Name: ____________________ Teacher: ____________________

SYDNEY TECHNICAL HIGH SCHOOL

Trial Higher School Certificate

August 2010

Mathematics-Extension 2

Time allowed : 180 minutes

Reading Time: 5 minutes

Instructions

- Use black or blue pen.
- Approved calculators may be used.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks awarded are shown on each question.
- Total marks — 120
- Attempt all questions.
- Start each question on a new page.
- A table of *Standard Integrals* is attached.

Question	1	2	3	4	5	6	7	8	Total /120
Marks /15									

Question 1 **Marks 15**

a) Find $\int x^2\cos(x^3 - 1)dx$ 2

b) Using integration by parts, or otherwise, evaluate $\int \cos^{-1}x\,dx$ 3

c) Using the substitution $u^2 = e^x + 1$, or otherwise, find $\int \frac{e^{2x}dx}{\sqrt{e^x + 1}}$ 4

d) Using partial fractions, or otherwise, find $\int \frac{dx}{4x^2 - 1}$ 3

e) Evaluate $\int_3^4 \frac{x^2 + x - 4}{x - 2}\,dx$ 3

Question 2 **Marks 15**

a) Let $A = 2 - i$ and $B = 3 + 4i$.

Find, in the form $x + iy$

(i) $A - iB$ 1

(ii) $\bar{A}B$ 1

(iii) $\frac{5}{A}$ 2

b) If $z = \sqrt{3} + i$

(i) Express z in modulus–argument form 2

(ii) Hence find z^4 in $x + iy$ form. 2

c) On an Argand diagram, clearly show the region where the inequalities

$2 < |z| \leq 4$ and $\frac{-\pi}{4} \leq arg\ z \leq \frac{\pi}{2}$ hold simultaneously. 3

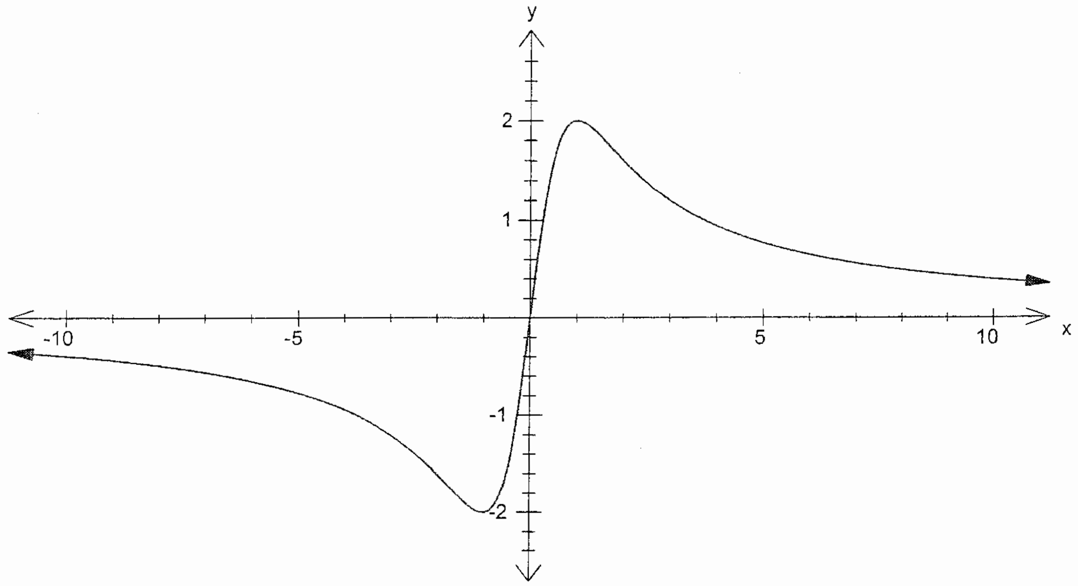
d) (i) With the aid of a diagram, describe the locus of Z on the Argand diagram if 2

$arg\,(z - 2k) - arg\,(z) = \frac{\pi}{2}$, $k > 0, k \in R(real\ numbers)$.

(ii) What is the Cartesian equation of this locus? 2

Question 3 Marks 15

a) The diagram below shows the graph of $y = f(x)$, which is an odd function.



Draw neat separate sketches showing all necessary detail of the following:

(i) $y = f(-x)$ 1

(ii) $y = [f(x)]^2$ 1

(iii) $y = \frac{1}{f(x)}$ 2

(iv) $y = x + f(x)$, showing any asymptotes. 2

(v) $y = f'(x)$ 2

b) Sketch the graph of $y = \frac{x-2}{x^2-4}$, clearly indicating any special features. 3

c) Consider the function $y = tan^{-1}x - x + \frac{1}{3}x^3$.

(i) Show that $\frac{dy}{dx} > 0$ for all values of $x > 0$. 2

(ii) Show that $tan^{-1} x > x - \frac{1}{3}x^3$ for all values of $x > 0$. 2

Question 4 **Marks 15**

a) An ellipse, E can be described as the locus of a point moving so that the sum of its distances from two fixed points (foci) is constant.

