Name:_________________________ Teacher:_________________________

SYDNEY TECHNICAL HIGH SCHOOL

Trial Higher School Certificate

August 2011

MATHEMATICS EXTENSION 2

Time Allowed: **180 minutes**

Reading Time: 5 minutes

Instructions:

- Use black or blue pen
- Approved calculators may be used
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks awarded are shown on each question
- Total marks – 120
- Attempt all questions
- **Start each question on a new page**
- A table of *Standard Integrals* is attached.

Question	1	2	3	4	5	6	7	8	Total/120
Marks /15									

Question 1 **Marks**

a) Find $\int x^3 e^{x^4+7} dx$ 1

b) (i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form 3

$\frac{Ax+B}{x^2+1}$ + $\frac{C}{x+1}$ where A,B and C are constants.

(ii) Hence find $\int \frac{x^2+x+2}{(x^2+1)(x+1)} dx$ 2

c) Evaluate $\int_0^1 tan^{-1} x dx$ 3

d) (i) Use the substitution $t = tan\frac{x}{2}$ to show

$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{2+cosx} dx = \frac{2\pi\sqrt{3}}{9}$ 3

(ii) Using the substitution $u = 4\pi - x$, evaluate

$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{x}{2+cosx} dx$ 3

Question 2

a) $\frac{4+3i}{1+\sqrt{2}i} = a + ib$ for a , b real. 2

Find the exact values of a and b.

b) (i) Solve $z^4 + 1 = 0$, giving your answers in mod-arg form. 3

(ii) Plot these roots on an Argand diagram. 1

(iii) Find the exact area of the quadrilateral that they form. 1

c) $z_, = 4cis\frac{\pi}{12}$ and $z_2 = 2cis\ \frac{5\pi}{12.}$

(i) On an Argand diagram, draw the vectors OA, OB, OC representing

$z_1, z_2, z_1 + z_2$ respectively (not to scale). 2

(ii) Hence or otherwise find $|z_1 + z_2|$ in simplest exact form 2

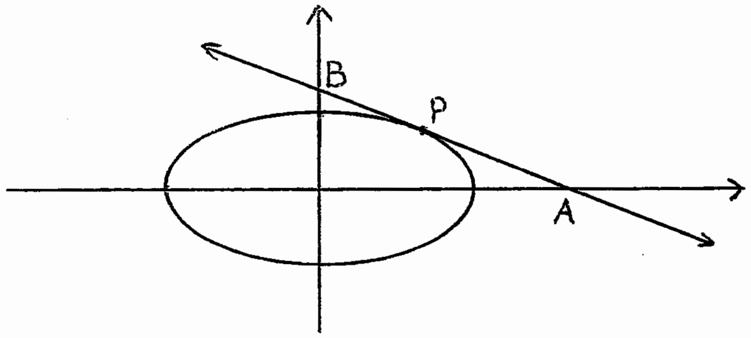
Marks

d) $\arg(z-2) = \arg(z+2) + \frac{\pi}{4}$ is the locus of the point P representing z on an Argand diagram.

(i) Show with a diagram why this locus is an arc of a circle 2

(ii) Find the centre and radius of this circle 2

Question 3



a) In the diagram above, $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The tangent at P cuts the x-axis at A and the y-axis at B.

(i) Derive the parametric equation of the tangent at P in any form and find the coordinates of A and B in parametric form. 4

(ii) Show that $\frac{PA}{PB} = \tan^2\theta$ 2

b) The rectangular hyperbola H has equation $x^2 - y^2 = 8$. Write down:

(i) The eccentricity 1

(ii) The coordinates of the foci 1

(iii) The equations of the directrices 1

(iv) The equations of the asymptotes 1

(v) Sketch the curve showing the above 1

If this curve is rotated through 45° in an anticlockwise direction the equation takes the form $xy = 4$.

(vi) Find the equation of the normal to this hyperbola at the point $\left(2p, \frac{2}{p}\right)$. 2

(vii) If this normal meets the hyperbola again at $\left(2q, \frac{2}{q}\right)$, prove that $q = \frac{-1}{p^3}$ 2

