overleaf ...

QUESTION SIX

Let $f(x)$ be an odd function and $g(x)$ be an even function.
Which of the following statements is NOT true?

(A) $g(g(x))$ is an even function.

(B) $g(f(x))$ is an even function.

(C) $f(g(x))$ is an odd function.

(D) $f(f(x))$ is an odd function.

QUESTION SEVEN

A particle of mass m kg has acceleration $a\,\mathrm{m/s^2}$ proportional to the square of its velocity v m/s. Hence $ma = -kv^2$, where k is some positive constant. Which of the following integrals will find a relation between the time t seconds and v?

(A) $t = \displaystyle\int \frac{-m}{kv^2}\,dv$

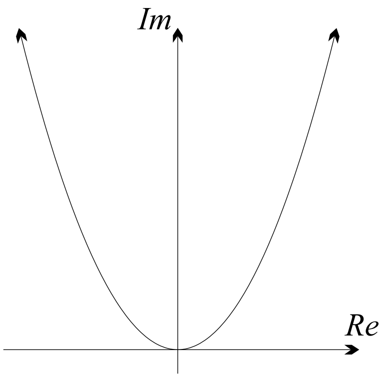
(B) $t = \displaystyle\int \frac{m}{kv^2}\,dv$

(C) $t = \displaystyle\int \frac{-k}{mv^2}\,dv$

(D) $t = \displaystyle\int \frac{k}{mv^2}\,dv$

QUESTION EIGHT

The diagram shows the locus of points representing the complex number z on the Argand plane, being a parabola with focus at the point represented by the number i.



Which equation in z best matches this locus?

(A) $|z + i| = \mathrm{Re}(z) - 1$

(B) $|z + i| = \mathrm{Im}(z) - 1$

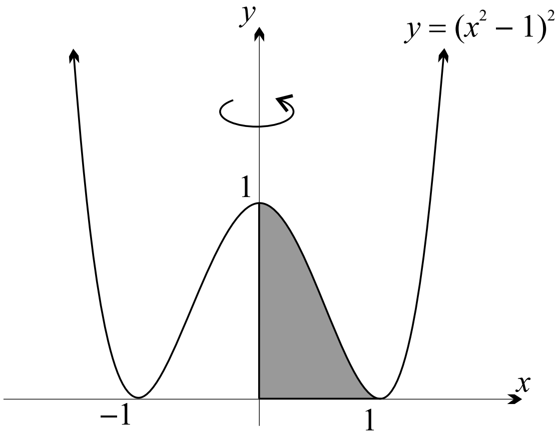
(C) $|z - i| = \mathrm{Re}(z) + 1$

(D) $|z - i| = \mathrm{Im}(z) + 1$

Examination continues next page ...

QUESTION NINE

The region enclosed by the curve $y = (x^2 - 1)^2$ and the axes in the first quadrant is rotated about the y-axis, as shown below.



By slicing perpendicular to the axis of rotation, which of the following gives the volume V of the resulting shape?

(A) $V = \pi \int_0^1 (1 - \sqrt{y})^2 \, dy$

(B) $V = \pi \int_0^1 (1 + \sqrt{y})^2 \, dy$

(C) $V = \pi \int_0^1 (1 - \sqrt{y}) \, dy$

(D) $V = \pi \int_0^1 (1 + \sqrt{y}) \, dy$

QUESTION TEN

The following formula, called a convolution, is used in signal processing to describe the impact of the signal f on the signal g over a period of t seconds:

$$(f * g)(t) = \int_0^t f(s)g(t - s)ds\,.$$

Which of the following is true if $f(x) = 1$ and $g(x) = e^{-x}$?

(A) $(f * g)(t) = e^{-s} - 1$

(B) $(f * g)(t) = 1 - e^{-s}$

(C) $(f * g)(t) = e^{-t} - 1$

(D) $(f * g)(t) = 1 - e^{-t}$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - 3i$ and $w = \sqrt{3} + i$.

(i) Express w in modulus–argument form. **2**

(ii) Express zw in modulus–argument form. **2**

(b) A curve is defined by the pair of parametric equations $x = 5\cos\theta$, $y = 3\sin\theta$.

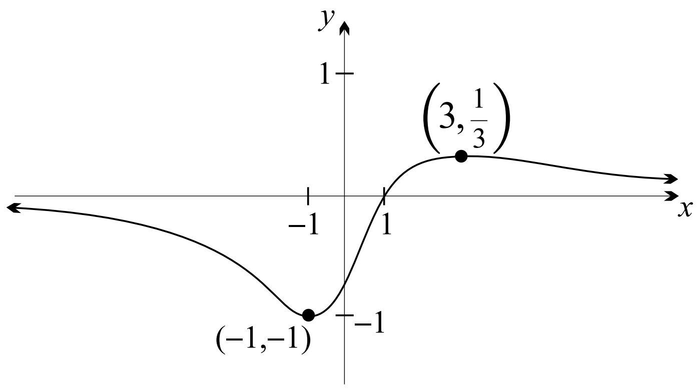
(i) Find the Cartesian equation of the curve. **1**

(ii) Find a focus and directrix of the curve. **2**

(iii) Sketch the curve, showing the features found in (ii). **2**

(c) Two complex numbers z and w have equal moduli and the angle subtended at the origin by the interval between them is a right angle. Show that $z^2 + w^2 = 0$. **2**

(d) The diagram below shows the graph of $y = f(x)$.



Draw a separate half-page diagram for each of the following functions, showing any intercepts, turning points and vertical asymptotes.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Expand $(\cos\theta + i\sin\theta)^3$. **1**

(ii) Use De Moivre's Theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

(b) A particle of unit mass is dropped from rest and experiences air resistance proportional to the square of its velocity $v\,\text{ms}^{-1}$. Its equation of motion is $\ddot{x} = 10 - kv^2$, where x m is the distance it falls from the point it is dropped, $10\,\text{ms}^{-2}$ is the acceleration due to gravity and k is some positive constant.

(i) Show that $x = \dfrac{1}{2k}\log_e\left(\dfrac{10}{10 - kv^2}\right)$. **3**

(ii) Show that its terminal velocity $V\,\text{ms}^{-1}$ is given by $V = \sqrt{\dfrac{10}{k}}$. **2**

(iii) The particle reaches half its terminal velocity when it has fallen 250 metres. Find k. **2**

(c) The polynomial $P(x) = 3x^4 - 4x^3 - 6x^2 + 20x - 16$ has a zero at $x = 1 + i$.

(i) Find a quadratic factor of $P(x)$. **2**

(ii) Factorise $P(x)$ into linear factors. **3**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$. **3**

(b) Let $\omega \neq 1$ be a complex cube root of unity.

(i) Show that $\omega^2 + \omega + 1 = 0$. **1**

(ii) Hence, or otherwise, evaluate the expression $(\omega + 1)(4\omega^2 + \omega + 1)^2$. **2**

(c) The polynomial $Q(x) = 2x^3 - x^2 + 4x - 5$ has zeros α, β and γ. Find a polynomial with zeros $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$. You may leave your answer in unsimplified form. **2**

(d) Consider the complex numbers $z_k = 2\,\text{cis}\left(\frac{(4k-1)\pi}{10}\right)$, where k is an integer.

