SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2002

MATHEMATICS EXTENSION 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplied at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value

Name :________________

Class :________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

Question 1 15 marks | Marks

a) Find $\int \sin^3 x \, dx$	2
b) Find $\int \frac{dx}{\sqrt{2x - x^2}}$	2
c) Use partial fractions to find $\int \frac{5}{(x-3)(2x-1)} \, dx$	3
d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}$ using the substitutuion $t = \tan \frac{x}{2}$	4
e) Find $\int e^x \sin x \, dx$	4

Question 2 15 marks (Use a new page) Marks

a) Express $\dfrac{\overline{2+3i}}{1+i}$ in the form $x+iy$. 2

b) Given $\omega = -1 + i\sqrt{3}$ find

i) $|\omega|$ 1

ii) $\arg(\omega)$ 1

iii) $\omega^5 + 16\omega$ in the form $x+iy$ 2

c) On an Argand diagram shade the region specified by

i) $|z-1| \le |z-i|$ 1

ii) $\text{Re}(\dfrac{2}{z}) \le 1$ 3

d) One of the square roots of $a+3i$ is equal to $3+bi$ where a and b are real. Find the value of a and b. 2

e) Sketch the region where both the following inequalities hold. 3

$|z-3| \le 3$ and $0 \le \arg(z) \le \dfrac{\pi}{2}$

Question 3 15 marks (Use a new page) Marks

a) Consider the function $f(x) = \frac{x}{\ln x}$

i) State the natural domain of $f(x)$ 1

ii) Show that the curve $y = f(x)$ 2

has a minimum turning point at the point (e, e).

iii) Find the point of inflexion of the curve $y = f(x)$. 1

iv) Sketch the curve $y = f(x)$ showing the important features. 2

b) $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ where $p, q > 0$ are two distinct points on

the hyperbola H with equation $xy = c^2$.

i) Show that the equation of the tangent to H at $P(cp, \frac{c}{p})$ 2

is given by $x + p^2 y = 2cp$.

ii) The tangents to H at $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ meet at T. 2

Find the coordinates of T.

iii) Find the equation of the chord PQ. 2

iv) Given that the chord PQ passes through the point $(2c, 0)$ 1

find a relationship between the parameters p and q.

v) Find the equation of the locus of T as P and Q move about H 2

according to the restriction in part iv).

Give a complete description of this locus.

Question 4 15 marks (Use a new page) Marks

a) The area bounded by $y = \dfrac{1}{x+1}$, the x axis 4

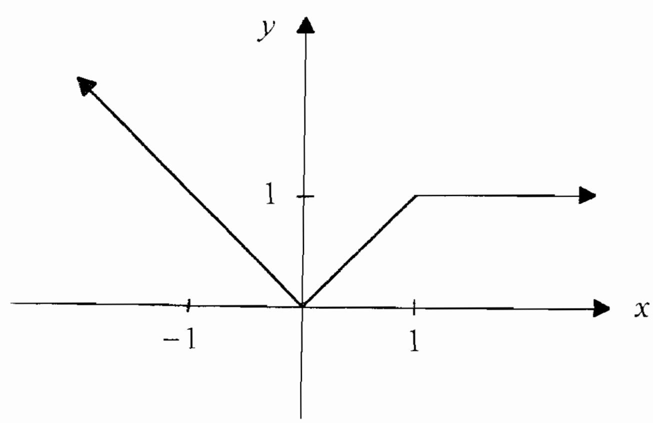
and the lines $x = 0$ and $x = 2$ is rotated about the line $x = 2$.

Use cylindrical shells to find the volume of the solid of revolution formed.

b) Find the equation of the ellipse with centre the origin, 3

which has a focus at $(2,0)$ and the corresponding directrix is $x = 4$.

c)



The diagram shows the graph of the function $y = f(x)$.