Question 4

a) Consider the function $y = cos^{-1}(sinx)$. Given the domain and range are

D: all real x

R: $0 \leq y \leq \pi$

(i) Find the period of this function. 1

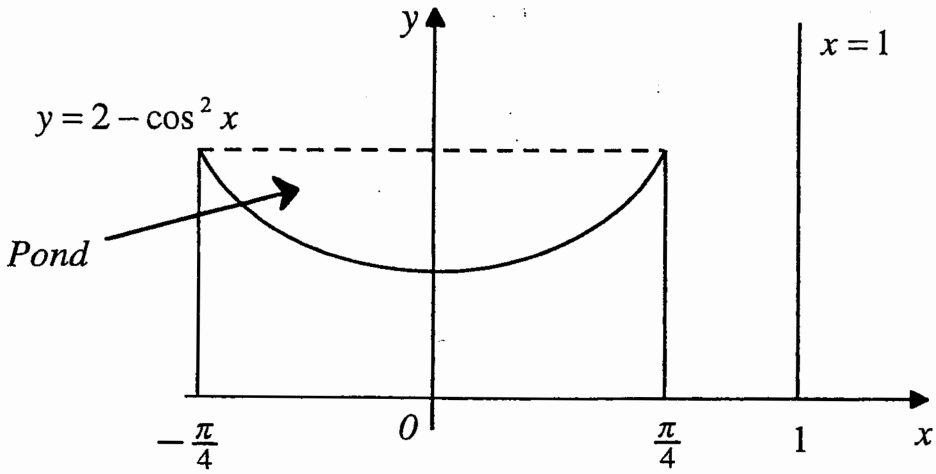
(ii) Sketch the graph of the function over the domain $-2\pi \leq x \leq 2\pi$ 2

b) Given the derivative of the function $f(x) = x - \frac{3sinx}{2+cosx}$ is $f'(x) = \left[\frac{1-cosx}{2+cosx}\right]^2$, 3

show that $x > \frac{3sinx}{2+cosx}$ for $x > 0$.

Marks

c)



A mould for an annular fish pond is made by rotating the region bounded by the curve $y = 2 - cos^2x$ and $y = \frac{3}{2}$ between $x = \frac{-\pi}{4}$ and $x = \frac{\pi}{4}$ through one complete revolution about the line $x = 1$. All measurements are in metres.

(i) Use the method of cylindrical shells to show that the volume of the fish pond is given by $V = \pi \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} (1 - x)cos2xdx$ 3

(ii) Find the capacity of the fish pond correct to the nearest litre. 3

d) The deck of a boat was 2.4m below the level of a wharf at low tide and 0.6m above the wharf at high tide. Low tide was at 11.30am and high tide at 5.35pm. Find when the deck was level with the wharf assuming the tidal motion is simple harmonic. Give your answer correct to the nearest minute. 3

Question 5

a) When a polynomial P(x) is divided by $(x-2)$ and $(x-4)$ the remainders are 11 and 15 respectively. Determine the remainder when P(x) is divided by $(x-2)(x-4)$. 2

b) (i) If $\propto$ is a multiple root of $P(x)=0$, prove that $P'(\propto)=0$. 2

(ii) If $ax^4+4bx+c=0$ has a double root, prove that $ac^3-27b^4=0$. 3

c) If $P(x)=x^4-2x^3\text{-}x^2+6x-6$ has a zero 1-i, factorise P(x) fully over the real field. 2

d) (i) If $I_n=\int cosec^n x dx$ for $n\geq 0$ show that 3

$I_n=\frac{-cotx cosec^{n-2}x}{n-1}+\frac{n-2}{n-1}I_{n-2}$

given $\frac{d}{dx}(cotx)=-cosec^2x.$

(ii) Hence show that if $I_n=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cosec^n x dx$ for $n\geq 0$, then 1

$I_n=\frac{(\sqrt{2})^{n-2}}{n-1}+\frac{n-2}{n-1}I_{n-2}$ for $n\geq 2$.

(iii) Deduce that $I_6=\frac{28}{15}$ 2

Question 6

(a) The equation $x^3+2x+1=0$ has roots $\propto,\beta$ and γ.

(i) Find the monic cubic equation with roots $\frac{1}{\propto},\frac{1}{\beta}$ and $\frac{1}{\gamma}$. 2

(ii) Find the monic cubic equation with roots

$\frac{\beta+\gamma}{\propto^2},\frac{\gamma+\propto}{\beta^2}$ and $\frac{\propto+\beta}{\gamma^2}$. 3

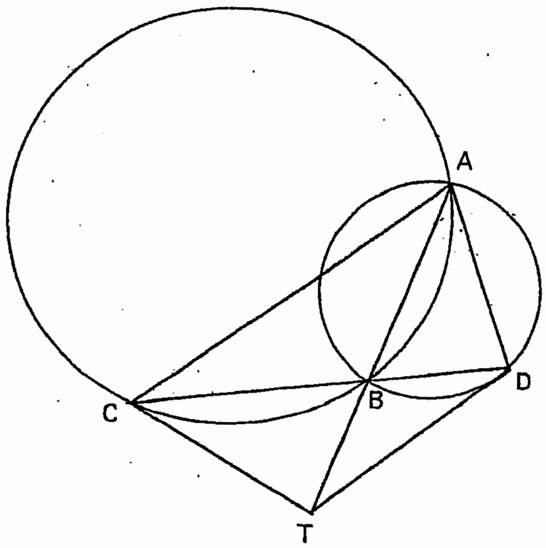
b) (i) If $4-tan\theta=5sin\theta cos\theta$, show that $x=tan\theta$ is a root of the equation 2