(i) On an Argand diagram, carefully plot z_k for $k = -2, -1, 0, 1$ and 2. **2**

(ii) Hence deduce the value of a and b if the complex numbers z_k are solutions to the equation $z^5 = a + ib$. **1**

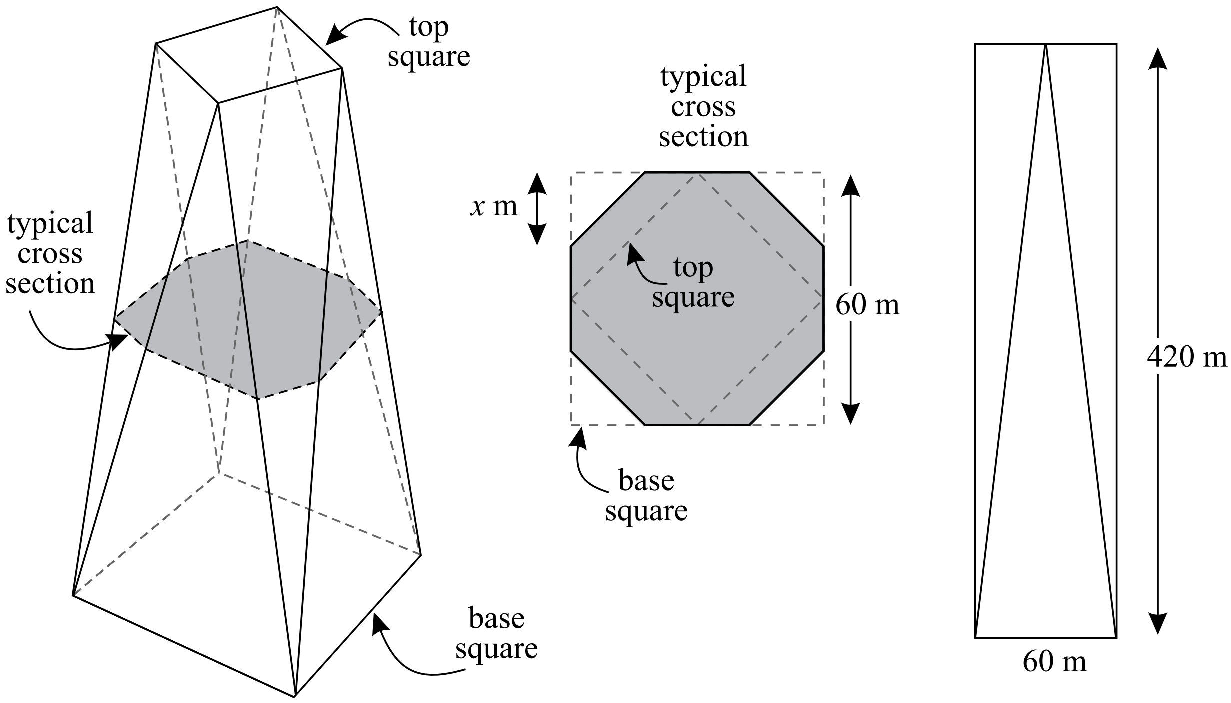
(e) (i) By substituting $u = z + \dfrac{1}{z}$ into the equation $3z^4 + 10z^3 + 9z^2 + 10z + 3 = 0$, show that $3u^2 + 10u + 3 = 0$. **2**

(ii) Hence, or otherwise, solve $3z^4 + 10z^3 + 9z^2 + 10z + 3 = 0$. **2**

Examination continues overleaf ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Below is a diagram of the One World Trade Centre building in New York.



The building is 420 metres tall with a square base of side length 60 metres. The top of the tower is a square parallel to the base with a diagonal of length 60 metres oriented at 45 degrees to the base.

The typical cross section of the tower at h metres above the ground is an octagon inscribed in a square the same size as the base of the building. Let x metres be the distance from any vertex of the octagonal cross section to the closest vertex of the circumscribed square, as shown in the diagram above.

(i) By finding the relationship between x and h, show that the area of the cross section is given by $A = 3600 - \dfrac{1}{98}h^2$ square metres. **2**

(ii) Hence find the exact volume of the tower. **2**

QUESTION FOURTEEN (Continued)

(b) For positive integers n, show $(1+\cos 2\theta + i\sin 2\theta)^n = 2^n \cos^n\theta\Big(\cos(n\theta)+i\sin(n\theta)\Big)$. **2**

(c) Evaluate $\displaystyle\int_{-1}^{0} \frac{x+8}{(x+2)(1-x)}\,dx$. **4**

(d) The formula $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$ was discovered by John Machin in 1706 and was used to calculate π before computers were invented.

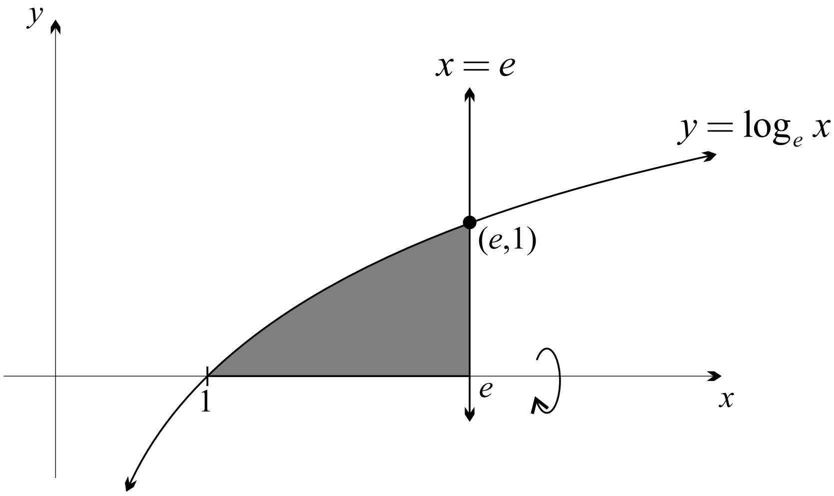
(i) Show that $\tan\left(2\tan^{-1}\dfrac{1}{5}\right) = \dfrac{5}{12}$. **1**

(ii) Show that $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$. **2**

(iii) Given $\tan^{-1}x \doteqdot x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$ for $x < 1$, use Machin's formula to generate an approximation of π, rounding your answer to five decimal places. **2**

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The region enclosed by the curve $y = \log_e x$, the line $x = e$ and the x-axis is shown in the diagram. **3**



Find the volume of the solid formed by rotating this region about the x-axis.

(b) Let $I_n = \displaystyle\int \operatorname{cosec}^n x \, dx$, for integers $n \geq 0$.

(i) Show that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. **1**

(ii) Show that $I_n = \dfrac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ for integers $n \geq 2$. **3**

(iii) Hence, or otherwise, find $\displaystyle\int_{\pi/6}^{5\pi/6} \operatorname{cosec}^4 x \, dx$. **3**

(c) The polynomial $P(x) = x^3 - 3x^2 + 4x - 1$ has zeros α, β and γ.

(i) Show that $Q(x) = x^3 - x^2 + 10x - 1$ has zeros α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find a polynomial with zeros $1 - \alpha$, $1 + \alpha$, $1 - \beta$, $1 + \beta$, $1 - \gamma$ and $1 + \gamma$. **3**

Examination continues next page ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Show the following:

(i) $\tan^2 \dfrac{\pi}{8} + 2\tan\dfrac{\pi}{8} = 1$ **1**

(ii) $\left(2+2\sqrt{2}\right)\tan^2\dfrac{\pi}{8} = 2\tan\dfrac{\pi}{8}$ **2**

(b) Consider the function $f(x) = x + \dfrac{1}{x}$.

