Sydney Technical High School

TRIAL HIGHER SCHOOL CERTIFICATE

2007

MATHEMATICS EXTENSION 2

General Instructions

- Reading time - 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplies at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________

Class: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

QUESTION 1 (15 Marks) — Marks

a) Find by using a suitable substitution or otherwise

i) $\displaystyle\int \frac{dx}{\sqrt{9-16x^2}}$ — 2

ii) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$ — 2

iii) $\displaystyle\int \sec^3 x \tan x \, dx$ — 2

b) Using the substitution $x = 3\tan\theta$ or otherwise find $\displaystyle\int \frac{dx}{(9+x^2)^{\frac{3}{2}}}$ — 4

c) i) Show that $\displaystyle\frac{d}{dx}\left[\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)\right] = \frac{1}{x^2-a^2}$ — 2

ii) Hence by using the substitution $x = u^2$ or otherwise find $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$ — 3

Question 2 (15 marks)

a) Find d if $(3+2i)(4-di)$ is wholly imaginary — 2

b) If $\alpha = -2+2\sqrt{3}\,i$ and $\beta = 1-i$

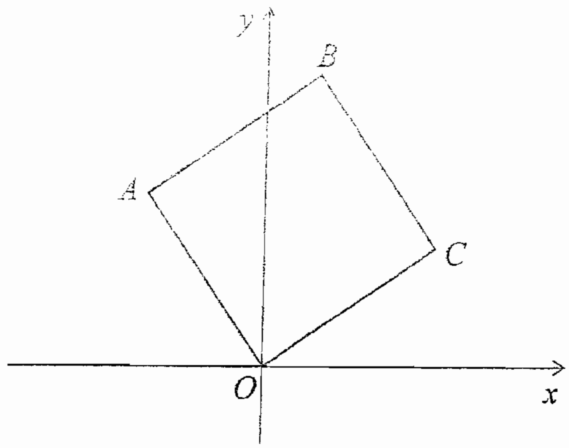
i) Find $\dfrac{\alpha}{\beta}$ in the form $x+iy$ — 1

ii) Express α in modulus – argument form — 1

iii) Given $\beta = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)$ find the modulus- argument form of $\dfrac{\alpha}{\beta}$ — 2

iv) Hence find the exact value of $\cos\left(\frac{\pi}{12}\right)$ — 2

c) **Marks**



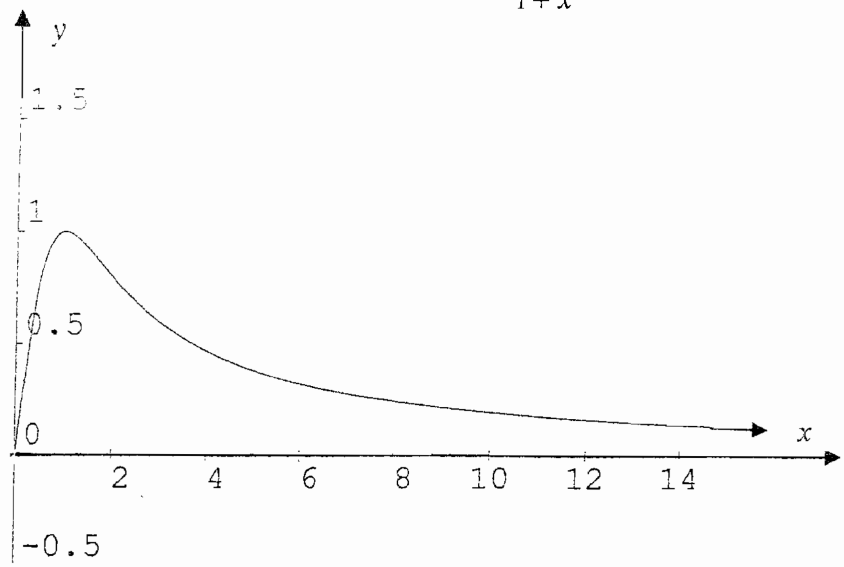
On the Argand diagram above, $OABC$ is a square. If B represents the complex number $4+6i$ find the complex number represented by C. 3

d) i) Sketch the region in the complex number plane where the inequalities $|z-1| \leq |z-i|$ and $|z-2-2i| \leq 1$ hold simultaneously 2

ii) If P is a point on the boundary of this region representing the complex number z, find the values of z in the form $x+iy$ where $\arg(z-1)=\frac{\pi}{4}$ 2