$x^3-4x^2+6x-4=0$

(ii) Solve the equation $4-tan\theta=5sin\theta cos\theta$ for $0°\leq\theta\leq 360°$ giving answers correct to the nearest degree. 3

Marks

c)



BAC, BAD are two circles such that tangents at C and D meet at T on AB produced. If CBD is a straight line prove that:

(i) TC=TD 1

(ii) $\angle$TAC=$\angle$TAD 2

(iii) TCAD is a cyclic quadrilateral 2

Question 7

a) P represents the complex number z, where z satisfies

$|z-2| = 2$ and $0 < arg z < \frac{\pi}{2}$

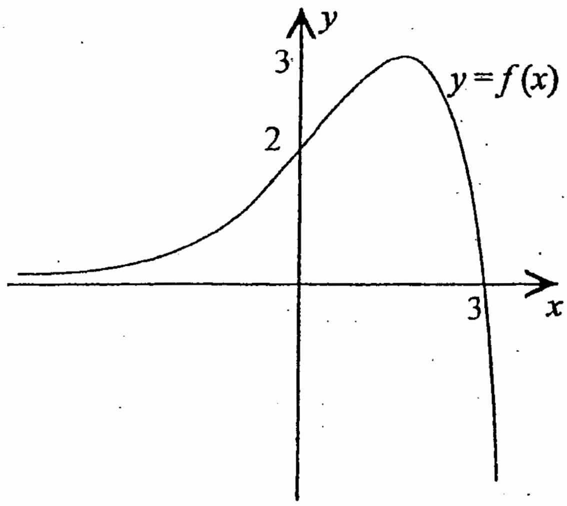
(i) Show that $|z^2 - 2z| = 2|z|$ 2

(ii) Find the value of k (a real number) if $\arg(z-2) = k \arg(z^2 - 2z)$ 3

b) Two sides of a triangle are in the ratio of 3:1 and the angles opposite these sides differ by $\frac{\pi}{6}$. Show that the smaller of the two angles is $tan^{-1}(\frac{1}{6-\sqrt{3}})$. 3

Marks

c)



Consider the above sketch of $y = f(x)$. On separate diagrams draw sketches of

(i) $y = f(x) - 1$ 1

(ii) $y = [f(x)]^3$ 2

(iii) $y = f(|x|)$ 2

(iv) $y = e^{f(x)}$ 2

Question 8

a) Prove that the y axis is a tangent to the curve $\sqrt{\frac{x}{u}} + \sqrt{\frac{y}{v}} = 1$ 2

(u and v are positive constants).

b) Gas is escaping from a spherical balloon. Find the radius of the balloon when the rate of decrease in the volume and the rate of decrease in the surface area are numerically equal. 2

Marks

c) A particle of mass mkg is dropped from rest in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the velocity of the particle is vms^{-1}. After t seconds the particle has fallen x metres and has velocity vms^{-1} and acceleration a ms^{-2}. Take the acceleration due to gravity as $10ms^{-2}$.

(i) Draw a diagram showing forces acting on the particle. Hence show that $a = \frac{100-v^2}{10}$. 1

(ii) Show that $t = \frac{1}{2}In\left[\frac{10+v}{10-v}\right]$ 3

(iii) Find expressions in terms of t for v and x. 4

(iv) Show that the terminal velocity is $10ms^{-1}$. 1

(v) Find the exact time taken and the exact distance fallen by the particle In reaching a speed equal to 80% of its terminal velocity. 2