(i) Without using calculus, sketch $y = f(x)$, showing any asymptotes. **2**

(ii) Consider a point $P\left(p, p+\dfrac{1}{p}\right)$ on the curve $y = f(x)$ in the first quadrant. Let S be the point $\left(k\tan\dfrac{\pi}{8}, k\right)$ and let Q be the foot of the perpendicular from P to the line $x\tan\dfrac{\pi}{8} + y - k = 0$, where $k^2 = 2+2\sqrt{2}$. **3**

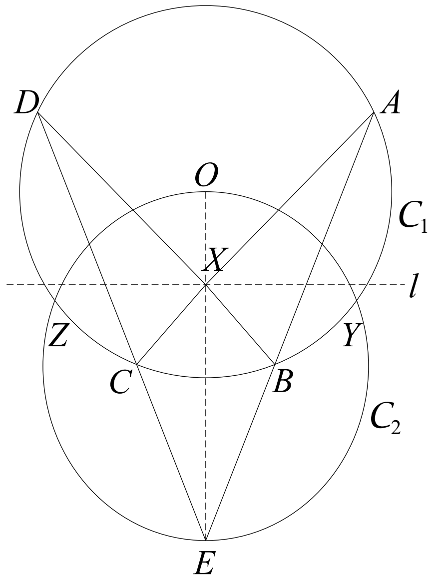
Using part (a), show that $PS^2 = PQ^2\sec^2\dfrac{\pi}{8}$.

(iii) Hence justify that the curve $y = x + \dfrac{1}{x}$ is a hyperbola. **1**

QUESTION SIXTEEN (Continued)

(c) NOTE: You do NOT need to copy the diagram below. It has been reproduced on a separate Working Sheet. Write your candidate number on the Working Sheet.

Write your solution in your Writing Booklet NOT on the Working Sheet, and insert the Working Sheet in your Writing Booklet for Question Sixteen.



In the diagram above, AB and CD are chords of equal length in circle $\mathcal{C}_1$ with centre O. When produced, AB and CD meet at E. Chords AC and BD both intersect OE at X.

Let $\mathcal{C}_2$ be the circle with diameter OE, and let $\mathcal{C}_2$ intersect $\mathcal{C}_1$ at Y and Z.

You may assume that the whole configuration is symmetric in the line OE.

(i) Prove that $DOBE$ is a cyclic quadrilateral. $\boxed{3}$

(ii) Consider the line l passing through X perpendicular to OE. Let the chord in $\mathcal{C}_1$ on l have length $2x$ and let the chord in $\mathcal{C}_2$ on l have length $2y$. $\boxed{3}$
Show that Y and Z lie on l.

End of Section II

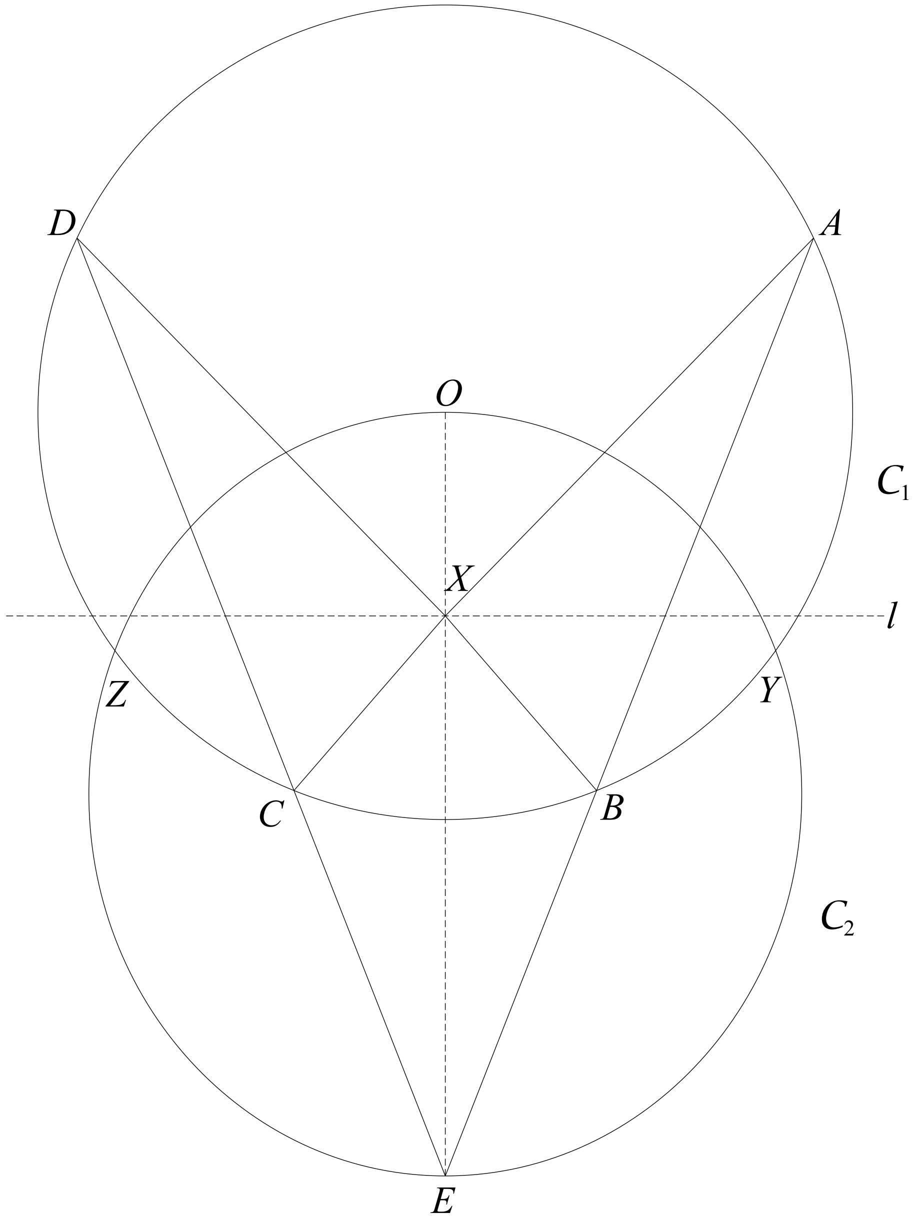
END OF EXAMINATION

CANDIDATE NUMBER

Insert this sheet in your Writing Booklet for Question Sixteen.
Use this sheet to mark angles and any other working you need.
Write your solution in your Writing Booklet, NOT on this sheet.

QUESTION SIXTEEN: Working Sheet

(c)



CANDIDATE NUMBER

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2019
Trial Examination
FORM VI
MATHEMATICS EXTENSION 2
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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

SGS Trial 2019 - Form VI Mathematics Extension 2 - Solutions

(1) $\int x e^x \, dx = xe^x - \int e^x \, dx$ $\qquad u = x \quad dv = e^x dx$ $\quad du = dx \quad v = e^x$

$= xe^x - e^x + C$

(A) Correct.

(2) $z\bar{z} = |z|^2$ is a known identity

so (B) Correct

OR let $z = x + iy$, $z\bar{z} - |z|^2 = (x+iy)(x-iy) - \sqrt{x^2+y^2}^{\,2}$

$= x^2 + y^2 - x^2 - y^2 = 0$

CHECKING DISTRACTORS

(C) $(z+\bar{z})(z-\bar{z}) = z^2 - \bar{z}^2 = x^2 + 2ixy - y^2 - (x^2 - 2ixy - y^2)$

$= 4ixy$

(D) $(z+iz)(z-iz) = z^2(1+i)(1-i) = 2z^2$

(3) $P(x) = x^4 + x^2 - 12 = (x^2+4)(x^2-3) = (x+2i)(x-2i)(x+\sqrt{3})(x-\sqrt{3})$ $\therefore$ (B) correct

(4) Let $F(x)$ be the primitive function of $f(x)$

$\int_0^{2a} f(2a-x) = \left[-F(2a-x)\right]_0^{2a} = -F(0) - -F(2a) = F(2a) - F(0)$