(i) If the two fixed points are $S\,(4,0)$ and $S'\,(-4,0)$ and the sum of the distances of $P(x,y)$ from these points is *10 units*, show that the equation of E is given by $\frac{x^2}{25}+\frac{y^2}{9}=1$ *[You may use the standard ellipse equation]* 2

(ii) Verify that $x = 5\cos\theta$ and $y = 3\sin\theta$ are the parametric equations of E. 1

(iii) Find the equation of the normal to E at the point where $\theta = \frac{\pi}{6}$. 3

(iv) Determine the eccentricity of **E** and, hence, the equations of the directrices. 2

b) Given that α, β and γ are the roots of $3x^3 + 4x^2 - 2x - 1 = 0$, find the values of:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ 1

(iii) $\alpha^2 + \beta^2 + \gamma^2$ 1

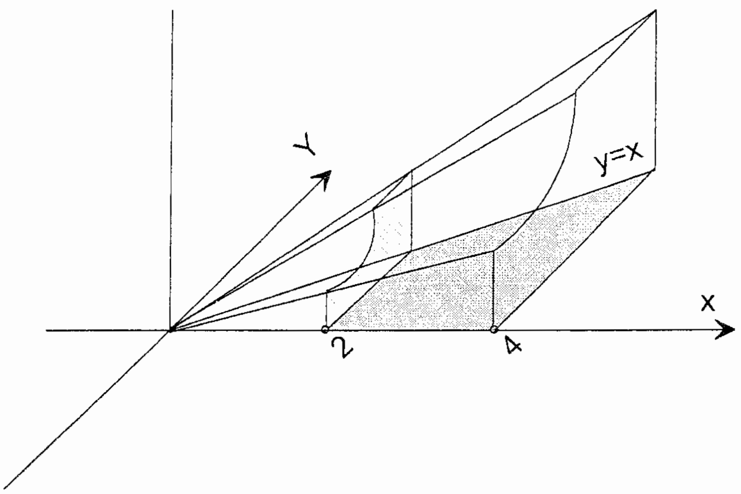
(iv) $\alpha^3 + \beta^3 + \gamma^3$ 2

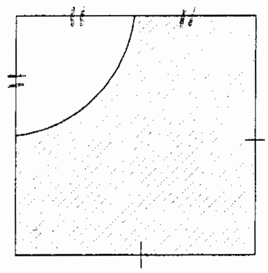
c) Factorise $x^4 - 2x^2 - 15$ over the rational and complex fields. 2

Question 5 **Marks 15**

a) The solid below has its base defined by the x-axis, the line $y = x$ and the lines $x = 2$ and $x = 4$ *(metres)*. Cross-sections consist of a square with a quarter circle (quadrant) removed (as shown). The radius of the circle is half of the side length of the square. **3**

Using the slicing technique, calculate the volume of this solid to the nearest cubic metre.





b) (i) Show that, if $y = px + q$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $p^2a^2 - b^2 = q^2$ **3**

(ii) Hence or otherwise, find the equations of the tangents from the point *(1,3)* to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1.$ **3**

c) If $I_n = \int_0^1 x^n e^{-x} dx$

(i) Show that $I_n = -\frac{1}{e} + nI_{n-1}$ **3**

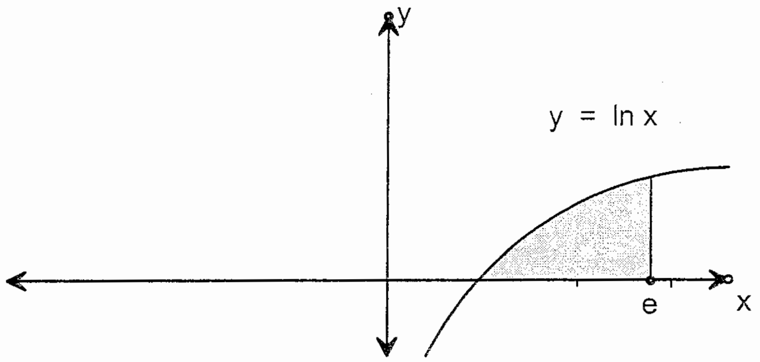
(ii) Hence find the exact value of $\int_0^1 x^3 e^{-x} dx$. **3**

Question 6

Marks 15

a) The region bounded by $y = ln\ x$, $x = e$ and the x-axis is rotated about the y-axis. **4**

Use the cylindrical shells method to find the volume of the solid formed.

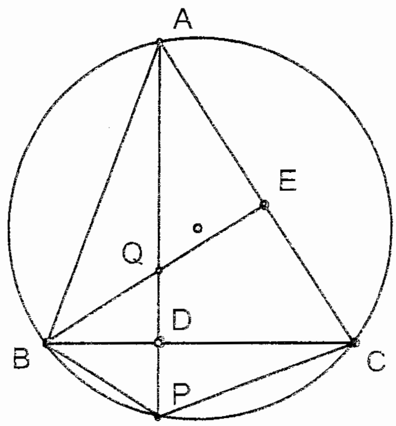


b) The angles A, B and C are consecutive terms in an arithmetic sequence.

(i) Show that $A + C = 2B$ **1**

(ii) Hence, show that $cosAcosC - cos^2B = sinAsinC - sin^2B$. **2**

(iii) ABC is an acute angled triangle inscribed in a circle. AP is perpendicular to BC. Q is the point on AP such that $DQ = DP$. BQ produced meets AC at E.



(i) Copy the diagram showing the above information. **1**

(ii) Show that $\Delta BDP \equiv \Delta BDQ$. **2**

(iii) Show that $BDEA$ is a cyclic quadrilateral. **4**

(iv) Show that BE is perpendicular to AC. **1**

Question 7 **Marks 15**

a) A particle is allowed to fall under gravity from rest in a medium which exerts a resistance proportional to the speed (v) of the particle.