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Question 1

a) $\int x^3 e^{x^4+7}\,dx$

$\frac{1}{4}\int 4x^3 e^{x^4+7}\,dx$

$= \frac{1}{4}e^{x^4+7} + c$ ②

b)(i) $\frac{x^2+x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$

$x^2+x+2 = (Ax+B)(x+1) + C(x^2+1)$

$x^2+x+2 = (A+C)x^2 + (A+B)x + (B+$

$\therefore A+C = 1$ $\quad B+C=+2$

$A+B = 1$ $\quad \therefore 2C = +2$

$C = B$

$C = +1$

$B = +1$

$A = 0$

$\Rightarrow \frac{1}{x^2+1} + \frac{1}{x+1}$

(ii) $\int \frac{1}{x^2+1} + \frac{1}{x+1}\,dx$

$= \log_e(x+1) + \tan^{-1}x + c$ ②

c) $\int_0^1 \tan^{-1}x\,dx$

$\int_0^1 \frac{d}{dx}(x)\tan^{-1}x\,dx$

$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$ ①

$= \frac{\pi}{4} - \frac{1}{2}\left[\log_e(1+x^2)\right]_0^1$ ①

$= \frac{\pi}{4} - \frac{1}{2}\left[\log_e 2 - \log_e 1\right]$

$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$ ①

d)(i) $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{2+\cos x}\,dx$

$t = \tan\frac{x}{2}$

$\int_{-1}^{1} \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$ ①

$\int_{-1}^{1} \frac{2dt}{2+2t^2+1-t^2}$

$\int_{-1}^{1} \frac{2dt}{t^2+3}$

$\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_{-1}^{1}$ ①

$\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}\frac{-1}{\sqrt{3}}\right]$

$\frac{2}{\sqrt{3}}\left[\frac{\pi}{6} - -\frac{\pi}{6}\right] = \frac{2\pi\sqrt{3}}{9}$

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(ii) $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{x}{2+\cos x} \, dx$

$u = 4\pi - x$

$du = -dx$

$\Rightarrow \int_{\frac{5\pi}{2}}^{\frac{3\pi}{2}} \frac{4\pi - u}{2+\cos(4\pi - u)} \times -du$

$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{4\pi - u \; du}{2+\cos u}$ (1)

$= \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{4\pi}{2+\cos u} \, du - \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{u}{2+\cos u} \, du$

$= \frac{1}{2} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{4\pi}{2+\cos u} \, du$ (1)

$= 2\pi \times \frac{2\pi\sqrt{3}}{9}$ from (i)

$= \frac{4\pi^2\sqrt{3}}{9}$ (1)

Question 2

a) $\frac{4+3i}{1+\sqrt{2}i} \times \frac{1-\sqrt{2}i}{1-\sqrt{2}i}$

$= \frac{4 - 4\sqrt{2}i + 3i + 3\sqrt{2}}{3}$

$= \frac{(4+3\sqrt{2})}{3} + \frac{i(3-4\sqrt{2})}{3}$

$a = \frac{4+3\sqrt{2}}{3}$ (1) $\quad b = \frac{3-4\sqrt{2}}{3}$ (1)

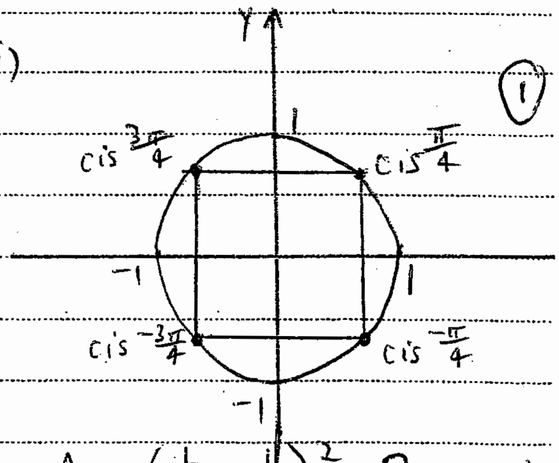
b)(i) $z^4 = -1$

$z^4 = \text{cis}(\pi + 2k\pi)$ (1)

(1) $z = \text{cis}\left(\frac{\pi + 2k\pi}{4}\right)$ $k = 0, \pm$

$z = \text{cis}\frac{\pi}{4}, \text{cis}\frac{3\pi}{4}, \text{cis}\left(\frac{-\pi}{4}\right), \text{cis}\left(\frac{-3\pi}{4}\right)$ (1)

(ii) (1)



(iii) $A = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 = 2 \text{ units}^2$ (1)

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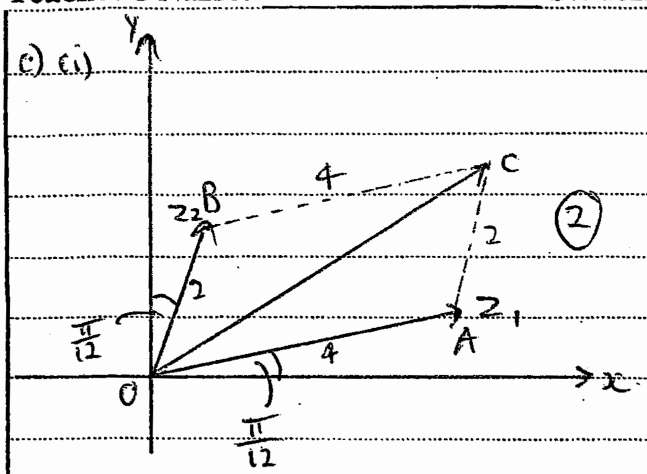
c) (i)