$= \int_0^{2a} f(x)\,dx$ (A) is correct ✓

$\int_0^{2a} f(a-x)\,dx = \left[-F(a-x)\right]_0^{2a} = -F(-a) - -F(a) = F(a) - F(-a)$

$= \int_{-a}^{a} f(x)\,dx$ B not correct

$2\int_0^{a} f(2a-x)\,dx = 2\left[-F(2a-x)\right]_0^{a} = 2\left[-F(a) - -F(2a)\right]$

$= 2\int_a^{2a} f(x)\,dx$ C not correct

$2\int_0^{a} f(a-x)\,dx = 2\left[-F(a-x)\right]_0^{a} = 2\left[-F(0) - -F(a)\right] = 2\int_0^{a} f(x)\,dx$

D not correct

5) (A) is $|y| = \ln|x|$

(B) is $y = \ln|x|$ $\therefore$ B is correct

(C) is $y = \ln x$ and $y = -\ln(-x)$

(D) is $|y| = |\ln|x||$

6) (A) $g(g(-x)) = g(g(x))$ $\therefore$ even $\Rightarrow$ TRUE

(B) $g(f(-x)) = g(-f(x)) = g(f(x))$ $\therefore$ even $\Rightarrow$ TRUE

C $f(g(-x)) = f(g(x))$ $\therefore$ even $\Rightarrow$ NOT TRUE

(D) $f(f(-x)) = f(-f(x)) = -f(f(x))$ $\therefore$ odd $\Rightarrow$ TRUE

7)

$$-kv^2 = ma$$

$$\frac{dv}{dt} = \frac{-kv^2}{m}$$

$$t = \int \frac{m}{-kv^2}\,dv$$

$\therefore$ option A

(B) integrand needs a negative

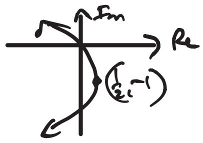
(C) and (D) have k and m reversed.

8) The curve is $4y = x^2$, which has focus $(0,1)$ and directrix $y=-1$

or rather, the distance from i will be equal to the $\operatorname{Im}(z)+1$.

$|z-i| = \operatorname{Im}(z)+1$ $\therefore$ option D

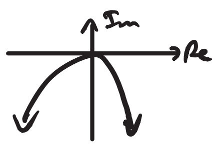
A would be a parabola



but actually LHS ≥ 0 while RHS < 0

$\therefore$ no points satisfy this equation.

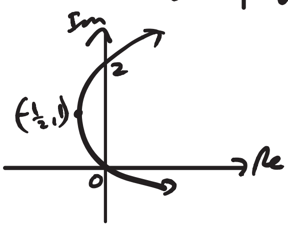
B would be a parabola



but again LHS ≥ 0 while RHS < 0

so no points satisfy this equation

C is parabola



(9) Slicing perpendicular gives $V = \pi \int_0^1 x^2 \, dy$

Now $y = (x^2-1)^2$

$x^2 - 1 = \pm\sqrt{y}$

$x^2 = 1 \pm \sqrt{y}$

for $0 < x < 1$,

$\underline{x^2 = 1 - \sqrt{y}}$

($x^2 = 1-\sqrt{y}$ gives the curve in $-1 \le x \le 1$

$x^2 = 1+\sqrt{y}$ gives the curve in $x \le -1, 1 \le x$)

$\therefore V = \pi \int_0^1 (1-\sqrt{y}) \, dy$ $\therefore$ option (C) $= \frac{\pi}{3}$

(10) $(f * g)\,t = \int_0^t 1 \cdot e^{-(t-s)} \, ds$

$= e^{-t} \int_0^t e^s \, ds$

$= e^{-t}(e^t - 1)$

$= \underline{1 - e^{-t}}$ $\therefore$ (D) is correct

NB The answer will certainly be in terms of t and not s so (A) and (B) are easily eliminated

11 (a) (i) $w = \sqrt{3} + i = 2\operatorname{cis}\frac{\pi}{6}$ ✓ ✓

(ii) $z = 3\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

$\therefore zw = 6\sqrt{2}\operatorname{cis}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = 6\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$ ✓ ✓

(b) $x = 5\cos\theta$, $y = 3\sin\theta$

(i) $\cos^2\theta + \sin^2\theta = 1$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$ ✓

(ii) This is an ellipse with $a = 5, b = 3$

so $b^2 = a^2(1 - e^2)$

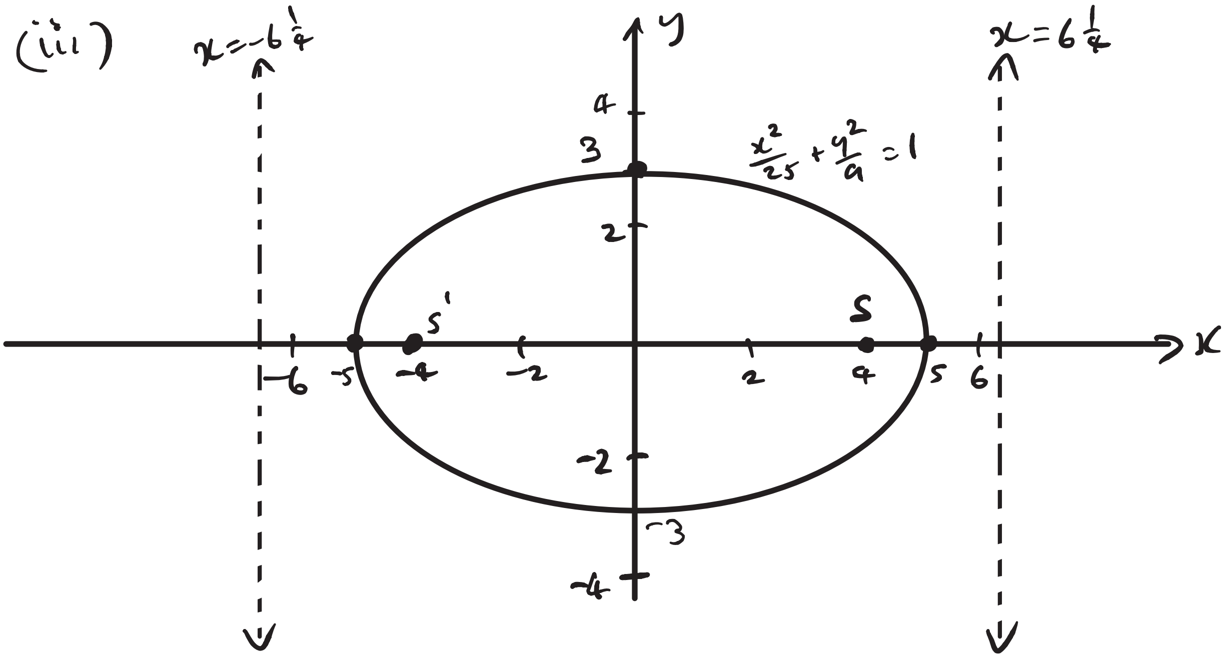
$\frac{9}{25} = 1 - e^2$

$e = \frac{4}{5}$

Focus is $(ae, 0) = (4, 0)$ ✓

Directrix is $x = \frac{a}{e} = \frac{5}{\frac{4}{5}} = 6\frac{1}{4}$. ✓

(iii)