Draw separate sketches of the following

i) $y = f(-x)$ 1

ii) $y = f|x|$ 1

iii) $y = \ln(f(x))$ 2

iv) $x = f(y)$ 1

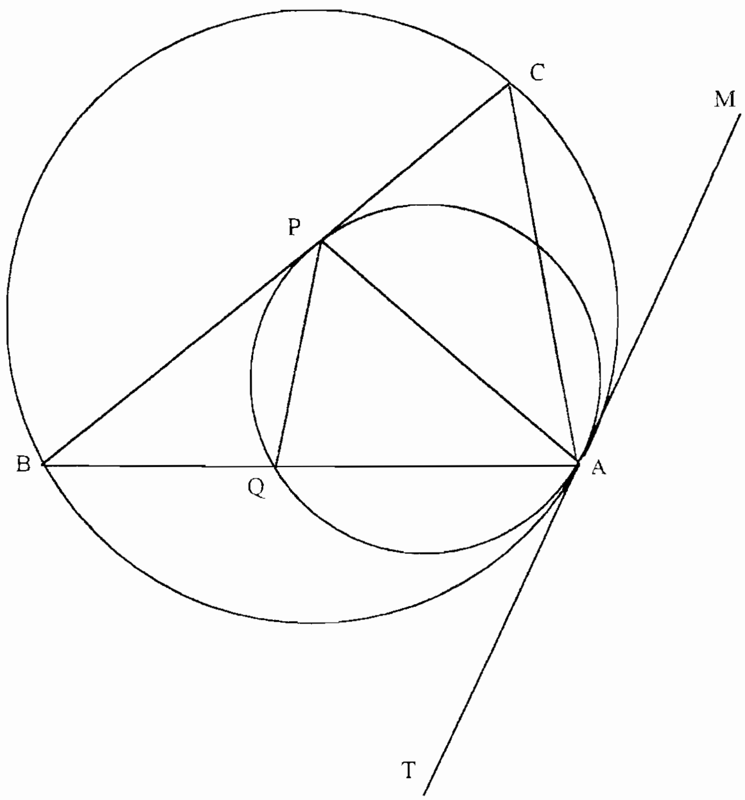
d) If ω is a complex root of $z^3 = 1$ find the value of

i) $1 + \omega + \omega^2$ 1

ii) $\dfrac{1}{3 + 5\omega + 3\omega^2} + \dfrac{1}{7 + 7\omega + 9\omega^2}$ 2

Question 5 15 marks (Use a new page) Marks

a)



Two circles touch internally at A. MT is a common tangent.

A tangent to the inner circle at P cuts the outer circle at B and C.

The interval AB cuts the inner circle at Q.

The intervals PA, CA and PQ have been drawn.

Neatly draw the diagram on your answer sheet.

Prove that PA bisects $\angle BAC$. 5

Question 5 (Continued) Marks

b) The equation $2x^4 - 3x^2 - 2x + k = 0$ has a triple root. 4

Find the value of k.

c) Solve $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ over the complex field 4

given that $1 + i$ is a solution.

d) On an Argand diagram ΔPOQ is a right angled isosceles triangle.

O is the origin, P lies in the first quadrant and Q lies in the second.

$\angle POQ = 90°$.

Given that OP represents the complex number $a + ib$,

Write down the complex number represented by

i) OQ 1

ii) PQ 1

Question 6 15 marks (Use a new page) Marks

a) The function f is given by

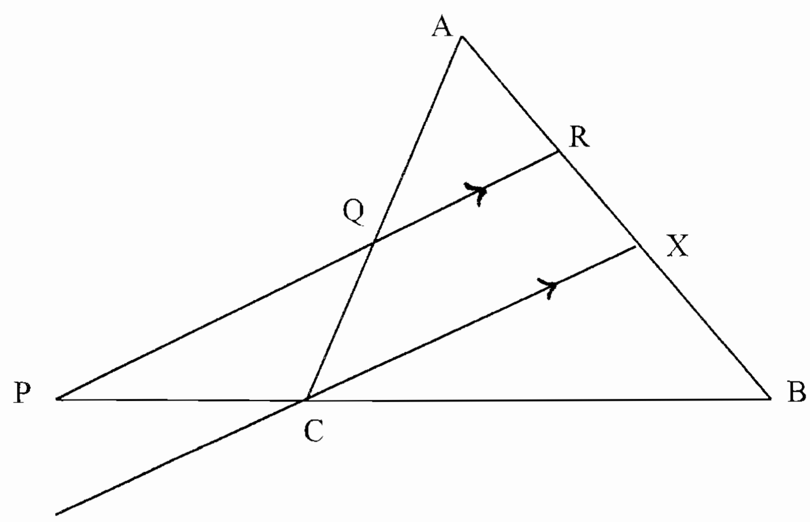
$$f(x) = e^{\frac{x}{1+kx}} \text{ where } k \text{ is a positive constant.}$$

i) Given that the tangent to $y = f(x)$ at the point $(a, f(a))$ passes through the origin show that 3

$$k^2a^2 + (2k-1)a + 1 = 0.$$

ii) Deduce that there is no such tangent as in part i) if $4k > 1$. 2

b)



ABC is a triangle. The line RQ produced meets BC produced at P.