(2)

(ii) $|z_1 + z_2|$

$\angle OAC = \frac{2\pi}{3}$ (cointerior angles)

By cosine rule

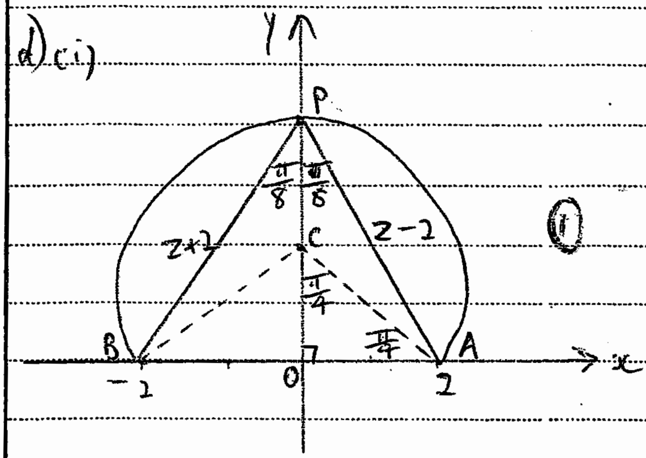
$OC^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos\frac{2\pi}{3}$ (1)

$= 20 - 16 \times -\frac{1}{2}$

$OC^2 = 28$

$\therefore OC = 2\sqrt{7}$ (1)

d) (i)



(1)

$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$ means angle between PA and PB is $\frac{\pi}{4}$ anywhere along APB (1)

$\Rightarrow$ APB is circular (angles in same segment are equal).

(ii) Let P be on imaginary axis $\therefore$ centre is on imaginary axis. Angle at centre is twice the angle at the circumference $\therefore \angle OCA$ is $\frac{\pi}{4} \Rightarrow \triangle OCA$ is isosceles $\therefore OC = 2$ units

$\Rightarrow$ centre is $(0, 2)$ (1) and radius $2\sqrt{2}$ units (1)

Question 3

a) $P(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \frac{b\cos\theta}{-a\sin\theta}$

$\therefore y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$ (2)

To find A, $y = 0$

$\therefore 0 - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$

$ab\sin^2\theta = b\cos\theta\, x - ab\cos^2\theta$

$ab = b\cos\theta\, x$ $\rightarrow x = a\sec\theta$ $\therefore A(a\sec\theta, 0)$ (1)

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To find B let $x=0$

$y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(-a\cos\theta)$

$y = \frac{b\cos^2\theta}{\sin\theta} + b\sin\theta$

$y = b\left(\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right)$

$\therefore$ B is $\left(0, b\left(\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right)\right)$ (1)

but $\sin\theta + \frac{\cos^2\theta}{\sin\theta} = \sin\theta(1+\cot^2\theta)$

$= \sin\theta \times \csc^2\theta$

$= \csc\theta$

$\therefore$ B is $(0, b\csc\theta)$

(ii) $\frac{PA}{PB}$

drop a perpendicular from P to x axis call it C.

$\therefore \frac{PA}{PB} = \frac{CA}{OC}$ (Ratio of intercepts) (1)

$= \frac{a\sec\theta - a\cos\theta}{a\cos\theta}$

$= \frac{1-\cos^2\theta}{\cos^2\theta}$

$= \tan^2\theta$ (1)

b)(i) $b^2 = a^2(e^2-1)$

$e^2 = 2$ as $a=b$

$\therefore e = \sqrt{2}$ (1)

(ii) $S(\pm ae, 0)$

$S(\pm\sqrt{8}\times\sqrt{2}, 0)$

$S(\pm 4, 0)$ (1)