Focus directrix and intercepts

Ellipse oriented correctly

(c) $|z| = |w|$ and one is a rotation of $\frac{\pi}{2}$ of the other, so

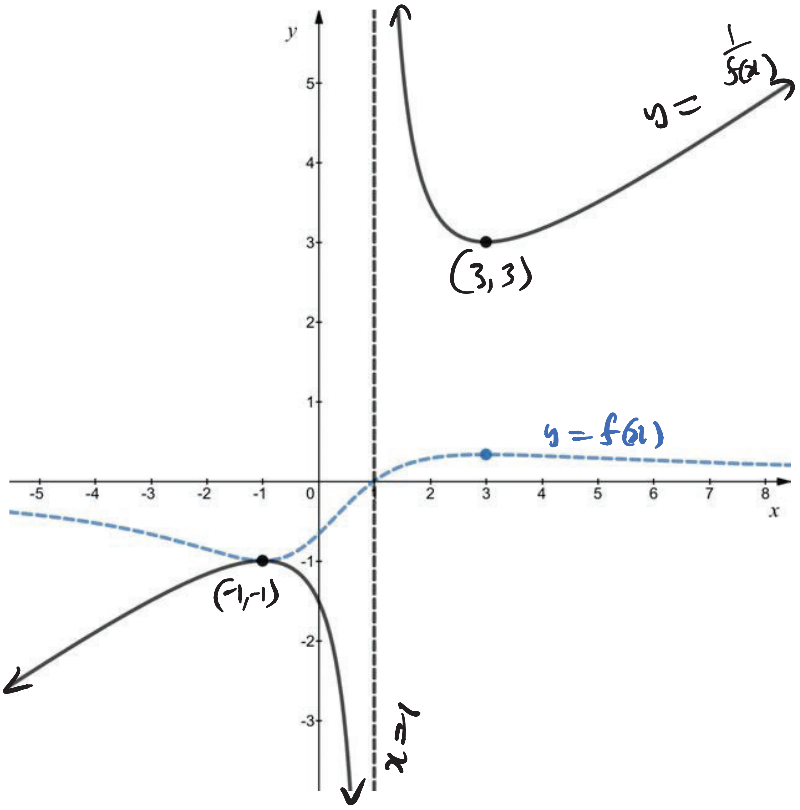
$z = \pm iw$ ✓ recognising $z = iw$

$z^2 = -w^2$ ✓ result

$z^2 + w^2 = 0$

NB it's possible $z = -iw$, i.e. $iz = w$

(1) (d) (i)



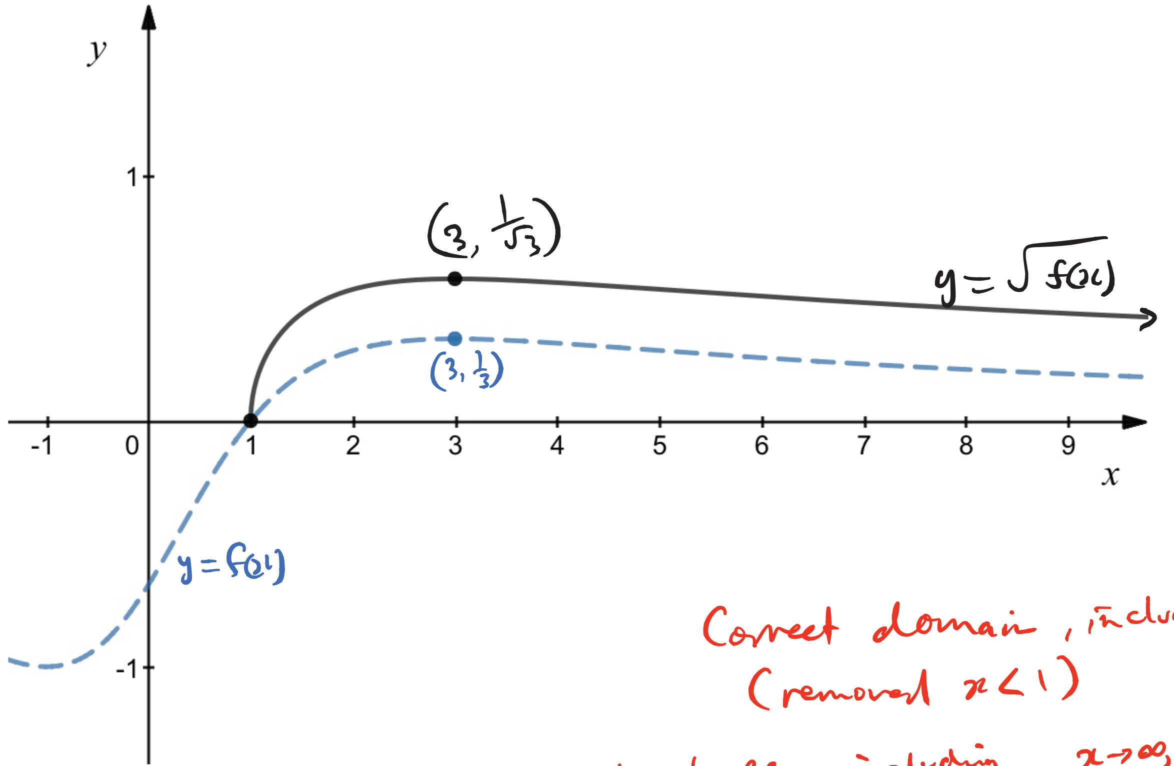
shape and asymptote $x = 1$

as $x \to \infty$, $y \to \infty$

as $x \to -\infty$, $y \to -\infty$

turning points (-1, -1) (3, 3)

(ii)



Correct domain, including (1, 0) (removed $x < 1$)

Correct general shape including $x \to \infty$, $y \to 0^+$ and max at $\left(3, \frac{1}{\sqrt{3}}\right)$ shown.

NB vertical tangent at (1, 0)

(12) (a) (i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

(ii) By DMT, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Equating imaginary parts of both expressions gives

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \quad (\text{NB } (\text{cis}\,\theta)^3 = \text{cis}\,3\theta)$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

(b) (i) $\ddot{x} = 10 - kv^2 \Rightarrow v \cdot \frac{dv}{dx} = 10 - kv^2$

$$\frac{dx}{dv} = \frac{v}{10-kv^2}$$
$$x = -\frac{1}{2k}\int \frac{-2kv}{10-kv^2}\,dv$$
$$= -\frac{1}{2k}\ln(10-kv^2) + c_1$$

$x=0, v=0 \quad c_1 = \frac{1}{2k}\ln 10$

$$\therefore x = \frac{1}{2k}\ln\left(\frac{10}{10-kv^2}\right)$$

(ii) $$e^{2kx} = \frac{10}{10-kv^2}$$
$$10-kv^2 = 10e^{-2kx}$$
$$kv^2 = 10 - 10e^{-2kx}$$

now as $x \to \infty$, $kv^2 \to 10$

$\therefore V^2 = \frac{10}{k}$

$V = \sqrt{\frac{10}{k}}$ ($V > 0$ since downwards is positive)

OR $v \to V$ as $\ddot{x} \to 0$, $10 - kv^2 \to 0$, $v^2 \to \frac{10}{k}$ $\therefore V = \sqrt{\frac{10}{k}}$

(iii) $x = 250$, $v = \frac{V}{2} = \frac{1}{2}\sqrt{\frac{10}{k}}$

$$250 = \frac{1}{2k}\ln\left(\frac{10}{10 - k\cdot\frac{1}{4}\cdot\frac{10}{k}}\right)$$

$$500k = \ln\frac{10}{10 - 2\frac{1}{2}}$$

$$k = \frac{1}{500}\ln\left(\frac{4}{3}\right).$$

(12) (c) (i) since coefficients are real, $1-i$ is also a root

$(x-(1+i))(x-(1-i))$ is a factor

$= x^2-2x+2$

(ii)

$$\begin{array}{r} 3x^2+2x-8 \\ x^2-2x+2 \,\big)\, 3x^4-4x^3-6x^2+20x-16 \\ 3x^4-6x^3+6x^2 \\ \hline 2x^3-12x^2 \\ 2x^3-4x^2+4x \\ \hline -8x^2+16x \\ -8x^2+16x-16 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-1+i)(x-1-i)(3x^2+2x-8)$

$= (x-1+i)(x-1-i)(3x-4)(x+2)$

OR $P(2) = 48-32-24+40-16 \neq 0$

$P(-2) = 48+32-24-40-16 = 0$

$\therefore (x+2)$ is a factor. product of roots $= \frac{-16}{3} = -2 \times 2\alpha \Rightarrow \alpha = \frac{4}{3}$

$\therefore (3x-4)$ is the last factor

$P(x) = (x-1+i)(x-1-i)(3x-4)(x+2)$

NB $P(x)$ is not monic!