Question 3 **(15 marks)** **Marks**

a) The diagram shows the graph of $f(x) = \dfrac{2x}{1+x^2}$ *for* $x \geq 0$



For each of the following draw a one-third page sketch:

i) Sketch the graph of $y = \dfrac{2x}{1+x^2}$ for all real x 1

ii) Use your completed graph in (i) to help sketch the graphs of

α) $y = \dfrac{|2x|}{1+x^2}$ 2

β) $y^2 = \dfrac{2x}{1+x^2}$ 2

γ) $y = \log_e\left[\dfrac{2x}{1+x^2}\right]$ 2

iii) Sketch $y = \dfrac{1+x^2}{2x}$ clearly showing and stating the equations of any asymptotes. 2

iv) Find the value(s) of A so that the graphs of

$y = \dfrac{Ax}{1+x^2}$ and $y = \dfrac{1+x^2}{Ax}$ have no points in common. 2

b) The area between the curve $y = \dfrac{2x}{1+x^2}$ and the x-axis for $0 \leq x \leq 1$ is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the resulting solid of revolution 4

Question 4 (15 marks) **Marks**

a) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos(-\theta), b\sin(-\theta))$ are the extremities of the latus rectum, $x = ae$, of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

i) Draw a neat diagram, marking the points P and Q and clearly showing the angle θ. 1

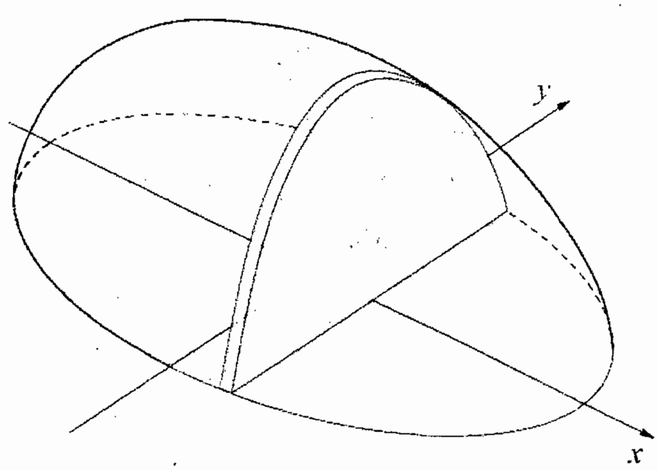
ii) Show that $\cos\theta = e$ 1

iii) Show that the length of PQ is $\frac{2b^2}{a}$ 2

b) Show that the area enclosed between the parabola $x^2 = 4ay$ and its latus rectum is $\frac{8a^2}{3}$ units^2 3

c) A solid figure has as its base, in the xy plane, the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Cross-sections perpendicular to the x-axis are parabolas with latus rectums in The xy plane



i) Show that the area of the cross-section at $x = h$ is $\frac{16 - h^2}{6}$ units^2. 3

$[\textit{use your answer to part}(b)]$

ii) Hence, find the volume of this solid. 2

d) Over the complex field $P(x) = 2x^3 - 15x^2 + Cx - D$ has a zero $x = 3 - 2i$

i) Determine the other two zeros 2

ii) Find the value of D 1

Question 5 (15 marks)