(iii) Directrices $x = \pm\frac{a}{e}$

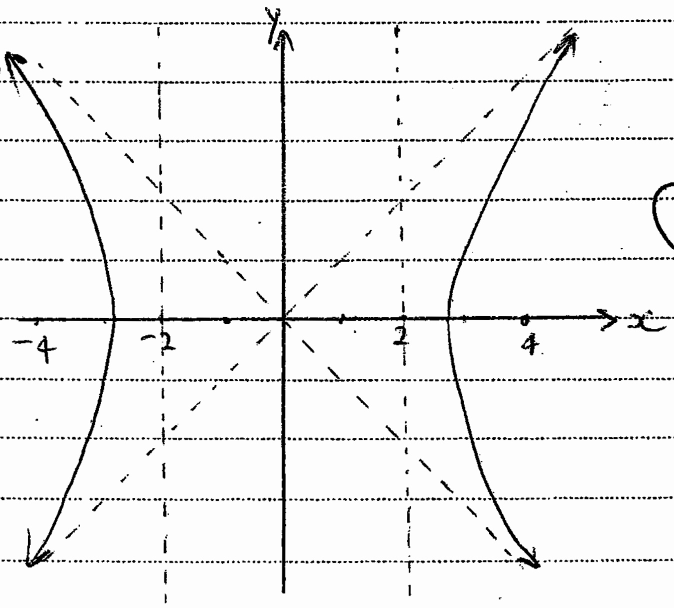
$x = \pm\frac{\sqrt{8}}{\sqrt{2}}$

$x = \pm 2$ (1)

(iv) Asymptotes

$y = \pm x$ (1)

(v)



(1)

(vi) $y = \frac{4}{x}$

$\frac{dy}{dx} = \frac{-4}{x^2}$

At $x = 2p$

$\frac{dy}{dx} = \frac{-4}{4p^2}$

$= \frac{-1}{p^2}$ (1)

$\therefore y - \frac{2}{p} = p^2(x-2p)$

$py - 2 = p^3(x-2p)$

(1)

Normal has gradient p^2

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(vii) $(2q, \frac{2}{q})$ lies on normal

$\therefore \quad p \times \frac{2}{q} - 2 = p^3(2q - 2p)$

$\frac{2p}{q} - 2 = 2p^3(q - p)$

$\frac{p}{q} - 1 = p^3(q - p)$ ①

$\frac{p}{q} - \frac{q}{q} = p^3(q - p)$

$\frac{(p - q)}{q} = p^3(q - p)$

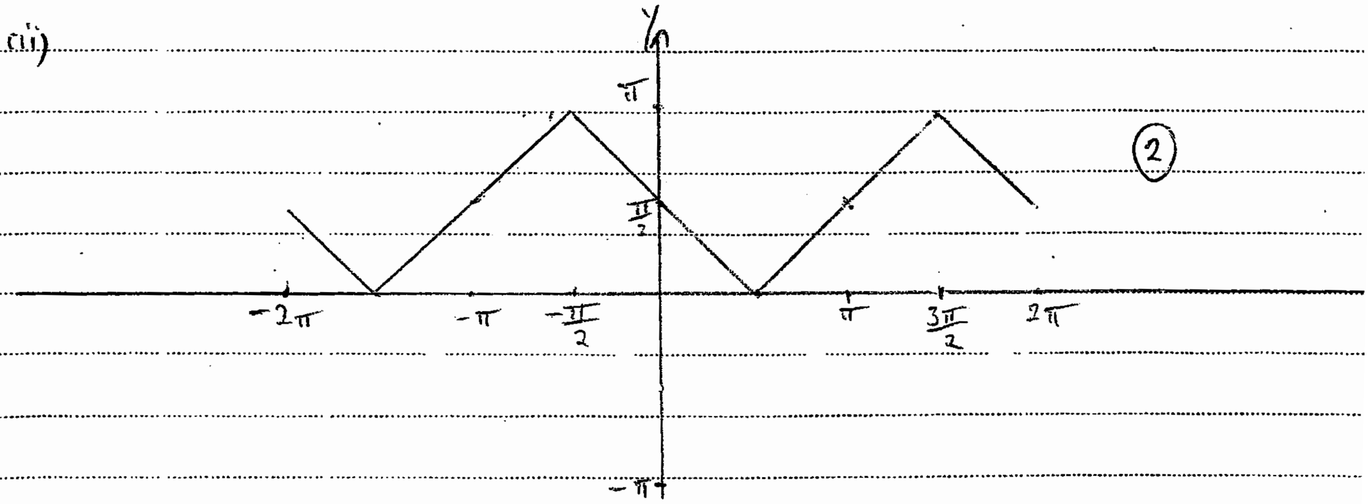
$-\frac{1}{q} = p^3$

$q = -\frac{1}{p^3}$ ①

Question 4

(i) Period is 2π (same as $\sin x$). ①

(ii)



②

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b) $f'(0) = 0 \Rightarrow$ a stationary point at $x = 0$ and $x = 2n\pi$ (n integral)