(i) Show that the particle reaches a terminal velocity, T given by $T = \frac{g}{k}$ (where k is a positive constant). **2**

(ii) Show that the distance fallen to reach half its terminal velocity $(\frac{T}{2})$ is given by **4**

$$x = \frac{T^2}{g} \ln 2 - \frac{T^2}{2g}$$

(iii) Determine an expression for the time taken to reach a speed of $\frac{T}{2}$. **3**

b) Consider the curve given by the equation $x^2 - y^2 + xy + 5 = 0$.

(i) Show that $\frac{dy}{dx} = \frac{2x+y}{2y-x}$ **2**

(ii) Hence or otherwise, find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = x$. **2**

c) By taking the logarithms of both sides of $y = U(x).V(x)$, verify the Product Rule for differentiation. **2**

Question 8 **Marks 15**

a) (i) If α is a double root of a polynomial $P(x)$, show that α is a zero of $P'(x)$. **2**

(ii) Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$. **3**

b) (i) Find the five roots of $z^5 = 1$ and write them in mod-arg form. **3**

(ii) Show these roots on an Argand diagram and find the area (in exact form) of the pentagon formed by them. **2**

(iii) Factorise $z^5 - 1$ over the real field. **2**

c) The lengths of the sides of a triangle are the first three terms of an arithmetic sequence, with the first term equal to 1 and the common difference d. Find the set of possible values of d. **3**

End of paper

SOLUTIONS — EXTENSION 2 TRIAL HSC — 2010

Q1 a) $\frac{d}{dx}\sin(x^3-1) = 3x^2\cos(x^3-1)$

$\therefore \int x^2\cos(x^3-1)\,dx = \frac{1}{3}\sin(x^3-1)+c$

b) $\int \cos^{-1}x\,dx = x\cos^{-1}x - \int \frac{-x}{\sqrt{1-x^2}}\,dx$

let $u = x$, $du = dx$, $v = \cos^{-1}x$, $dv = \frac{-1}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x - \frac{1}{2}\int \frac{dw}{\sqrt{w}}$

let $w = 1-x^2$, $dw = -2x\,dx$

$= x\cos^{-1}x - \frac{1}{2}\times 2w^{\frac{1}{2}} + c$

$= x\cos^{-1}x - \sqrt{1-x^2} + c$

c) $\int \frac{e^{2x}\,dx}{\sqrt{e^x+1}} = 2\int e^x\cdot\frac{e^x\,dx}{2\sqrt{e^x+1}}$

$u^2 = e^x+1$

$u = (e^x+1)^{\frac{1}{2}}$

$du = \frac{1}{2}(e^x+1)^{-\frac{1}{2}}\cdot e^x$

$= \frac{e^x}{2\sqrt{e^x+1}}\,dx$

$= 2\int e^x\,du$

$= 2\int (u^2-1)\,du$

$= 2\left(\frac{u^3}{3}-u\right)+c$

$= 2\left[\frac{(e^x+1)^{\frac{3}{2}}}{3} - \sqrt{e^x+1}\right]+c$

$= \frac{2}{3}\left[(e^x+1)^{\frac{3}{2}} - 3\sqrt{e^x+1}\right]+c$

$= \frac{2}{3}\sqrt{e^x+1}\left((e^x+1)-3\right)+c = \frac{2}{3}\sqrt{e^x+1}\,(e^x-2)+c$

d) $\int \frac{dx}{4x^2-1} = \int\left(\frac{-\frac{1}{2}}{2x+1}+\frac{\frac{1}{2}}{2x-1}\right)dx$

Let $\frac{A}{2x+1}+\frac{B}{2x-1} = \frac{1}{4x^2-1}$

$\therefore A(2x-1)+B(2x+1) = 1$

$= -\frac{1}{4}\ln(2x+1)+\frac{1}{4}\ln(2x-1)+c$

$= \frac{1}{4}\ln\left(\frac{2x-1}{2x+1}\right)+c$

Let $x = \frac{1}{2}$, $\therefore 2B = 1$, $\therefore B = \frac{1}{2}$

Let $x = -\frac{1}{2}$, $\therefore -2A = 1$, $\therefore A = -\frac{1}{2}$

e) $\displaystyle\int_3^4 \frac{x^2+x-4}{x-2}\,dx$

$$\begin{array}{r} x+3 \\ x-2\,\overline{)\,x^2+x-4} \\ \underline{x^2-2x} \\ 3x-4 \\ \underline{3x-6} \\ 2 \end{array}$$

$= \displaystyle\int_3^4 \left(x+3+\frac{2}{x-2}\right)dx$

$= \left[\frac{x^2}{2}+3x+2\ln(x-2)\right]_3^4$

$= \left[(8+12+2\ln 2)-\left(\frac{9}{2}+9+2\ln 1\right)\right]$

$= 20+2\ln 2-13\frac{1}{2}$

$= 6\frac{1}{2}+2\ln 2$

Q2 a) (i) $A - iB = 2 - i - i(3+4i)$
$= 2 - i - 3i + 4$
$= 6 - 4i$

(ii) $\bar{A}B = (2+i)(3+4i)$
$= 6 + 8i + 3i - 4$
$= 2 + 11i$

(iii) $\dfrac{5}{2-i} = \dfrac{5(2+i)}{(2-i)(2+i)}$
$= \dfrac{5(2+i)}{4+1}$
$= 2+i$