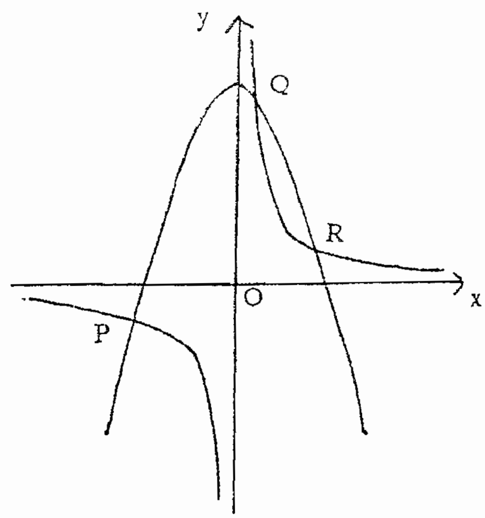
a) The roots of the equation $z^5 - 1 = 0$ are $1,\ w,\ w^2,\ w^3,\ w^4$

i) Mark this information on an Argand diagram 1

ii) Find a real quadratic equation with roots $w + w^4$ and $w^2 + w^3$ 2

iii) Hence find the value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$ 2

b)



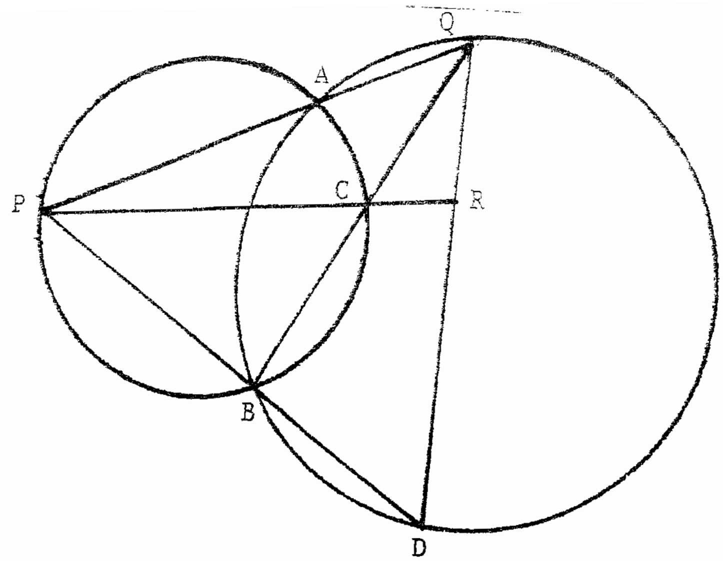
The curves $y = k - x^2$, for some real number k, and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha,\ x = \beta$ and $x = \gamma$.

i) Show that the monic cubic equation with coefficients in terms of k whose roots are $\alpha^2,\ \beta^2$ and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$ 3

ii) Find the monic cubic equation with coefficients in terms of k whose roots are $\frac{1}{\alpha^2},\ \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$ 2

iii) Hence show that $OP^2 + OQ^2 + OR^2 = k^2 + 2k$, where O is the origin 2

c) **Marks**

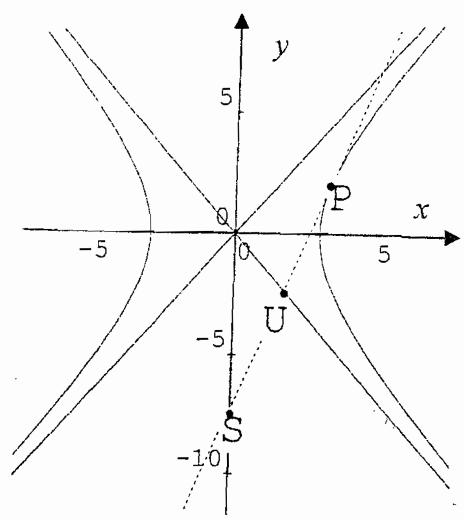


i) Copy the diagram onto your page.

ii) Prove $BCRD$ is a cyclic quadrilateral (Hint: let $\angle D = \theta$) 3

Question 6 (15 marks)

a)



Consider the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

i) Write down the equation of each asymptote 1

ii) By differentiation find the gradient of the tangent to the hyperbola at $P(3\sec\theta,\ 4\tan\theta)$ 1

iii) Show that the equation of the tangent at P is $4x = 3\sin\theta y + 12\cos\theta$ 2

iv) Find the x-coordinate of U, the point where the tangent meets the asymptote (as shown on the diagram). 2

v) Using the x-values only, find the value for θ such that U is the mid point of PS. 2