$f'(x) \geq 0 \Rightarrow f(x)$ is increasing for $x > 0$ except $x = 2n\pi$ ①

$\therefore f(x) > f(0)$ for $x > 0$

But $f(0) = 0 - \frac{0}{2+1} = 0$

Hence $f(x) > 0$ for $x > 0$ ①

$\therefore x - \frac{3\sin x}{2+\cos x} > 0$

$x > \frac{3\sin x}{2+\cos x}$ for $x > 0$. ①

c)(i) $\delta V = 2\pi \times (1-x) \times \left(\frac{3}{2} - (2 - \cos^2 x)\right)\delta x$ ① as $y = \frac{3}{2}$ when $x = \pm\frac{\pi}{4}$

$= 2\pi \times (1-x) \times \left(-\frac{1}{2} + \frac{\cos 2x + 1}{2}\right)\delta x$

$= \pi(1-x)\cos 2x \,\delta x$

$V = \sum_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}} \lim_{\delta x \to 0} \pi(1-x)\cos 2x \,\delta x$ ①

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi(1-x)\cos 2x \, dx$ ①

c(ii) $= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \, dx - \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\cos 2x \, dx$

$= \frac{\pi}{2}\left[\sin 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - 0$ (as $x\cos 2x$ is odd) ①

$= \frac{\pi}{2}(1 - -1)$

$= \pi \text{ m}^3$

$=$ 3142 L to nearest litre ①

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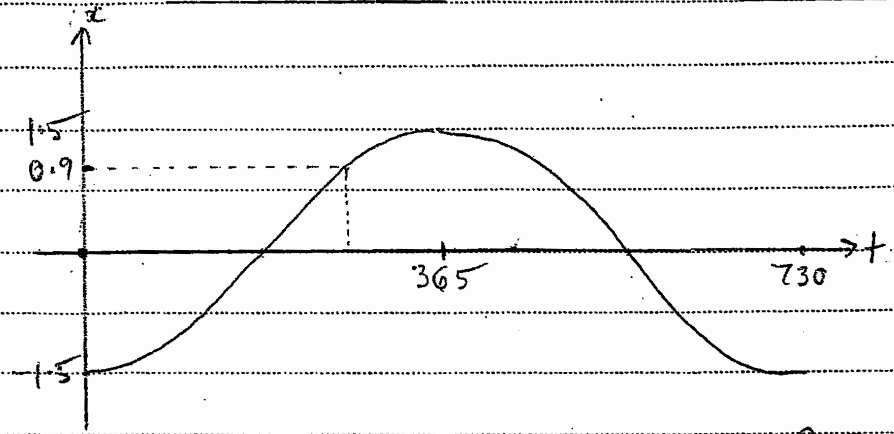
d)



Period is 730 $\therefore 730 = \frac{2\pi}{n}$ $n = \frac{2\pi}{730} = \frac{\pi}{365}$

Equation of tidal motion is

$x = -1 \cdot 5 \cos \frac{\pi}{365} t$ (1)

Need to solve this when $x = 0 \cdot 9$ (0·6m below high tide)

$0 \cdot 9 = -1 \cdot 5 \cos \frac{\pi}{365} t$ (1)

$-0 \cdot 6 = \cos \frac{\pi}{365} t$

$\frac{\pi}{365} t = 2 \cdot 214$,

$t = 257$ minutes

$\therefore$ 11:30 am plus 257 minutes

<u>3:47 pm</u> (1)

<u>Question 5</u>

a) $P(x) = Q(x)(x-2) + 11$

$P(x) = R(x)(x-4) + 15$

$P(x) = S(x)(x-2)(x-4) + (ax+b)$

$P(2) = 2a + b = 11$

$P(4) = 4a + b = 15$

$2a = 4$

$a = 2 \quad \therefore b = 7$ (1)

$\therefore$ Remainder is $\underline{2x+7}$ (1)

b)(i) $P(x) = (x-2)^n Q(x)$

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$P'(x) = n(x-\alpha)^{n-1}Q(x) + (x-\alpha)^n Q'(x)$ (1)

$P'(\alpha) = 0 + 0$

$\underline{P'(\alpha) = 0}$ (1)

(ii) $P(x) = ax^4 + 4bx + c = 0$

$P'(x) = 4ax^3 + 4b = 0$

when $x = \sqrt[3]{\frac{-b}{a}}$ now sub this into $P(x)$ (1)

$$a\left(\sqrt[3]{\frac{-b}{a}}\right)^4 + 4b\left(\sqrt[3]{\frac{-b}{a}}\right) + c = 0$$

$$\sqrt[3]{\frac{-b}{a}}\left[a\left(\frac{-b}{a}\right) + 4b\right] + c = 0$$

$$\sqrt[3]{\frac{-b}{a}}(-b + 4b) = -c \quad (1)$$

$$\sqrt[3]{\frac{-b}{a}}(3b) = -c$$

$$-\frac{b}{a}(27b^3) = -c^3$$

$$-27b^4 = -ac^3$$

$$\underline{ac^3 - 27b^4 = 0} \quad (1)$$

c) $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$

zeros at $1 \pm i$ (coefficients are real)

quadratic factor is $x^2 - 2x + 2$ (1)