b) $z = \sqrt{3}+i$

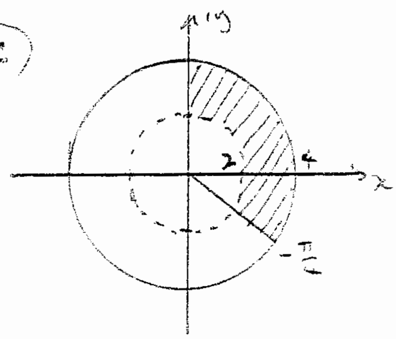
(i) $|z| = \sqrt{3+1}$
$= 2$

$\arg z = \tan^{-1}\dfrac{1}{\sqrt{3}}$
$= \dfrac{\pi}{6}$

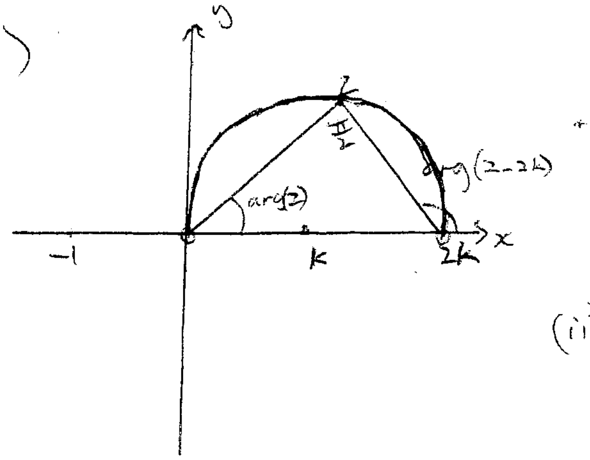
$\therefore\ z = 2\,\mathrm{cis}\,\dfrac{\pi}{6}$

(ii) $z^4 = \left(2\,\mathrm{cis}\,\dfrac{\pi}{6}\right)^4$
$= 2^4\,\mathrm{cis}\,\dfrac{4\pi}{6}$
$= 16\cos\dfrac{4\pi}{6} + i\,16\sin\dfrac{4\pi}{6}$
$= -8 + 8\sqrt{3}\,i$

c)



d) (i)

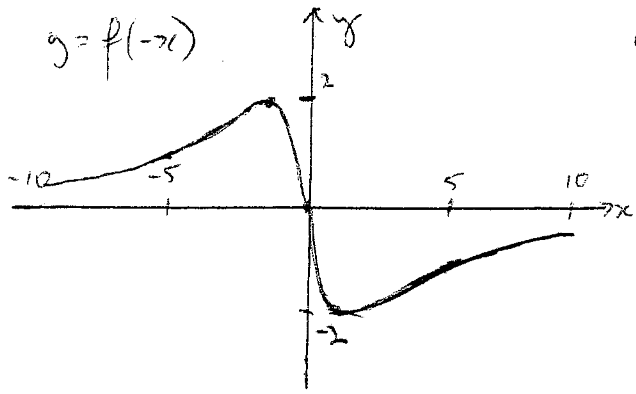


$\therefore$ locus of z is a semi-circle centre $(k, 0)$ radius k.