Marks

b) i) Show that $\int_0^{\frac{\pi}{4}} \tan\theta \, d\theta = \frac{1}{2}\log_e 2$ 2

ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta$ show that for $n \geq 2$ 3

$$I_n + I_{n-2} = \frac{1}{n-1}$$

iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^5\theta \, d\theta$ 2

Question 7 (15 marks)

a) i) Show that $\tan^{-1}(3) - \tan^{-1}(\frac{1}{2}) = \frac{\pi}{4}$ 2

ii) Prove by mathematical induction that

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right) = \tan^{-1}(2n+1) - \frac{\pi}{4}$$ 4

is true for all integral values of n for $n \geq 1$

b) A particle is moving in a straight line. After time t seconds it has displacement x metres from a fixed point O on the line, velocity $v = \frac{1-x^2}{2}\ ms^{-1}$ and acceleration $a\,ms^{-2}$. Initially the particle is at O.

i) Find an expression for a in terms of x 1

ii) Show that $\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}$ and hence find an expression for x in terms of t. 3

iii) Describe the motion of the particle, explaining whether it moves to the left or right of O, whether it slows down or speeds up, and where its limiting position is. 2

c) i) Differentiate $x^3 + y^3 = 6xy$ to find $\frac{dy}{dx}$. 1

ii) Find the x value(s) of the point(s) where $\frac{dy}{dx} = 0$ 2

Question 8 (15 marks) **Marks**

a) i) If $S = 1 - x + x^2 - x^3 + \ . \ . \ .$ where $|x| < 1$, find an expression for S, the limiting sum, of the series. 1

ii) By integrating both sides of this expression and then making a substitution for x show that $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ . \ .$ 2

b) i) Show that $\int x \tan^{-1} x \, dx = \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2}x + c$ 3

ii) If $I_n = \int_0^1 x^n \tan^{-1} x \, dx$ for $n \geq 2$ show that

$$I_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} I_{n-2}$$ 4

c) i) Write the general solution to $\cos 5\theta = \cos A$ 1

ii) Hence or otherwise find the total number of solutions to the equation $\cos 5\theta = \sin \theta$ for $0 \leq \theta \leq 10\pi$ 4

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

1 a) $\int \frac{1}{\sqrt{9-16x^2}}\,dx = \frac{1}{4}\sin^{-1}\frac{4x}{3} + c$

b) $\int \frac{dx}{\sqrt{(x+3)^2+4}} = \ln\left(x+3+\sqrt{(x+3)^2+4}\right)+c$

c) $\int \sec^3 x \tan x = \int \sec^2 x \frac{d}{dx}(\sec x)$

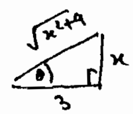
$= \frac{1}{3}\sec^3 x + c$

b) $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$\int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{(9+9\tan^2\theta)^{3/2}}$

on simplification $= \frac{1}{9}\int \cos\theta\, d\theta$

$= \frac{1}{9}\sin\theta$



$= \frac{1}{9}\frac{x}{\sqrt{x^2+9}}$

c) $\frac{d}{dx}\left[\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)\right] = \frac{1}{2a}\frac{d}{dx}\left[\log(x-a)-\log(x+a)\right]$

$= \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$

on simplification $= \frac{1}{x^2-a^2}$

d) $x = u^2 \quad dx = 2u\,du$

$\int \frac{\sqrt{x}}{x-1}\,dx = \int \frac{u \cdot 2u}{u^2-1}\,du$

$= 2\int \frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\,du$

$= 2\left[u + \frac{1}{2}\log_e\left(\frac{u-1}{u+1}\right)\right]+c$

$= 2\sqrt{x} + \log_e\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+c$

Question 2

a) $(3+2i)(4-di) = (12+2d)$

$\therefore 12+2d = 0$

$d = -$

b) i) $\frac{\alpha}{\beta} = (-1-\sqrt{3}) + i(\sqrt{3} -$

ii) $\alpha = 4\operatorname{cis} 2\pi/3$

iii) $\frac{\alpha}{\beta} = \frac{4\operatorname{cis} 2\pi/3}{\sqrt{2}\operatorname{cis}(-\pi/4)}$