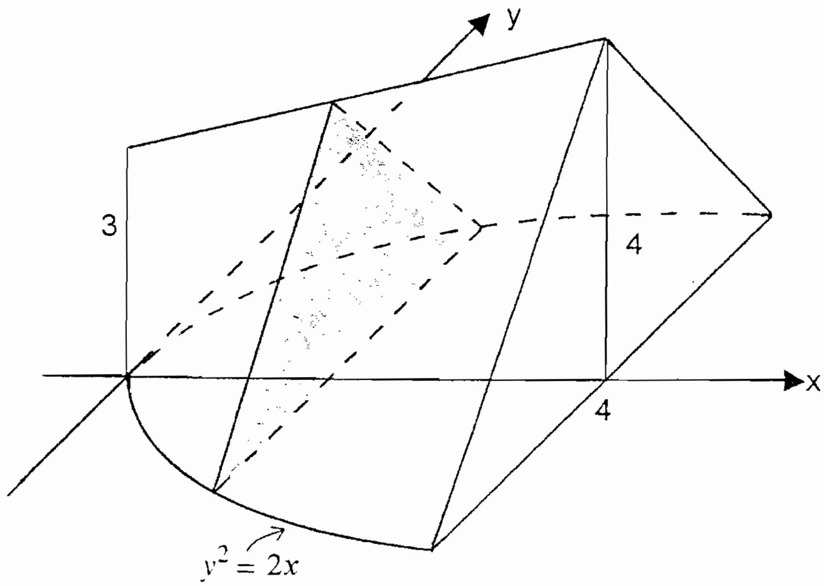
CX is drawn parallel to QR.

Show that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1$ 4

(This is known as Menelaus' Theorem.)

Question 6 (Continued) Marks

c)



The base of the above solid is the area enclosed by $y^2 = 2x$ and $x = 4$.

Each cross section of the solid by planes perpendicular to the x axis is an isosceles triangle with the equal sides meeting above the base of the solid. A typical cross section of thickness Δx has been shaded.

The perpendicular height of the solid at $x = 4$ is 4 and the perpendicular height of the solid at $x = 0$ is 3.

A straight line joins the vertex of the triangle at $x = 4$ to the point 3 above $x = 0$.

i) Show that the perpendicular height of the solid as a function of x is given by $\frac{1}{4}x + 3$ — 2

ii) Find the volume of the solid. — 4

Question 7 15 marks (Use a new page) Marks

a) Given that $I_n = \int x^n e^{2x} dx$

i) Show that $I_n = \frac{x^n e^{2x}}{2} - \frac{n}{2} I_{n-1}$ 3

ii) Use the above result to find $\int x^2 e^{2x} dx$. 2

b) A particle of mass $2kg$ is propelled from the origin along the x axis with an initial velocity of $Q\,m/s$.

The only forces acting on the body in the direction of the x axis are friction which is a constant 16 Newtons and air resistance which equals v^2 Newtons where v is the velocity of the particle t seconds after leaving the origin.

i) Explain why $\frac{dv}{dt} = -8 - \frac{1}{2}v^2$. 1

ii) Show that $t = \frac{1}{2}\tan^{-1}\left(\frac{4Q - 4v}{16 + Qv}\right)$ 5

iii) By using $\frac{dv}{dt} = v\frac{dv}{dx}$ 4

find an expression for v in terms of x.

Question 8 15 marks (Use a new page) Marks

a) Solve the equation 4

$$\sin(x+10°) = \cos(4x) \quad \text{for} \quad 0° \leq x \leq 180°.$$

b) Use Mathematical Induction to prove that 4

$$\tan\theta + 2\tan 2\theta + \cdots\cdots\cdots + 2^{n-1}\tan(2^{n-1}\theta) = \cot\theta - 2^n\cot(2^n\theta)$$

for all positive integers n.

c) The function $f(x)$ is given, for $x > 0$, by

$$f(x) = 2\log_e x - \frac{x^2-1}{x}.$$

i) Show that the only zero of $f(x)$ occurs at $x = 1$. 3

Justify your answer.

ii) Let $g(x) = \dfrac{x\log_e x}{x^2-1}$, for $x > 0$ and $x \neq 1$. 4

Show that $0 < g(x) < \dfrac{1}{2}$.

End of Test.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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SOLUTIONS

Q1

a) $\int \sin^3 x \, dx = \int \sin x \,(1-\cos^2 x)\, dx$ let $u = \cos x$, $du = -\sin x \, dx$

$= \int u^2 - 1 \, du$

$= \frac{1}{3}u^3 - u + c$

$= \frac{1}{3}\cos^3 x - \cos x + c$

b) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

$= \sin^{-1}(x-1) + c$

(or $-\sin^{-1}(1-x) + c$)

c) $\frac{5}{(x-3)(2x-1)} = \frac{A}{x-3} + \frac{B}{2x-1}$

$\therefore 5 = A(2x-1) + B(x-3)$

$x = 3$: $5 = 5A \Rightarrow A = 1$

$x = \frac{1}{2}$: $5 = -\frac{5}{2}B \Rightarrow B = -2$

$\therefore \int \frac{5}{(x-3)(2x-1)} dx = \int \frac{1}{x-3} - \frac{2}{2x-1} dx$