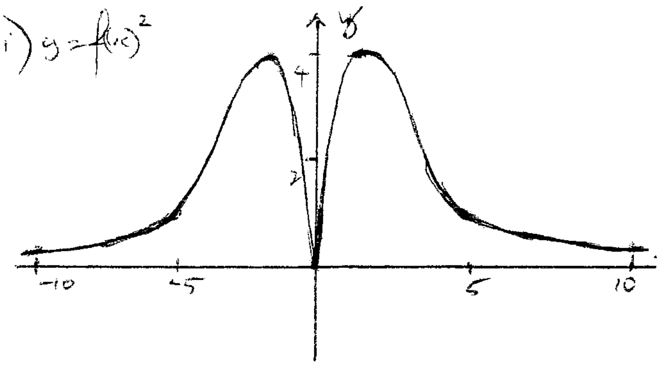
(ii) cartesian equation of locus

$$(x-k)^2 + y^2 = k^2$$

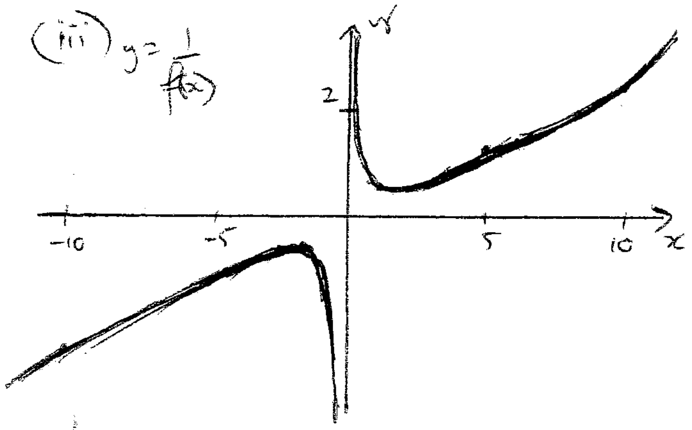
Q3 a) (i) $y = f(-x)$



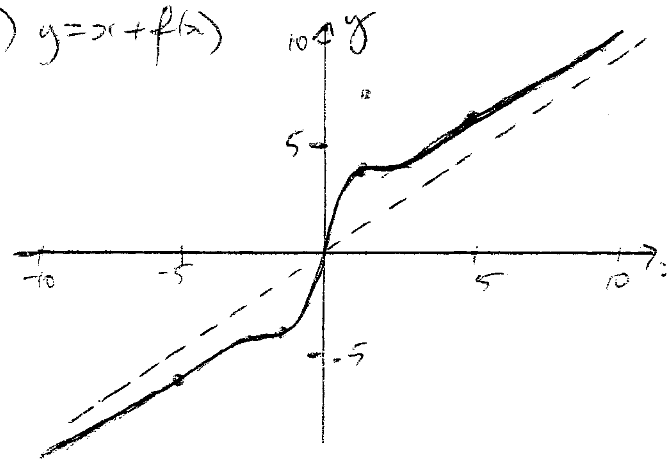
(ii) $y = f(x)^2$



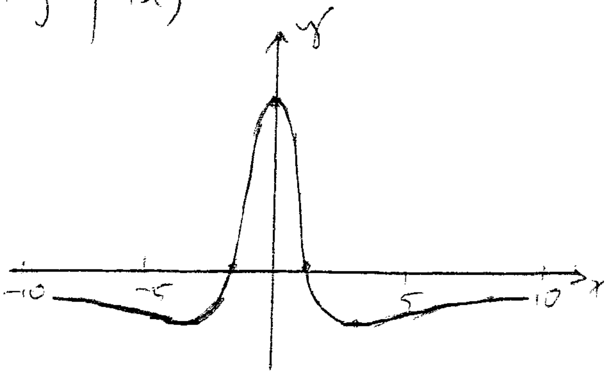
(iii) $y = \frac{1}{f(x)}$



(iv) $y = x + f(x)$



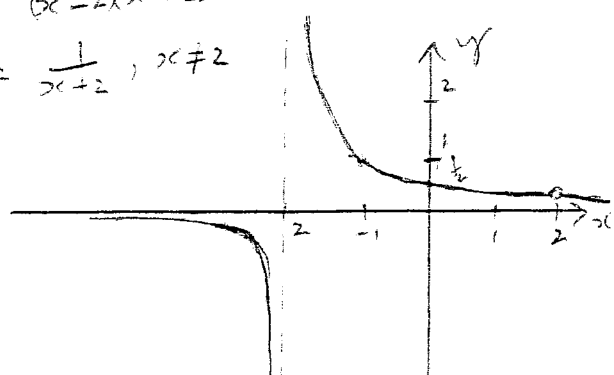
(v) $y = f'(x)$



b) $y = \frac{x-2}{x^2-4}$

$= \frac{x-2}{(x-2)(x+2)}$

$= \frac{1}{x+2}, \quad x \neq 2$



c) $y = \tan^{-1} x - x + \frac{x^3}{3}$

(i) $\frac{dy}{dx} = \frac{1}{1+x^2} - 1 + x^2$

$= \frac{1 + (x^2-1)(1+x^2)}{1+x^2}$

$= \frac{1 + x^2 + x^4 - 1 - x^2}{1+x^2}$

$= \frac{x^4}{1+x^2}$

$> 0 \quad \forall x > 0.$

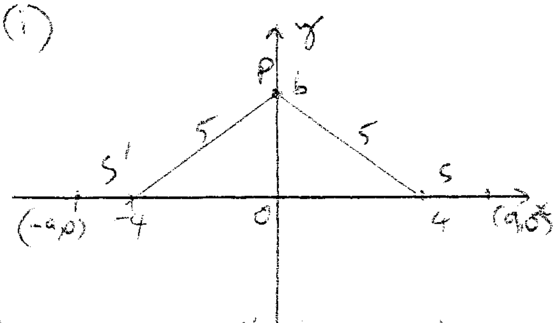
(ii) When $x = 0$, $y = 0$
and y is increasing for all $x > 0$
$\therefore y > 0 \quad \forall x > 0.$

$\therefore \tan^{-1} x - x + \frac{x^3}{3} > 0$ for $x > 0$

$\therefore \tan^{-1} x > x - \frac{x^3}{3}$ for $x > 0$

QED.

Q4 a) (i)



P moves such that $PS + PS' = 10$
$PO = 3$ (Pythag. Thm)
also $a = 5$ as $2a = 10$.
$\therefore$ equation of ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

(ii) $x = 5\cos\theta \qquad y = 3\sin\theta$

$\therefore \cos\theta = \frac{x}{5} \qquad \therefore \sin\theta = \frac{y}{3}$

Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ (reversed)

$\therefore \frac{x^2}{25} + \frac{y^2}{9} = 1$

$\therefore x = 5\cos\theta$ and $y = 3\sin\theta$ are the parametric equations for the ellipse.

(iii) $\frac{dx}{d\theta} = -5\sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= \frac{3\cos\theta}{-5\sin\theta}$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{3 \times \frac{\sqrt{3}}{2}}{-5 \times \frac{1}{2}}$

$= -\frac{3\sqrt{3}}{5}$

$\therefore$ grad normal $= \frac{5}{3\sqrt{3}}$

when $\theta = \frac{\pi}{6}$, $x = \frac{5\sqrt{3}}{2}$, $y = \frac{3}{2}$

$\therefore$ eqtn of normal is

$y - \frac{3}{2} = \frac{5}{3\sqrt{3}}\left(x - \frac{5\sqrt{3}}{2}\right)$

$\therefore 5x - 3\sqrt{3}y - 8\sqrt{3} = 0$

(iv) $S = ae$

$\therefore 4 = 5e$

$\therefore e = \frac{4}{5}$

directrices $\Rightarrow x = \pm \frac{a}{e}$

$= \pm 5 \times \frac{5}{4}$

$= \pm \frac{25}{4}$

b) $3x^3 + 4x^2 - 2x - 1 = 0$

(i) $\sum \alpha = \frac{-b}{a}$

$= -\frac{4}{3}$

$\sum \alpha\beta = \frac{c}{a}$

$= \frac{-2}{3}$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= \frac{16}{9} - 2 \times \frac{-2}{3}$

$= \frac{28}{9}$

(iii) $\alpha^3 + \beta^3 + \gamma^3$

As α, β and γ are roots, we know...