$= 2\sqrt{2}\operatorname{cis}(11\pi/12)$

iv) $2\sqrt{2}\cos 11\pi/12 = -1 -$

$\cos 11\pi/12 = -\frac{1-}{2}$

$\therefore \cos \pi/12 = \frac{1+\sqrt{3}}{2\sqrt{2}}$

c) $C = x+iy$

$\therefore A = i(x+iy)$

$= ix - y$

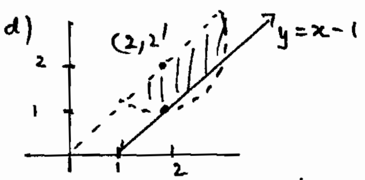
$B = C + A$

$4+6i = x-y + i(x+$

$\therefore x = 5, y = 1$

$A = 5+i$

d)



ii P is point of interse

$y = x-1$ and $(x-2)^2+(y-$

$\therefore (x-2)^2+(x-3)^2 = 1$

$x^2-5x+6 = 0$

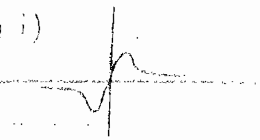
$(x-3)(x-2)$

$x=2, y=1 \qquad x=3, y=$

$\therefore$ P is $2+i$ or $3+2$

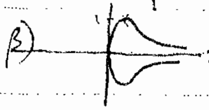
...stion 3

i)

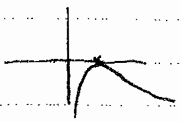


ii) α)

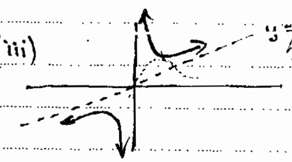
β)



γ)



iii)



$\frac{Ax}{1+x^2} = \frac{1+x^2}{Ax}$

$\frac{Ax}{1+x^2} = \pm 1$

$x^2 - Ax + 1 = 0$ or $x^2 + Ax + 1 = 0$

point of intersection $\Delta < 0$

solving $A^2 < 4$

$\therefore -2 < A < 2 \quad , A \neq 0$

Volume $= \lim_{\Delta x \to 0} \sum_0^1 2\pi x y \, \Delta x$

$= 4\pi \int_0^1 \frac{x^2}{1+x^2} \, dx$

$= 4\pi \int_0^1 \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \, dx$

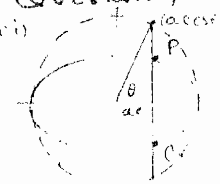
$= 4\pi \left[x - \tan^{-1} x \right]_0^1$

$= 4\pi \left[1 - \frac{\pi}{4} \right]$

$= 4\pi - \pi^2$

Question 4

a) i)



ii) [illegible]

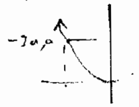
iii) [illegible]

$= 2b\sqrt{1-e^2}$ $\qquad b^2 = a^2(1-e^2)$

$= 2b \cdot \frac{b}{a}$

$= \frac{2b^2}{a}$

b) i)



$x^2 = 4ay$

$\text{Area} = 4a^2 - \int_{-2a}^{2a} \frac{x^2}{4a} \, dx$

$= 4a^2 - \frac{1}{2a} \left[\frac{x^3}{3} \right]_0^{2a}$

$= \frac{8a^2}{3}$

c ii) at $x = h$.