$= \ln(x-3) - \ln(2x-1) + c$

$= \ln\left(\frac{x-3}{2x-1}\right) + c$

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d) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}$

$= \int_0^1 \frac{2dt}{3+t^2}$

$= \left[\frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right)$

$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$

$= \frac{\pi}{3\sqrt{3}}$

e) $\int e^x \sin x \, dx$ $u = e^x$, $v = -\cos x$; $u' = e^x$, $v' = \sin x$

$= -e^x \cos x + \int e^x \cos x \, dx$ $u = e^x$, $v = \sin x$; $u' = e^x$, $v' = \cos x$

$= -e^x \cos x + \left[e^x \sin x - \int e^x \sin x \, dx\right]$

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$\therefore 2\int e^x \sin x \, dx = e^x(\sin x - \cos x)$

$\therefore \int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + c$

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Q2

a) $\frac{\overline{2+3i}}{1+i} = \frac{2-3i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{2-2i-3i-3}{2}$

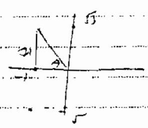
$= \frac{-1-5i}{2}$

$= -\frac{1}{2} - \frac{5}{2}i$

b) i) $|w| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= 2$

ii) $\arg(w) = \frac{2\pi}{3}$



iii) $w^5 + 16w$

$= \left(2\,\text{cis}\,\frac{2\pi}{3}\right)^5 + 16\left(2\,\text{cis}\,\frac{2\pi}{3}\right)$

$= 32\,\text{cis}\,\frac{10\pi}{3} + 32\,\text{cis}\,\frac{2\pi}{3}$

$= 32\,\text{cis}\left(-\frac{2\pi}{3}\right) + 32\,\text{cis}\,\frac{2\pi}{3}$

$= 32\left[\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= 64\cos\frac{2\pi}{3}$

$= -32$

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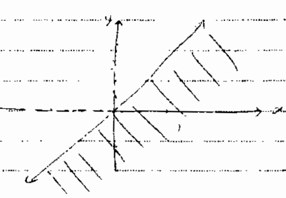
c) i)



ii) $\text{Re}\left(\frac{2}{z}\right) \le 1$ $\frac{2}{z} = \frac{2}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{2(x-iy)}{x^2+y^2}$

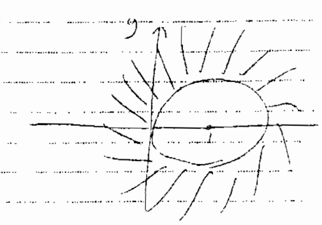
$\therefore \text{Re}\left(\frac{2}{z}\right) \le 1$

$\frac{2x}{x^2+y^2} \le 1$

$2x \le x^2+y^2$

$x^2 - 2x + y^2 \ge 0$

$(x-1)^2 + y^2 \ge 1$



d) $\sqrt{a+3i} = 3+bi$

$a+3i = (3+bi)^2$

$= 9 - b^2 + 6bi$

equating real + imaginary parts

$9 - b^2 = a$ and $6b = 3$

$\Rightarrow b = \frac{1}{2}$

$\therefore 9 - \frac{1}{4} = a$

$a = 8\frac{3}{4}$

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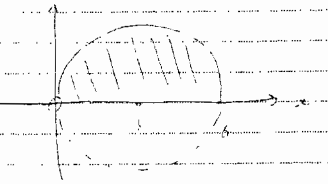
c)



Q3

a) i) $x > 0,\ x \neq 1$.

ii) $y = \frac{x}{\ln x}$

$y' = \frac{(\ln x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2}$

$= \frac{\ln x - 1}{(\ln x)^2}$

st. pt when $y' = 0$

$\therefore \ln x - 1 = 0$

$\ln x = 1$

$x = e$

when $x = e$, $y = e$

$y'' = \frac{(\ln x)^2\left(\frac{1}{x}\right) - (\ln x - 1)(2\ln x)\frac{1}{x}}{(\ln x)^4}$

$= \frac{2 - \ln x}{x(\ln x)^3}$

test st pt: when $x = e$ $y''(e) = \frac{2 - \ln e}{e(\ln e)^3}$

$= +ve$

$\therefore$ min at (e, e)

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iii) inflexion when $y'' = 0$