$3\alpha^3 + 4\alpha^2 - 2\alpha - 1 = 0$
$3\beta^3 + 4\beta^2 - 2\beta - 1 = 0$
$3\gamma^3 + 4\gamma^2 - 2\gamma - 1 = 0$

$\therefore 3(\alpha^3+\beta^3+\gamma^3) + 4(\alpha^2+\beta^2+\gamma^2) - 2(\alpha+\beta+\gamma) - 3 = 0$

$\therefore 3(\alpha^3+\beta^3+\gamma^3) + 4 \times \frac{28}{9} - 2 \times -\frac{4}{3} - 3 = 0$

$\therefore 3(\alpha^3+\beta^3+\gamma^3) = -\frac{109}{9}$

$\therefore \alpha^3+\beta^3+\gamma^3 = -\underline{109}$

c) $x^4 - 2x^2 - 15 = (x^2 - 5)(x^2 + 3)$ over $\mathbb{Q}$
$= (x - \sqrt{5})(x + \sqrt{5})(x^2 + 3)$ over $\mathbb{R}$
$= (x - \sqrt{5})(x + \sqrt{5})(x - \sqrt{3}i)(x + \sqrt{3}i)$ over $\mathbb{C}$

Q5

a) Area of ~~cross~~section, A

$= x^2 - \frac{1}{4}\pi\left(\frac{x}{2}\right)^2$

$= x^2 - \frac{\pi x^2}{16}$

$\therefore V = \int_2^4 x^2\left(\frac{16-\pi}{16}\right) dx$

$= \frac{16-\pi}{16}\left[\frac{x^3}{3}\right]_2^4$

$= \frac{16-\pi}{16}\left[\frac{64}{3} - \frac{8}{3}\right]$

$= \frac{16-\pi}{16} \times \frac{56}{3}$

$= \frac{7}{6}(16-\pi)$ m^3

$\doteqdot 15$ m^3

b) (i) Solve $y = px + q$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ simultaneously.

$\therefore \frac{x^2}{a^2} - \frac{(px+q)^2}{b^2} = 1$

$\therefore b^2x^2 - a^2(px+q)^2 = a^2b^2$

$\therefore b^2x^2 - a^2(p^2x^2 + 2pqx + q^2) = a^2b^2$

$\therefore (b^2 - a^2p^2)x^2 - 2a^2pqx - a^2q^2 - a^2b^2 = 0$

Because $y = px + q$ is tangential,

$\exists$ only one solution to this quadratic.