$\frac{h^2}{16} + \frac{y^2}{4} = 1$

$y = \pm 2\sqrt{1 - \frac{h^2}{16}}$

$\therefore 2a = 2\sqrt{1 - \frac{h^2}{4}}$

$a = \sqrt{1 - \frac{h^2}{16}}$

$\therefore \text{Area} = \frac{8}{3} \cdot \left(1 - \frac{h^2}{16}\right)$

$= \frac{16 - h^2}{6}$

iii) $\text{Volume} = \lim_{\Delta h \to 0} \sum_{-4}^{4} \frac{16 - h^2}{6} \cdot \Delta h$

$= \frac{2}{6} \int_0^4 16 - h^2 \, dh$

$= \frac{1}{3} \left[16h - \frac{h^3}{3} \right]_0^4$

$= \frac{128}{9} u^3$

d i) $x = 3 + 2i$ (conjugate root theorem)

$3 + 2i + 3 - 2i + \alpha = 15/2$ $(\Sigma \alpha)$

$\therefore \alpha = 3/2$

ii) $\Sigma \alpha\beta\gamma = (3+2i)(3-2i) \times \frac{3}{2} = \frac{k}{2}$

$k = $

Teacher's Name .... Student's Name/Nº ....

$$I_n = \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2}\left[I_n + I_{n-2}\right] + \frac{(n-1)}{2}\int_0^1 x^{n-1}\,dx$$

$$I_n = \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2}\left[I_n + I_{n-2}\right] + \frac{(n-1)}{2}\left[\frac{x^n}{n}\right]_0^1$$

$$2I_n = \frac{\pi}{2} - 1 - (n-1)I_n - (n-1)I_{n-2} + \frac{n-1}{n}$$

$$(n+1)I_n = \frac{\pi}{2} - \frac{1}{n} - (n-1)I_{n-2}$$

$$\therefore I_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}I_{n-2}$$

c) i) $\cos 5\theta = \cos A$

$5\theta = 2n\pi \pm A$

ii) $\cos 5\theta = \cos(\pi/2 - \theta)$

$5\theta = 2n\pi \pm (\pi/2 - \theta)$

A | B

$\therefore 5\theta = 2n\pi + \pi/2 - \theta$ OR $5\theta = 2n\pi - \pi/2 + \theta$

$6\theta = \frac{4n\pi + \pi}{2}$ | $4\theta = \frac{4n\pi - \pi}{2}$

$\theta = \frac{\pi(4n+1)}{12}$ | $\theta = \frac{\pi(4n-1)}{8}$

Now $0\pi \le \frac{\pi(4n+1)}{12} \le 10\pi$ | $0\pi \le \frac{\pi(4n-1)}{8} \le 10\pi$

$0 \le 4n+1 \le 120$ | $0 \le 4n-1 \le 80$

$4n \le 119$ | $4n \le 81$

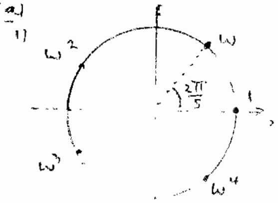
$n \le 29\frac{3}{4}$ | $n \le 20\frac{1}{4}$

$\therefore n = 29$ | $n = 20$

$\therefore$ 49 solutions (there are no common solutions)

Q.5a)

i)



ii) $(\omega+\omega^4)+(\omega^2+\omega^3) = -1$ sum of roots

$(\omega+\omega^4)(\omega^2+\omega^3) = \omega^3+\omega^4+\omega^6+\omega^7$
$= \omega^3+\omega^4+\omega+\omega^2$
$= -1$

$\therefore x^2+x-1=0$

iii $(\omega+\omega^4)+(\omega^2+\omega^3) = 2\cos 2\pi/5 + 2\cos 4\pi/5$

$\therefore \cos 2\pi/5 + \cos 4\pi/5 = -1/2$

b)

i) $k - x^2 = \frac{1}{x}$

$\therefore x^3 - kx + 1 = 0$ has roots α, β, γ

~~let~~ $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$.