$\therefore 2 - \ln x = 0$

$\ln x = 2$

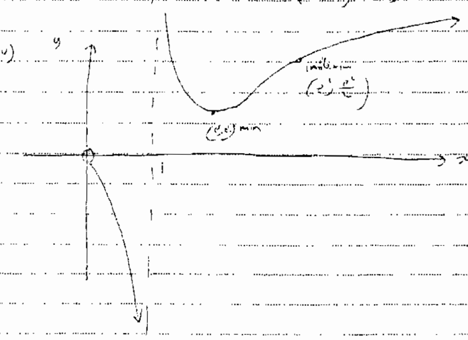
$x = e^2$

when $x = e^2$ $y = \frac{e^2}{\ln e^2}$

$= \frac{e^2}{2}$

$\therefore$ pt of inflexion: $\left(e^2, \frac{e^2}{2}\right)$

iv)



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Q3 b)

i) $y = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

when $x = cp$

$m_T = \frac{-c^2}{c^2 p^2}$

$= -\frac{1}{p^2}$

$\therefore$ equation of tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$

$x + p^2 y = 2cp$ //

ii) $x + p^2 y = 2cp$ } solve simultaneously
$x + q^2 y = 2cq$

$p^2 y - q^2 y = 2cp - 2cq$

$y = \frac{2c(p-q)}{(p-q)(p+q)}$

$= \frac{2c}{p+q}$

$\therefore x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$x + \frac{2cp^2}{p+q} = 2cp$

$x = 2cp - \frac{2cp^2}{[illegible]}$

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iii) $m_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$

$= \frac{-1}{pq}$

equation

$y - \frac{c}{p} = \frac{-1}{pq}(x - cp)$

$pqy - cq = -x + cp$

$x + pqy = c(p+q)$

iv) Sub $(2c, 0)$ into chord

$2c + 0 = c(p+q)$

$p + q = 2$

v) $T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

$x = \frac{2cpq}{p+q}$ $y = \frac{2c}{p+q}$

but $p + q = 2$

$\therefore$ locus $y = c$ but tangents must meet outside of hyperbola in 1st quadrant

$\therefore 0 < x < c$

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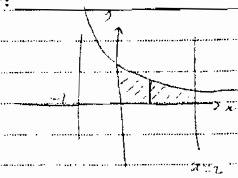
Q4

a)



$\Delta V = 2\pi$ radius × height × thickness

$= 2\pi(2-x) \times y \times \delta x$

$= 2\pi(2-x)\frac{1}{x+1}\,\delta x$

$\therefore V = \int_0^2 2\pi(2-x) \times \frac{1}{x+1}\,dx$

$= 2\pi\int_0^2 \frac{2-x}{x+1}\,dx$

$= 2\pi\int_0^2 \frac{2}{x+1} - \frac{x}{x+1}\,dx$

$= 2\pi\int_0^2 \frac{3}{x+1} - \frac{x+1}{x+1}\,dx$

$= 2\pi\left[3\ln(x+1) - x\right]_0^2$

$= 2\pi\left[(3\ln 3 - 2) - (3\ln 1 - 0)\right]$

$= 2\pi\left[3\ln 3 - 2\right]$ cu units

b) $ae = 2$ $\quad \frac{a}{e} = 4$

$\Rightarrow a = 4e$

$\therefore 4e(e) = 2$

$e^2 = \frac{1}{2}$

$e = \frac{1}{\sqrt{2}}$

$\therefore a = \frac{4}{\sqrt{2}}$ $\quad \therefore$ equation $\frac{x^2}{8} + \frac{y^2}{4} = 1$

$= 2\sqrt{2}$

$b^2 = a^2(1-e^2)$

$= 8(1-\frac{1}{2})$

$= 4$

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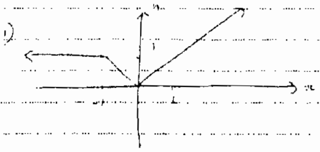
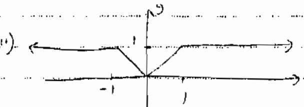
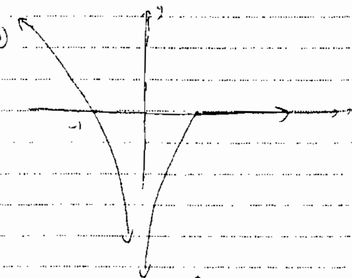
c) i)



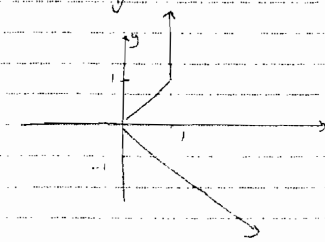
ii)



iii)



iv)



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(Q 4 d) i) If w is a root so is w^2