---

(13) (a) $I = \int_0^1 \frac{\sqrt{x}}{1+x}\,dx$

$x = u^2$ $\sqrt{x} = u$ (NB $0 \le x \le 1$)

$dx = 2u\,du$

$= \int_0^1 \frac{u}{1+u^2}\, 2u\,du$

$= 2\int_0^1 \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right)du$

$= 2\left[u - \tan^{-1}(u)\right]_0^1$

$= 2\left[\left(1-\frac{\pi}{4}\right)-(0-0)\right]$

$= 2-\frac{\pi}{2}$

OR $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$

$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{1+\tan^2\theta} \cdot 2\tan\theta\sec^2\theta\,d\theta$

$= 2\int_0^{\pi/4} \tan^2\theta\,d\theta$

$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta$

$= 2\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$

$= 2\left[\left(1-\frac{\pi}{4}\right)-(0-0)\right]$

$= 2-\pi/2$.

13 (b) (i) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$(\omega-1)(\omega^2+\omega+1) = 0$ ✓

but since $\omega \neq 1$, $\omega^2+\omega+1 = 0$

OR $z^3 = 1 \Rightarrow z = 1, \text{cis}\frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right)$

$= 1, \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(\frac{4\pi}{3}\right)$

$= 1, \omega, \omega^2$

$\therefore$ by sum of roots

$\alpha+\beta+\gamma = \frac{-b}{a}$

$1+\omega+\omega^2 = 0$

(ii) $(\omega+1)(4\omega^2+\omega+1)^2$

$= (4\omega^3+\omega^2+\omega+4\omega^2+\omega+1)(4\omega^2+\omega+1)$

$= (5\omega^2+2\omega+5)(4\omega^2+\omega+1)$

$= 20\omega^4 + (5+8)\omega^3 + (20+2+5)\omega^2 + (2+5)\omega + 5$ $\quad \omega^3 = 1$

$= 20\omega + 13 + 27\omega^2 + 7\omega + 5$ (1)

$= 27\omega^2 + 27\omega + 18$

$= 27(\omega^2+\omega+1) - 9$ notice $\omega^3 - 1 = 0$

$= -9$ (1) $(\omega-1)(\omega^2+\omega+1) = 0$

but $\omega \neq 1$, $\omega^2+\omega+1 = 0$

OR (better) $(\omega+1)(4\omega^2+\omega+1)^2 = (\omega^2+\omega+1-\omega^2)(\omega^2+\omega+1+3\omega^2)^2$ (1)

$= -\omega^2(3\omega^2)^2$

$= -9\omega^6$

$= -9$ (1)

(c) $Q(x) = 2x^3 - x^2 + 4x - 5$ Now $\alpha+\beta+\gamma = \frac{1}{2}$

So Roots of $\alpha+\beta, \alpha+\gamma, \beta+\gamma$ are $\frac{1}{2}-\gamma, \frac{1}{2}-\beta, \frac{1}{2}-\alpha$ (1)

So $Q\left(\frac{1}{2}-x\right) = 2\times\frac{1}{8}(1-2x)^3 - \frac{1}{4}(1-2x)^2 + 2(1-2x) - 5$ (1)

$= \frac{1}{4}\left[(1-4x+4x^2)(1-2x-1)\right] - 4x - 3$

$= \frac{1}{2}(-x+4x^2-4x^3) - 4x - 3$

$= -2x^3 + 2x^2 - 4\frac{1}{2}x - 3$ has roots $\alpha+\beta, \alpha+\gamma, \beta+\gamma$

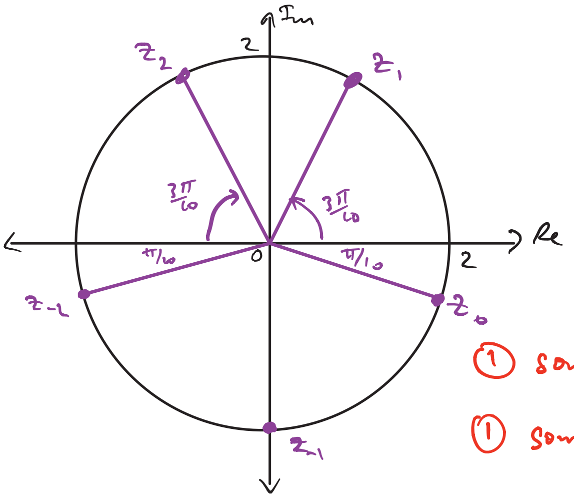
(or equivalent)

NB it's possible but very laborious to calculate $(\alpha+\beta)+(\alpha+\gamma)+(\beta+\gamma)$ and $(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)$ and $(\alpha+\beta)(\alpha+\gamma)+(\alpha+\beta)(\beta+\gamma)+(\alpha+\gamma)(\beta+\gamma)$

(13) (d) $z_k = 2\operatorname{cis}\left(\frac{4k-1}{10}\pi\right)$

$z_0 = 2\operatorname{cis}\left(\frac{-\pi}{10}\right)$, $z_1 = 2\operatorname{cis}\left(\frac{3\pi}{10}\right)$, $z_2 = 2\operatorname{cis}\left(\frac{7\pi}{10}\right)$

$z_{-1} = 2\operatorname{cis}\left(-\frac{\pi}{2}\right)$, $z_{-2} = 2\operatorname{cis}\left(\frac{-9\pi}{10}\right)$



(ii) $(z_0)^5 = 32\operatorname{cis}\frac{-\pi}{2}$

$= 32(-i)$

$\therefore a = 0,\ b = -32$ (1)

OR (EASIEST)

$(z_{-1})^5 = (-2i)^5$

$= -32i$

(1) some indication of modulus

(1) some indication of arguments

(e) $f(z) = 3z^4 + 10z^3 + 9z^2 + 10z + 3 = 0$) $\div z^2$ NB $z \neq 0$

(i) $3\left(z^2 + \frac{1}{z^2}\right) + 10\left(z + \frac{1}{z}\right) + 9 = 0$ (1)

$3\left(z^2 + 2 + \frac{1}{z^2}\right) + 10\left(z + \frac{1}{z}\right) + 3 = 0$ ← MUST SHOW THIS OR EQUIVALENT

$3u^2 + 10u + 3 = 0$ (1)

(ii) $(3u+1)(u+3) = 0$

$u = -\frac{1}{3}$ or -3

$z + \frac{1}{z} = -\frac{1}{3}$ or $z + \frac{1}{z} = -3$ (1)

$3z^2 + z + 3 = 0$

$z = \frac{-1 \pm i\sqrt{36-1}}{6}$

$= \frac{-1 \pm i\sqrt{35}}{6}$

$z^2 + 3z + 1 = 0$

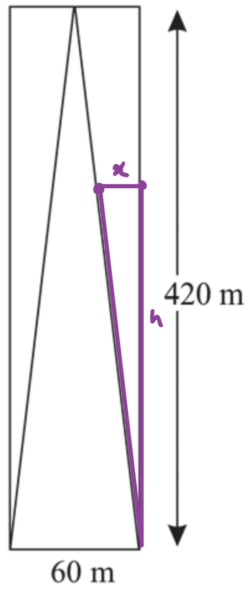
$z = \frac{-3 \pm \sqrt{9-4}}{2}$

$= \frac{-3 \pm \sqrt{5}}{2}$

$z = \frac{1}{6}\left(-1 \pm i\sqrt{35}\right),\ \frac{1}{2}\left(-3 \pm \sqrt{5}\right)$ (1)