$\therefore \Delta = 0$

$\therefore 4a^4p^2q^2 + 4(b^2 - a^2p^2)(a^2q^2 + a^2b^2) = 0$

$\therefore a^4p^2q^2 + a^2(b^2 - a^2p^2)(q^2 + b^2) = 0$

$\therefore a^2p^2q^2 + b^2q^2 + b^4 - a^2p^2q^2 - a^2b^2p^2 = 0$

$\therefore b^4 + b^2q^2 - a^2b^2p^2 = 0$

$\therefore b^2 + q^2 - a^2p^2 = 0$

$\therefore a^2p^2 - b^2 = q^2$ QED

(ii) $(1,3)$ satisfies tangent

$\therefore 3 = p + q$

$\therefore p = 3 - q$

Now $a^2 = 4$ and $b^2 = 15$

$\therefore q^2 = 4p^2 - 15$

$\therefore q^2 = 4(3-q)^2 - 15$

$\therefore q^2 - 4(9 - 6q + q^2) + 15 = 0$

$\therefore -3q^2 + 24q - 21 = 0$

$\therefore 3q^2 - 24q + 21 = 0$

$\therefore 3(q-7)(q-1) = 0$

$\therefore q = 7$ with $p = -4$

or $q = 1$ with $p = 2$

$\therefore$ eqtns are $y = -4x + 7$

and $y = 2x + 1$

c) (i) Let $u = x^n$ and $v' = e^{-x}$

$\therefore u' = nx^{n-1}$ and $v = -e^{-x}$

$\therefore I_n = \left[-e^{-x}x^n\right]_0^1 + \int_0^1 nx^{n-1}e^{-x}\,dx$

$= \left[-\frac{1}{e} - 0\right] + nI_{n-1}$

$= -\frac{1}{e} - nI_{n-1}$

(ii) $\int_0^1 x^3 e^{-x}\,dx = I_3$

$= -\frac{1}{e} + 3I_2$

$I_2 = -\frac{1}{e} + 2I_1$

$I_1 = -\frac{1}{e} + 1\,I_0$

$I_0 = \int_0^1 e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^1$

$= 1 - \frac{1}{e}$

$\therefore I_1 = 1 - \frac{2}{e}$

$\therefore I_2 = 2\left(1 - \frac{2}{e}\right) - \frac{1}{e}$

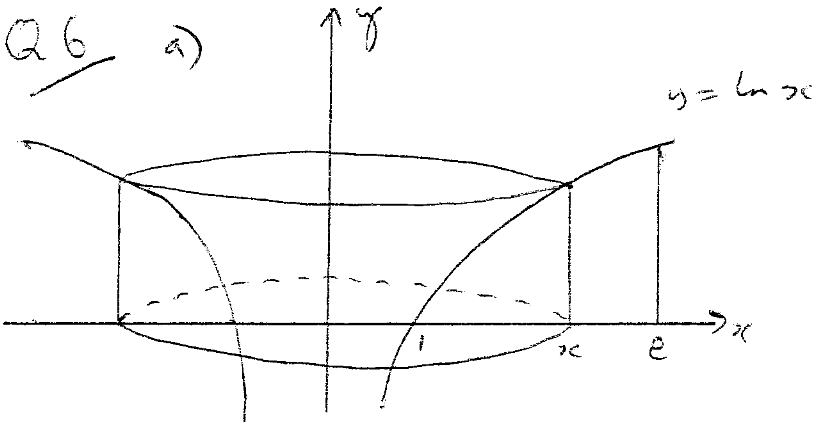
$= 2 - \frac{5}{e}$

$\therefore I_3 = 3\left(2 - \frac{5}{e}\right) - \frac{1}{e}$

$= 6 - \frac{16}{e}$

$\therefore \int_0^1 x^3 e^{-x}\,dx = 6 - \frac{16}{e}$

Q6 a)



$A_{shell} = 2\pi x \ln x$

$\therefore V = \int_1^e 2\pi x \ln x\,dx$

$= 2\pi \int_1^e x \ln x\,dx$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

let $v = \frac{x^2}{2}$

$\therefore dv = x\,dx$

$= 2\pi\left[\frac{x^2}{2}\ln x\right]_1^e - 2\pi\int_1^e \frac{x^2}{2}\cdot\frac{1}{x}$

$= 2\pi\left[\frac{e^2}{2} - 0\right] - \pi\left[\frac{x^2}{2}\right]_1^e$

$= \pi e^2 - \pi\left[\frac{e^2}{2} - \frac{1}{2}\right]$

$= \pi\left[\frac{e^2}{2} + \frac{1}{2}\right]$

$= \frac{\pi}{2}\left[e^2 + 1\right]\ u^3$

b) (i) $A, B, C \rightarrow AP$

$A,\ A+d,\ A+2d$

$A + C = 2A + 2d$

$2B = 2(A+d)$

$= 2A + 2d$

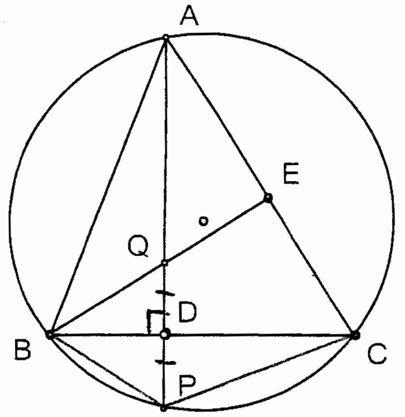
$\therefore A + C = 2B$

(ii) $\therefore \cos(A+C) = \cos 2B$

$\therefore \cos A\cos C - \sin A\sin C = \cos^2 B - \sin^2 B$

$\therefore \cos A\cos C - \cos^2 B = \sin A\sin C - \sin^2 B$

c) (i)



(ii) In $\triangle BDQ$ and $\triangle BDP$,

$DP = DQ$ (given)

BD is common

$B\hat{D}Q = B\hat{D}P = 90^\circ$ (given)

$\therefore \triangle BDQ \equiv \triangle BDP$ (SAS)

(iii) $Q\hat{B}D = P\hat{B}D$ (corresponding angles in congruent triangles)

$\therefore E\hat{B}D = P\hat{B}C$ (same angles)

Now $P\hat{A}C = P\hat{B}C$ (two angles on same arc)

$\therefore E\hat{B}D = D\hat{A}E$ ($D\hat{A}E$ same angle as $P\hat{A}C$)

$\therefore BDEA$ is a cyclic quadrilateral (two angles on same arc DE)

(iv) $B\hat{D}A = B\hat{E}A$ (two angles on same arc, AB)

$B\hat{D}A = 90^\circ$ (given)

$\therefore B\hat{E}A = 90^\circ$

$\therefore BE \perp AC$

Q7 a) (i) $\ddot{x} = g - kv \quad , k > 0$

terminal velocity, $\ddot{x} = 0$

$\therefore g - kv = 0$

$\therefore g = kv$

$\therefore v = \frac{g}{k}$

$\therefore$ terminal velocity, $T = \frac{g}{k}$

(ii) $\ddot{x} = g - kv$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - kv$