$\therefore x^3 - 2kx^2 + k^2x - 1 = 0$ (A)

has roots $\alpha^2, \beta^2, \gamma^2$

let $y = \frac{1}{\alpha}$ in A $\Rightarrow \alpha = \frac{1}{y}$

$\therefore x^3 - k^2x^2 + 2kx - 1 = 0$ B

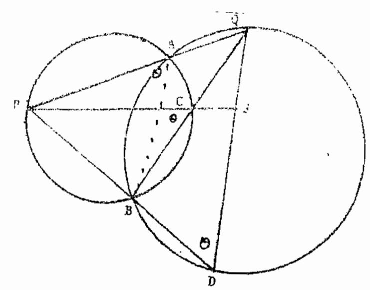
has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

ii) $OP^2+OQ^2+OR^2 = \frac{1}{\alpha^2}+\alpha^2+\frac{1}{\beta^2}+\beta^2+\frac{1}{\gamma^2}+\gamma^2$

$= \alpha^2+\beta^2+\gamma^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

$= 2K+K^2$ from A and B

c)



Join AB

$\angle PAB = \theta$ (exterior angle of cyclic quad)

$\angle PCB = \theta$ (angles in same segment subtended by arc BP)

$\therefore$ DBCQ is a cyclic quadrilateral (exterior $\angle$BC.. is interior opposite $\angle$BDQ)

Q 6 a)

i) $y = 4/3x,\ y = -4/3x$

ii) $\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{4\sec\theta}{3\tan\theta}$ $\left(\frac{4}{3\sin\ldots}\right.$

iii) $y - 4\tan\theta = \frac{4\sec\theta}{3\tan\theta}$ (...

$3\tan\theta y - 12\tan^2\theta = 4\sec\theta x$ ...

$4\sec\theta x = 3\tan\theta y + 12($...

$\times\cos\theta$ $4x = 3\sin\theta y + 12\cos$...

iv) Now $4x = -3y$

$\therefore -3y = 3\sin\theta y + 1$...

$\therefore y = \frac{-12\cos\theta}{3\sin\theta - 3}$

$\therefore x = \frac{3\cos\theta}{\sin\theta + 1}$

v) $\frac{3\sec\theta}{2} = \frac{3\cos\theta}{1+\sin\theta}$

Simplifies to

$6\sin^2\theta + 3\sin\theta - 3 =$ ...

$3(2\sin\theta - 1)(\sin\theta + 1)$ ...

$\therefore \sin\theta = 1/2$ or -1

$\sin\theta \ne -1$ sin $\frac{3\cos\theta}{1+\sin\theta}$ i...

$\therefore \theta = \pi/6$

b) $\int_0^{\pi/4} \tan\theta\,d\theta = \int_0^{\pi/4} \frac{\sin\theta}{\cos\theta}$ ...

$= -[\log_e(\cos$ ...

$= -[\log_e \frac{1}{\sqrt{\ldots}}$ ...

$= \frac{1}{2}\log_e$ ...

ii) $I_n = \int_0^{\pi/4} \tan^n\theta\,d\theta = \int_0^{\pi/4} \tan^{n-2}\theta.$ ...

$= \int_0^{\pi/4} \tan^{n-2}\theta\sec^2\theta$ ...

$\therefore I_n + I_{n-2} = \left[\frac{\tan^{n-1}}{n-1}\right]$ ...

$= \frac{1}{n-1}$

iii) $I_5 + I_3 = \frac{1}{4}$

$I_3 + I_1 = 1/2$

$\therefore I_5 - I_1 = -1/4$

$I_5 = -1/4 + I_1$ → from ...

$\int_0^{\pi/4} \tan^5\theta\,d\theta = \frac{1}{2}\log_e 2 - 1/4$

$[\tan^{-1}(3) - \tan^{-1}(1/2)]$

$\frac{3 - 1/2}{1 + 3 \times 1/2}$

$= 1$

$= \tan \pi/4$

$\tan^{-1}(3) - \tan^{-1}(1/2) = \pi/4$

$= 1$. statement is true

n(a)

assume true for $n = k$.