(14) (a) (i) By inspection, the relationship b/w h and x will be linear (observe the third diagram) and

x	0	30
h	0	420



$\therefore \; x = \frac{h}{14}$ (1)

Typical cross section area

$A = 60^2 - 4 \times \frac{1}{2} x^2$

$= 3600 - 2\left(\frac{h}{14}\right)^2$ (1)

$= 3600 - \frac{h^2}{98}$

(ii) $V = \int_0^{420} \left(3600 - \frac{h^2}{98}\right) dh$ (1)

$= \left(3600h - \frac{h^3}{3 \times 98}\right)_0^{420}$

$= 3600 \times 420 - \frac{420^3}{3 \times 98}$

$= 1260\,000 \text{ m}^3$ (1)

(b) $(1 + \cos 2\theta + i \sin 2\theta)^n = (1 + 2\cos^2\theta - 1 + i\, 2\sin\theta\cos\theta)^n$ ✓

$= (2\cos\theta(\operatorname{cis}\theta))^n$ ✓

$= 2^n \cos^n\theta\,(\cos(n\theta) + i\sin(n\theta))$

(c) $\frac{x+8}{(x+2)(1-x)} \equiv \frac{A}{x+2} + \frac{B}{1-x}$

$(-A+B)x + (A+2B) \equiv x+8$

$B = 1+A$, $\quad A + 2A + 2 = 8$

$3A = 6$

$A = 2$

$B = 3$ ✓

$\int_{-1}^{0} \frac{x+8}{(x+2)(1-x)}\, dx = \int_{-1}^{0} \frac{2}{x+2}\, dx + \int_{-1}^{0} \frac{3}{1-x}\, dx$ ✓

$= \Big[2\ln(x+2) - 3\ln(1-x)\Big]_{-1}^{0}$ ✓

$= (2\ln 2 - 0) - (0 - 3\ln 2)$

$= \underline{5\ln 2}$ ✓

(d) (i) $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)\right) = \dfrac{2\tan\left(\tan^{-1}\left(\frac{1}{5}\right)\right)}{1-\tan^2\left(\tan^{-1}\left(\frac{1}{5}\right)\right)} = \dfrac{\frac{2}{5}}{1-\frac{1}{25}}$ ✓

$$= \frac{2}{5}\times\frac{25}{24} = \frac{5}{12}$$

(ii) $\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right) = \tan\left(2\tan^{-1}\left(\frac{5}{12}\right)\right) = \dfrac{\frac{5}{6}}{1-\frac{25}{144}} = \frac{144}{6}\times\frac{5}{119} = \frac{120}{119}$ 1

so $\tan(\text{LHS}) = \dfrac{\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right)-\tan\left(\tan^{-1}\left(\frac{1}{239}\right)\right)}{1+\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{239}\right)\right)}$

$$= \frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}} \quad 1$$

$$= \frac{120\times239-119}{119\times239+120} = \frac{119\times239+239-119}{119\times239+120} = 1$$

so $\tan(\text{LHS}) = \tan(\text{RHS})$ since $\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right) \approx 1$

then clearly $4\tan^{-1}\left(\frac{1}{5}\right) \approx \frac{\pi}{4}$

so LHS and RHS of equation are

both in the first quadrant, so LHS = RHS

No marks deducted for not considering quadrants

(iii) so $\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right)$ 1

$$\approx 16\left(\frac{1}{5}-\frac{1}{3\times5^3}+\frac{1}{5\times5^5}\right) - 4\left(\frac{1}{239}-\frac{1}{3\times239^3}+\frac{1}{5\times239^5}\right)$$

$\approx \underline{3.14162}$ 1 (smaller than π by 0.000028)

(15) (a) By slicing perpendicular $V = \pi \int_1^e y^2 \, dx$

$$= \pi \int_1^e (\ln x)^2 \, dx$$

$u = (\ln x)^2 \qquad dv = dx$

$du = 2 \ln x \cdot \frac{1}{x} \, dx \qquad v = x$

$$\therefore V = \pi \left(x (\ln x)^2 \right)_1^e - \pi \int_1^e 2 \ln x \, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \, dx \qquad v = x$

$$= \pi(e - 0) - 2\pi \left[(x \ln x)_1^e - \int_1^e x \cdot \frac{1}{x} \, dx \right]$$

$$= \pi e - 2\pi \left[(e - 0) - [x]_1^e \right]$$

$$= -\pi e + 2\pi(e - 1)$$

$$= \underline{\pi(e - 2)} \cdot u^3$$

OR

By cylindrical shells $V = 2\pi \int_0^1 (e - x) y \, dy$

$$= 2\pi \int_0^1 (ey - ye^y) \, dy$$

$u = y \qquad dv = e^y dy$

$du = dy \qquad v = e^y$

$$= 2\pi \left[\left(e \frac{y^2}{2} \right)_0^1 - \left[(ye^y)_0^1 - \int_0^1 e^y \, dy \right] \right]$$

$$= 2\pi \left[\frac{e}{2} - (e - 0) + [e^y]_0^1 \right]$$

$$= 2\pi \left[-\frac{e}{2} + (e - 1) \right]$$

$$= \underline{\pi(e - 2)} \, u^3$$

(b) (i) $\frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$

$$= -\operatorname{cosec}^2 x$$

(15) (b) (ii) $I_n = \int \operatorname{cosec}^n x \, dx$

$u = \operatorname{cosec}^{n-2} x$

$du = (n-2)\operatorname{cosec}^{n-3} x \left(-\sin^{-2} x \times \cos x\right) dx$

$= -(n-2)\operatorname{cosec}^{n-2} x \cot x \, dx$

$dv = \operatorname{cosec}^2 x \, dx$

$v = -\cot x$ (from(i))

$$I_n = \left(-\cot x \operatorname{cosec}^{n-2} x\right) - \int (-\cot x)\left(-(n-2)\right)\cot x \operatorname{cosec}^{n-2} x \, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \cot^2 x \operatorname{cosec}^{n-2} x \, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\left(I_n - I_{n-2}\right)$$

$$I_n\left(1+(n-2)\right) = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

(iii) $$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \operatorname{cosec}^2 x \, dx = \left[-\cot x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \left(- -\sqrt{3} - (-\sqrt{3})\right) = 2\sqrt{3}$$

$$\int_{\frac{\pi}{6}}^{5\pi/6} \operatorname{cosec}^4 x \, dx = \left[\frac{-\cot x \operatorname{cosec}^2 x}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \operatorname{cosec}^2 x \, dx$$

$$= \tfrac{1}{3}\left(+\sqrt{3}\times 2^2 + \sqrt{3}\times 4\right) + \tfrac{2}{3}\times 2\sqrt{3}$$

$$= +\frac{8\sqrt{3}}{3} + \frac{4}{3}\sqrt{3} = 4\sqrt{3}.$$

(c) $P(x) = x^3 - 3x^2 + 4x - 1$ has zeros α, β and γ

(i) $P(\sqrt{x}) = 0$ has solns $\alpha^2, \beta^2, \gamma^2$

$$x\sqrt{x} - 3x + 4\sqrt{x} - 1 = 0$$

$$\sqrt{x}(x+4) = 3x+1$$

$$x\left(x^2+8x+16\right) = 9x^2+6x+1$$

$$x^3 - x^2 + 10x - 1 = 0$$

So $Q(x) = x^3 - x^2 + 10x - 1$ has zeros α^2, β^2 and γ^2

(ii) Now $S(x)$ has roots $1-\alpha, 1+\alpha$ etc.