$w^3 = 1 \quad \therefore (w^2)^3 = (w^3)^2 = 1^2 = 1$

$\therefore w^2$ is a root

$\therefore 1 + w + w^2 = 0$ (sum of roots)

ii) $\frac{1}{3+5w+3w^2} + \frac{1}{7+7w+9w^2}$

$= \frac{1}{3+3w+3w^2+2w} + \frac{1}{7+7w+7w^2+2w^2}$

$= \frac{1}{2w} + \frac{1}{2w^2}$

$= \frac{1}{2}\left[\frac{w+1}{w^2}\right]$

$= \frac{1}{2}\left[\frac{w^2+w}{w^3}\right]$ $\quad \left(\begin{matrix} w^3 = 1 \\ 1+w+w^2=0 \end{matrix}\right)$

$= \frac{1}{2}\left[\frac{-1}{1}\right]$

$= -\frac{1}{2}$

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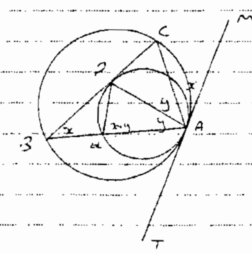
Q5

a)



let $\angle MAC = x$, $\angle PAC = y$

$\therefore \angle ABC = x$ (angle in alternate segment)

$\angle PQA = x+y$ (angle in alternate segment)

$\therefore \angle BPQ = y$ (ext angle of $\triangle$ equals sum of 2 remote interior angles)

$\therefore \angle PAQ = y$ (angle in alternate segment)

$\therefore \angle BAP = \angle CAP$

$\therefore$ PA bisects $\angle BAC$.

b) $2x^4 - 3x^2 - 2x + k = 0$ triple

$8x^3 - 6x - 2 = 0$ double

$24x^2 - 6 = 0$ single

solving

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

substitute $\pm\frac{1}{2}$ into 1st derivative

$8(\frac{1}{2})^3 - 6(\frac{1}{2}) - 2 \neq 0 \quad \therefore \frac{1}{2}$ is not root

$\therefore$ triple root must be $x = -\frac{1}{2}$

sub into equation

$2(-\frac{1}{2})^4 - 3(-\frac{1}{2})^2 - 2(-\frac{1}{2}) + k = 0$

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Q5

c) $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

$1+i$ is a root

$\therefore 1-i$ is a root

$\therefore (x-(1+i))(x-(1-i))$ is a factor

$\therefore x^2 - 2x + 2$ is a factor

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

$(x^2 - 2x + 2)(x^2 - 4x + 5) = 0$

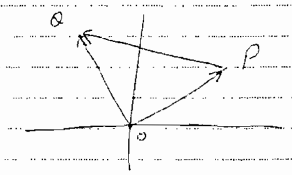
$x^2 - 4x + 5 = 0$

$x = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2}$

$= 2 \pm i$

$\therefore$ roots $2 \pm i$, $1 \pm i$

d)



i) $OQ = i(a+ib)$

$= -b + ia$

ii) $PQ = OQ - OP$

$= (-b+ia) - (a+ib)$

$= -(a+b) + i(a-b)$

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Q6

a) i) $f(x) = e^{\frac{x}{1+kx}}$

$f'(x) = e^{\frac{x}{1+kx}} \times \frac{(1+kx)(1) - (k)x}{(1+kx)^2}$

$= e^{\frac{x}{1+kx}} \times \frac{1}{(1+kx)^2}$

at $x = a$

$m_T = e^{\frac{a}{1+ka}} \times \frac{1}{(1+ka)^2}$

$\therefore$ equation of tangent is

$y - f(a) = \frac{e^{\frac{a}{1+ka}}}{(1+ka)^2}(x-a)$

sub $(0,0)$

$-f(a) = \frac{e^{\frac{a}{1+ka}}}{(1+ka)^2}(-a)$

$-e^{\frac{a}{1+ka}} = \frac{-a e^{\frac{a}{1+ka}}}{(1+ka)^2}$

$1 = \frac{a}{(1+ka)^2}$

$k^2a^2 + 2ka + 1 = a$

$k^2a^2 + a(2k-1) + 1 = 0$

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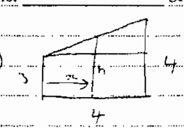
c) i)



as linear $h = mx + b$

when $x = 0$ $h = 3 \Rightarrow b = 3$

when $x = 4$ $h = 4$

$\therefore 4 = 4m + 3$

$m = \frac{1}{4}$

$\therefore h = \frac{1}{4}x + 3$

ii) $A(x) = \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 2y \times (\frac{1}{4}x + 3)$

$= \frac{1}{2} \times 2 \times \sqrt{2x}\,(\frac{1}{4}x + 3)$

$\therefore \Delta V = \sqrt{2x}\,(\frac{1}{4}x + 3)\,\Delta x$

$\therefore V = \int_0^4 \sqrt{2x}\,(\frac{1}{4}x + 3)\,dx$

$= \sqrt{2}\int_0^4 \frac{1}{4}x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\,dx$