$\therefore v\frac{dv}{dx} = g - kv$

$\therefore \frac{dv}{dx} = \frac{g - kv}{v}$

$\therefore \frac{dx}{dv} = \frac{v}{g - kv}$

$= \frac{1}{k}\left(\frac{v}{\frac{g}{k} - v}\right)$

$= \frac{1}{k}\left(\frac{v}{T - v}\right)$ as $T = \frac{g}{k}$

$= \frac{1}{k}\left(-1 + \frac{T}{T - v}\right)$

$\therefore x = \frac{1}{k}\left[-v - T\ln(T - v)\right] + c$

When $x = 0$, $v = 0$

$\therefore 0 = \frac{1}{k}\left[-T\ln T\right] + c$

$\therefore c = \frac{T\ln T}{k}$

$\therefore x = \frac{1}{k}\left[-v - T\ln(T - v) + T\ln T\right]$

$= \frac{T}{k} \cdot \ln\left(\frac{T}{T - v}\right) - \frac{v}{k}$

When $v = \frac{T}{2}$,

$x = \frac{T}{k} \cdot \ln 2 - \frac{T}{2k}$

But $k = \frac{g}{T}$

$\therefore x = \frac{T^2}{g} \cdot \ln 2 - \frac{T^2}{2g}$

(iii) $\ddot{x} = \frac{dv}{dt} = g - kv$

$\therefore \frac{dt}{dv} = \frac{1}{g-kv}$

$\therefore t = -\frac{1}{k}\ln(g-kv) + c$

When $t=0, v=0$

$\therefore 0 = -\frac{1}{k}\ln g + c$

$\therefore t = -\frac{1}{k}\ln(g-kv) + \frac{1}{k}\ln g$

$= \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$

when $v = \frac{T}{2}$

$= \frac{g}{2k}$ as $T = \frac{g}{k}$

$\therefore t = \frac{1}{k}\ln\left(\frac{g}{g-k\cdot\frac{g}{2k}}\right)$

$= \frac{1}{k}\ln\left(\frac{1}{1-\frac{1}{2}}\right)$

$= \frac{1}{k}\ln 2$

$= \frac{T}{g}\ln 2$ as $k = \frac{g}{T}$

b)(i) $x^2 - y^2 + xy + 5 = 0$

$\therefore 2x - 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

$2x + y - \frac{dy}{dx}(2y + x) = 0$

$\frac{dy}{dx}(2y+x) = 2x+y$

$\therefore \frac{dy}{dx} = \frac{2x+y}{2y+x}$

(ii) Tangent parallel to $y=x$.

$\therefore \frac{dy}{dx} = 1$

$\therefore 2x+y = 2y+x$

$\therefore y = 3x$

Sub into $x^2-y^2+xy+5=0$

$\therefore x^2 - 9x^2 + 3x^2 + 5 = 0$

$-5x^2 + 5 = 0$

$\therefore x^2 = 1$

$\therefore$ contact points are $(1,3)$ and $(-1,-3)$

c) $y = u(x)\,V(x)$

$\therefore \ln y = \ln[u(x)v(x)]$

$= \ln[u(x)] + \ln[v(x)]$

$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{V(x)}$

$\therefore \frac{1}{u(x)v(x)}\frac{dy}{dx} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}$

$\therefore \frac{dy}{dx} = \frac{u'(x)}{u(x)}\,u(x)v(x) + \frac{v'(x)}{V(x)}\,u(x)v(x)$

$= u'(x)\,v(x) + v'(x)\,u(x)$

QED

Q8 a) (i) $P(x) = (x-\alpha)^2 Q(x)$

$\therefore P'(x) = (x-\alpha)^2 Q'(x) + Q(x) \cdot 2(x-\alpha)$

$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$

$\therefore \alpha$ is a root of $P'(x)$, as $x-\alpha$ is a factor.

(ii) Let $P(x) = x^5 + 2x^2 + mx + n$

$\therefore P'(x) = 5x^4 + 4x + m$

Now $(x+1)^2$ is a factor of $P(x)$ and so $(x+1)$ is a factor of $P'(x)$

$\therefore (-1)^5 + 2(-1)^2 + m(-1) + n = 0$ and $5(-1)^4 + 4(-1) + m = 0$

$-1 + 2 - m + n = 0$ $\qquad 5 - 4 + m = 0$

$\therefore m - n = 1$ $\qquad \therefore \underline{m = -1}$

$\therefore \underline{n = -2}$

b(i) $z^5 = 1$

Let $z^5 = (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i \sin 5\theta$

$\therefore \cos 5\theta = 1$

$\therefore 5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$\therefore$ roots are $\cos 0 + i\sin 0 = 1$

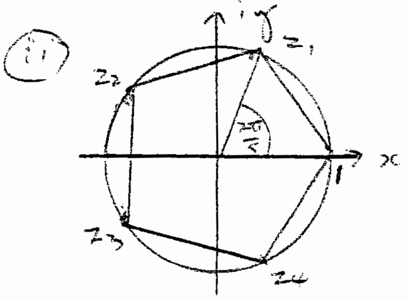
$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

(ii)



(iii) $z^5 - 1 = (z-1)\left(z - \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)\left(z - \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$
$\left(z - \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right)\left(z - \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

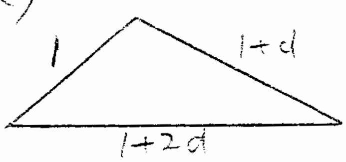
$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + \sin^2\frac{2\pi}{5} + \cos^2\frac{2\pi}{5}\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + \sin^2\frac{4\pi}{5} + \cos^2\frac{4\pi}{5}\right)$

$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$

$A = 5 \times \frac{1}{2} \times 1 \times 1 \times \sin\frac{2\pi}{5}$

$= \frac{5}{2}\sin\frac{2\pi}{5}$ u^2

c)



$2d > -1$

$\therefore d > -\frac{1}{2}$

Now, the sum of any 2 sides is greater than the third side.

$\therefore 1 + (1+d) > 1+2d$	$1 + (1+2d) > 1+d$	$(1+d) + (1+2d) > 1$
$\therefore 2 + d > 1 + 2d$	$\therefore 2 + 2d > 1 + d$	$\therefore 2 + 3d > 1$
$\therefore d < 1$	$\therefore d > -1$	$\therefore d > -\frac{1}{3}$

$\therefore -\frac{1}{3} < d < 1$ as all must be satisfied.