so $S(x) = \left(x-(1-\alpha)\right)\left(x-(1+\alpha)\right)\left(x-(1-\beta)\right)\left(x-(1+\beta)\right)\left(x-(1-\gamma)\right)\left(x-(1+\gamma)\right)$

$$= \left((x-1)^2-\alpha^2\right)\left((x-1)^2-\beta^2\right)\left((x-1)^2-\gamma^2\right)$$

So $S(x) = Q\left((x-1)^2\right)$

$$= \left((x-1)^2\right)^3 - \left((x-1)^2\right)^2 + 10(x-1)^2 - 1$$

$$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$
$$- x^4 + 4x^3 - 6x^2 + 4x - 1$$
$$+ 10x^2 - 20x + 10$$
$$- 1$$

$$= x^6 - 6x^5 + 14x^4 - 16x^3 + 19x^2 - 22x + 9$$

---

16 (a)

(i) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x} \Rightarrow \tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$$1 - \tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$
$$\underline{\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} = 1}$$

(ii) $\left(\tan\frac{\pi}{8}+1\right)^2 = 2$

$\tan\frac{\pi}{8} + 1 = \pm\sqrt{2}$

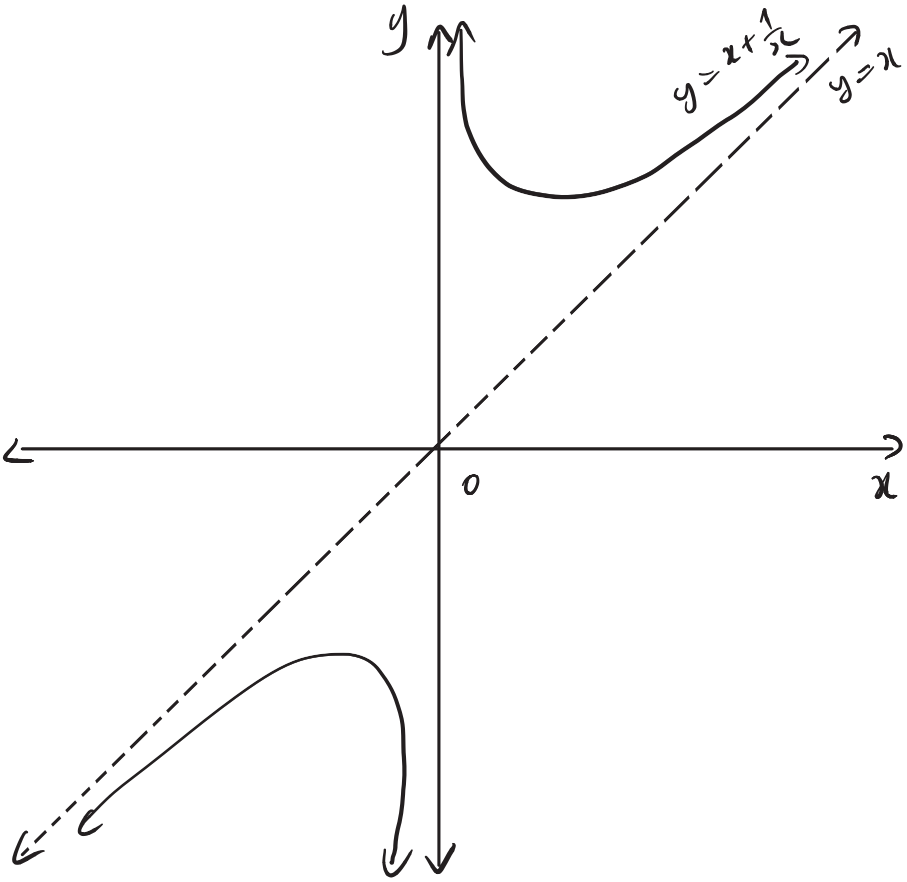
$\tan\frac{\pi}{8} = -1 \pm \sqrt{2}$ NB: $\tan\frac{\pi}{8} > 1$

$\therefore \tan\frac{\pi}{8} = \sqrt{2} - 1$

$$LHS = (2+2\sqrt{2})\tan^2\frac{\pi}{8} = (2+2\sqrt{2})(\sqrt{2}-1)^2$$
$$= 2(\sqrt{2}+1)(\sqrt{2}-1)(\sqrt{2}-1)$$
$$= 2\times 1\times(\sqrt{2}-1)$$
$$= 2\tan\left(\frac{\pi}{8}\right) = RHS$$

16 (b) $f(x) = x + \frac{1}{x}$

(i)



✓ asymptotes clear or implied

U shape (both arms)

(ii) RTP: $PS^2 = PQ^2 \sec^2 \frac{\pi}{8}$

$$LHS = \left(p - k\tan\tfrac{\pi}{8}\right)^2 + \left(\left(p + \tfrac{1}{p}\right) - k\right)^2$$

$$= p^2 - 2pk\tan\tfrac{\pi}{8} + k^2\tan^2\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)^2$$ from (a)(ii)

$$= p^2 - 2pk\tan\tfrac{\pi}{8} + 2\tan\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)^2$$

Expressions for PS^2 and PQ^2 correct (2)

Complete soln (3)

$$RHS = \frac{\left|p\tan\frac{\pi}{8} + \left(p + \frac{1}{p}\right) - k\right|^2}{\left(\sqrt{\tan^2\frac{\pi}{8} + 1}\right)^2} \times \sec^2\tfrac{\pi}{8}$$

$$= \left(p\tan\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)\right)^2$$

$$= p^2\tan^2\tfrac{\pi}{8} + 2p\left(p + \tfrac{1}{p} - k\right)\tan\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)^2$$

$$= p^2\left(\tan^2\tfrac{\pi}{8} + 2\tan\tfrac{\pi}{8}\right) + 2\tan\tfrac{\pi}{8} - 2pk\tan\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)^2$$ from (a)(i)

$$= p^2 - 2pk\tan\tfrac{\pi}{8} + 2\tan\tfrac{\pi}{8} + \left(p + \tfrac{1}{p} - k\right)^2$$

(iii) Here $\frac{PS}{PQ} = \sec\frac{\pi}{8}$, $\sec\frac{\pi}{8} \approx 1.08 > 1$

$\therefore$ $y = x + \frac{1}{x}$ a hyperbola with eccentricity $\sec\frac{\pi}{8}$,

focus $(k\tan\frac{\pi}{8}, k)$ and directrix $x\tan\frac{\pi}{8} + y - k = 0$

Must observe $\sec\frac{\pi}{8} > 1$

No need to list eccentricity, focus and directrix.

(c) (i) $\angle COB = 2\angle CDB$ ($\angle$ on arc CB of C_1 at centre and circumference)

$\angle EOB = \frac{1}{2}\angle COB$ (symmetry as given)

$= \angle CDB$

i.e. $\angle EOB = \angle EDB$ so by converse of theorem that $\angle$'s subtended at circumference on common arc (EB) are equal,

E, O, B and D are concyclic.

(ii) $x^2 = DX \cdot XB$ (intercepts of intersecting chords in C_1)

$= OX \cdot XE$ (intercepts of intersecting chords in circle DOBE)

$= y^2$ (intercepts of intersecting chords in C_2)

Hence $2x = 2y$

Since the chords on ℓ have equal length, the end-points are common to both circles.

Thus YZ is the common chord (also on ℓ).