$= \sqrt{2}\left[\frac{2}{20}x^{\frac{5}{2}} + 2x^{\frac{3}{2}}\right]_0^4$

$= \sqrt{2}\left[\left(\frac{2 \times 32}{20} + 8 \times 2\right) - (0)\right]$

$= \frac{96\sqrt{2}}{5}$ cu. units

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ii) no tangent if equation has no roots

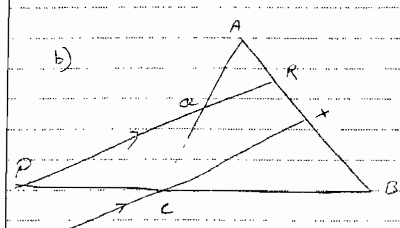
i.e. $\Delta < 0$

$(2k-1)^2 - 4(k^2)(1) < 0$

$4k^2 - 4k + 1 - 4k^2 < 0$

$4k > 1$

b)



$\frac{BP}{PC} = \frac{BR}{RX}$ ✱ (intercepts in proportion)

$\frac{QC}{AQ} = \frac{RX}{AR}$ (intercepts in proportion)

$RX = \frac{AR \times QC}{AQ}$ sub into ✱

$\frac{BP}{PC} = \frac{BR}{\frac{AR \times QC}{AQ}}$

$\therefore \frac{BP \times AR \times QC}{PC \times BR \times AQ} = 1$

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Q7

a) i) $I_n = \int x^n e^{2x}\,dx$ $\quad u = x^n \quad v = \frac{1}{2}e^{2x}$ $\quad u' = nx^{n-1} \quad v' = e^{2x}$

$\therefore I_n = (x^n)(\frac{1}{2}e^{2x}) - \int (nx^{n-1})(\frac{1}{2}e^{2x})\,dx$

$= \frac{x^n e^{2x}}{2} - \frac{n}{2}\int x^{n-1}e^{2x}\,dx$

$= \frac{x^n e^{2x}}{2} - \frac{n}{2} I_{n-1}$

ii) $I_2 = \int x^2 e^{2x}\,dx$

$= \frac{1}{2}x^2e^{2x} - I_1$

$= \frac{1}{2}x^2e^{2x} - [\frac{1}{2}xe^{2x} - \frac{1}{2}I_0]$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$

$= \frac{1}{2}e^{2x}(x^2 - x + \frac{1}{2}) + c$

b) i) $F = ma$

$\therefore 2 \times a = -16 - v^2$ (-ve as acting against motion)

$\therefore a = -8 - \frac{1}{2}v^2$

$\frac{dv}{dt} = -8 - \frac{1}{2}v^2$ as $a = \frac{dv}{dt}$

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ii) $\frac{dv}{dt} = -(8 + \frac{1}{2}v^2)$

$\frac{dv}{8 + \frac{1}{2}v^2} = -dt$

$\frac{2\,dv}{16 + v^2} = -dt$

$\therefore \int \frac{2\,dv}{16+v^2} = -\int dt$

$\frac{1}{2}\tan^{-1}\frac{v}{4} = -t + c$

when $t = 0$ $v = Q$

$\therefore c = \frac{1}{2}\tan^{-1}\frac{Q}{4}$

$\therefore t = \frac{1}{2}\tan^{-1}\frac{Q}{4} - \frac{1}{2}\tan^{-1}\frac{v}{4}$

$2t = \tan^{-1}\frac{Q}{4} - \tan^{-1}\frac{v}{4}$

take tan of both sides

$\tan 2t = \tan\left(\tan^{-1}\frac{Q}{4} - \tan^{-1}\frac{v}{4}\right)$

$= \frac{\tan\tan^{-1}\frac{Q}{4} - \tan\tan^{-1}\frac{v}{4}}{1 + (\tan\tan^{-1}\frac{Q}{4})(\tan\tan^{-1}\frac{v}{4})}$

$= \frac{\frac{Q}{4} - \frac{v}{4}}{1 + \frac{Qv}{16}}$

$= \frac{4Q - 4v}{16 + Qv}$

$2t = \tan^{-1}\left(\frac{4Q - 4v}{16 + Qv}\right)$

$t = \frac{1}{2}\tan^{-1}\left(\frac{4Q - 4v}{16 + Qv}\right)$

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iii) $v\frac{dv}{dx} = -8 - \frac{1}{2}v^2$