$\sum \tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2k+1) - \pi/4$ (A)

$\sum_{r=1}^{k+1} \tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2k+3) - \pi/4$

$\sum_{r=1}^{k+1} \tan^{-1}\frac{1}{2r^2} = \sum_{r=1}^{k} \tan^{-1}\frac{1}{2r^2} + \tan^{-1}\frac{1}{2(k+1)^2}$

$= \tan^{-1}(2k+1) - \pi/4 + \tan^{-1}\frac{1}{2(k+1)^2}$

is true if

$\tan^{-1}(2k+3) - \pi/4 = \tan^{-1}(2k+1) - \pi/4 + \tan^{-1}\frac{1}{2(k+1)^2}$

$\frac{1}{2(k+1)^2} = \tan(\tan^{-1}(2k+3) - \tan^{-1}(2k+1))$

$\tan[\tan^{-1}(2k+3) - \tan^{-1}(2k+1)]$

$\frac{2k+3-(2k+1)}{1+(2k+3)(2k+1)}$

$= \frac{2}{4k^2+8k+4}$

$= \frac{1}{2(k+1)^2}$

$\tan^{-1}(2k+3) - \tan^{-1}(2k+1) = \tan^{-1}\frac{1}{2(k+1)^2}$

true for $n = k+1$ if true $n = k$ and since true for 1 it is true for all integral values of n

b) i) $v = \frac{1}{2}(1 - x^2)$

$\frac{dv}{dx} = -x$

Now $a = v\frac{dv}{dx} = \frac{x^3 - x}{2}$

ii) $\frac{1}{1+x} + \frac{1}{1-x} = \frac{1-x+1+x}{(1+x)(1-x)} = \frac{2}{1-x^2}$

$\frac{dx}{dt} = \frac{1-x^2}{2}$

$\int_0^x \frac{2}{1-x^2}\,dx = \int_0^t dt$

$\int_0^x \frac{1}{1+x} + \frac{1}{1-x}\,dx = t$

$[\log_e(1+x) - \log(1-x)]_0^x = t$

$\therefore t = \log_e \frac{1+x}{1-x}$

$e^t = \frac{1+x}{1-x}$

$e^t(1-x) = 1+x$

$x = \frac{e^t - 1}{e^t + 1}$ $\left(\frac{1-e^{-t}}{1+e^{-t}}\right)$

iii) moves to right, slowing down. limiting position is $x = 1$

c) i) $x^3 + y^3 = 6xy$

$3x^2 + 3y^2\frac{dy}{dx} = 6[y.1 + x\frac{dy}{dx}]$

$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$

ii) $\frac{dy}{dx} = 0$ $\therefore 2y - x^2 = 0$

$y = \frac{x^2}{2}$

Sub in (i)

$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x.\frac{x^2}{2}$

$16x^3 - x^6 = 0$

$x^3(16 - x^3) = 0$

$x = 0$ or $\sqrt[3]{16}$

at $x = 0$ $\frac{dy}{dx}$ is undefined

$\therefore x = \sqrt[3]{16}$

a) i) $S = \frac{1}{1+x}$

ii) $\int \frac{1}{1+x}\,dx = \int 1 - x + x^2 - x^3\,dx$

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Let $x = 1$

$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - 1/4 + \ldots$

b) i) $\int x\tan^{-1}x\,dx = \int \tan^{-1}x \frac{d}{dx}(\frac{x^2}{2})\,dx$

$= \frac{x^2}{2}.\tan^{-1}x - \int \frac{x^2}{2}.\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}[x - \tan^{-1}x]$

$= \frac{1}{2}[x^2+1]\tan^{-1}x - 1/2\,x + c$

ii) $\int_0^1 x^n \tan^{-1}x\,dx = \int_0^1 x^{n-1}(x\tan^{-1}x)\,dx$

$= \int_0^1 x^{n-1}\frac{d}{dx}[\frac{1}{2}(x^2+1)\tan^{-1}x - 1/2\,x]\,dx$

$= [x^{n-1}[1/2(x^2+1)\tan^{-1}x - 1/2\,x]]_0^1 - \int_0^1 (n-1)x^{n-2}[\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x]$

$I_n = \frac{\pi}{4} - \frac{1}{2} - \frac{n-1}{2}\int_0^1 x^n\tan^{-1}x + x^{n-2}\tan^{-1}x - x^{n-1}\,dx$