$\frac{v\,dv}{16 + v^2} = -\frac{1}{2}dx$

integrating

$\int \frac{v\,dv}{16+v^2} = -\frac{1}{2}\int dx$

$\frac{1}{2}\ln(16 + v^2) = -\frac{1}{2}x + c$

when $t = 0$ $v = Q$ $x = 0$

$\therefore c = \frac{1}{2}\ln(16 + Q^2)$

$\therefore x = \ln(16 + Q^2) - \ln(16 + v^2)$

$x = \ln\left(\frac{16 + Q^2}{16 + v^2}\right)$

$e^x = \frac{16 + Q^2}{16 + v^2}$

$e^{-x} = \frac{16 + v^2}{16 + Q^2}$

$16 + v^2 = (16 + Q^2)e^{-x}$

$v^2 = (16 + Q^2)e^{-x} - 16$

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Q8

a) $\sin(x + 10°) = \cos 4x$

$\therefore \cos(90° - (x + 10°)) = \cos 4x$

$\therefore 90 - (x + 10) = n \times 360° \pm 4x$

$80° - x = n \times 360° + 4x$ $\qquad 80° - x = n \times 360° - 4x$

$-5x = n \times 360° - 80°$ $\qquad 3x = n \times 360° - 80°$

$x = -\frac{1}{5}(n \times 360° - 80°)$ $\qquad x = \frac{1}{3}(n \times 360° - 80°)$

$n = 0$ $x = 16°$ $\qquad n = 1$ $x = 93\frac{1}{3}°$

$n = -1$ $x = 88°$

$n = -2$ $x = 160°$

$\therefore$ Solutions: $16°, 88°, 93\frac{1}{3}°, 160°$

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b) **Step 1** test $n=1$ $\quad LHS = \tan\theta$

$$RHS = \cot\theta - 2\cot 2\theta \quad *$$
$$= \cot\theta - 2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)$$
$$= \cot\theta - \cot\theta + \tan\theta$$
$$= \tan\theta$$
$$= LHS$$

$\therefore$ true for $n=1$

Step 2 Assume true for $n=k$

i.e. $\tan\theta + \tan 2\theta + \dots + 2^{k-1}\tan(2^{k-1}\theta) = \cot\theta - 2^k\cot(2^k\theta)$

Step 3 show true for $n=k+1$

i.e. $S_{k+1} = S_k + T_{k+1}$

$$= \cot\theta - 2^k\cot(2^k\theta) + 2^k\tan(2^k\theta)$$
$$= \cot\theta - 2^k\cot(2^k\theta) + 2^k\left(\cot 2^k\theta - 2\cot(2^{k+1}\theta)\right)$$
(* from above)
$$= \cot\theta - 2^{k+1}\cot(2^{k+1}\theta)$$

$\therefore$ true for $n=k+1$ if true for $n=k$

Step 4 As true for $n=1$ it is also true for $n=1+1$ i.e. $n=2$
As true for $n=2$ it is also true for $n=2+1$ i.e. $n=3$
and so on for all positive integers n.

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c) $f(x) = 2\log_e x - \frac{x^2-1}{x}$

i) $f(1) = 2\times 0 - \frac{1-1}{1}$
$= 0$

$\therefore$ zero at $x=1$

$$f'(x) = \frac{2}{x} - 1 - x^{-2}$$
$$= \frac{2x - x^2 - 1}{x^2}$$
$$= \frac{-(x^2-2x+1)}{x^2}$$
$$= \frac{-(x-1)^2}{x^2}$$

which is negative for all values of $x>0$ except $x=1$
$\therefore$ curve always decreasing except at $x=1$
$\therefore$ Can only be one zero

ii) $g(x) = \frac{x\log_e x}{x^2-1}$

Consider $0<x<1$ $\qquad x\ln x < 0$
$\qquad x^2-1<0$
$\therefore g(x) > 0$

$x>1$ $\qquad x\ln x > 0$
$\qquad x^2-1>0$
$\therefore g(x)>0$

$\therefore g(x) > 0$ for all $x>0$, $x\neq 1$

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Q8 c) ii) (cont)

$$g(x) = \frac{x\ln x}{x^2-1}$$
$$= \frac{x}{x^2-1}(\ln x)$$
$$= \frac{x}{x^2-1}\cdot\frac{1}{2}(2\ln x)$$
$$= \frac{x}{x^2-1}\times\frac{1}{2}\left[f(x) + \frac{x^2-1}{x}\right]$$
$$= \frac{1}{2}\left[1 + \frac{x f(x)}{x^2-1}\right]$$

for $0<x<1$ $\qquad x^2-1<0$
$\qquad f(x)>0$ $\quad$ from part i)

$$\therefore \frac{x f(x)}{x^2-1} < 0$$

$$g(x) < \frac{1}{2}$$

for $x>1$ $\qquad x^2-1>0$
$\qquad f(x)<0$ $\quad$ from part i)

$$\therefore \frac{x f(x)}{x^2-1} < 0$$

$$\therefore g(x) < \frac{1}{2}$$

$\therefore$ Combining both results $\quad 0 < g(x) < \frac{1}{2}$

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