$P(x) = (x^2 - 2x + 2)(x^2 - 3)$

$= (x^2 - 2x + 2)(x - \sqrt{3})(x + \sqrt{3})$ (1)

d) (i) $I_n = \int \operatorname{cosec}^n x \, dx$

$= \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$

$= \int \operatorname{cosec}^{n-2} x \frac{d}{dx}(-\cot x) \, dx$ (1)

$= -\cot x \operatorname{cosec}^{n-2} x - \int -\cot x \times -(n-2)\operatorname{cosec}^{n-3} x \times \operatorname{cosec} x \cot x \, dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$

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5d(i) continued. (1)

$$= -\cot x \,\mathrm{cosec}^{n-2} x - (n-2)\int \mathrm{cosec}^n x\, dx + (n-2)\int \mathrm{cosec}^{n-2} x\, dx$$

$$(n-1)\int \mathrm{cosec}^n x\, dx = -\cot x \,\mathrm{cosec}^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{-\cot x\,\mathrm{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2} \quad (1)$$

(ii) $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{cosec}^n x\, dx = \left[\frac{-\cot x\,\mathrm{cosec}^{n-2} x}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{n-2}{n-1} I_{n-2}$$

$$= \left[0 + \frac{\cot\frac{\pi}{4}\,\mathrm{cosec}^{n-2}\frac{\pi}{4}}{n-1}\right] + \frac{n-2}{n-1} I \quad (1)$$

$$= \frac{1 \times (\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-}$$

(iii) $$I_6 = \frac{(\sqrt{2})^4}{6-1} + \frac{4}{5} I_4$$

$$= \frac{2^2}{5} + \frac{4}{5}\left[\frac{(\sqrt{2})^2}{4-1} + \frac{2}{3} I_2\right] \quad (1)$$

$$= \frac{4}{5} + \frac{4}{5}\times\frac{2}{3} + \frac{4}{5}\times\frac{2}{3}\left[\frac{\sqrt{2}^0}{2-1} + 0\right]$$

$$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15}\times\frac{1}{1}$$

$$= \frac{28}{15} \quad (1)$$

Question 6

a) (i) $x^3 + 2x + 1 = 0$

Replace x with $\frac{1}{x}$ (1)

$(\frac{1}{x})^3 + 2(\frac{1}{x}) + 1 = 0$

$1 + 2x^2 + x^3 = 0$

$x^3 + 2x^2 + 1 = 0$ (1)

(ii) $\frac{\beta+\gamma}{\alpha^2}$, $\frac{\gamma+\alpha}{\beta^2}$ and $\frac{\alpha+\beta}{\gamma^2}$

Since $\alpha + \beta + \gamma = 0$, th

roots become

$\frac{-\alpha}{\alpha^2}$, $\frac{-\beta}{\beta^2}$, $\frac{-\gamma}{\gamma^2}$ (1)

or $\frac{-1}{\alpha}$, $\frac{-1}{\beta}$, $\frac{-1}{\gamma}$ so replace

x with $(-x)$ in (i)

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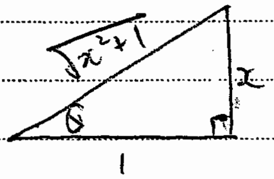
$(-x)^3 + 2(-x)^2 + 1 = 0$ (1)

$-x^3 + 2x^2 + 1 = 0$

$x^3 - 2x^2 - 1 = 0$ (1)

b(i) $4 - \tan\theta = 5\sin\theta\cos\theta$



(1)

$4 - x = 5 \times \frac{x}{\sqrt{x^2+1}} \times \frac{1}{\sqrt{x^2+1}}$

$(4-x)(x^2+1) = 5x$

$4x^2 + 4 - x^3 - x = 5x$

$x^3 - 4x^2 + 6x - 4 = 0$ (1)

(ii) Roots of polynomial

$(x-2)(x^2-2x+2) = 0$

Only one real root

$x = 2$ (1)

Now since $x = \tan\theta$

$\tan\theta = 2$ (1)

$\theta = 63°, 243°$ (1)

c)(i) $TC^2 = TB.TA$ (Square of tangent theorem)

$TD^2 = TB.TA$ (Square of tangent theorem) (1)

$\therefore TC = TD$

(ii) Since $TC = TD$

$\triangle TCD$ is isosceles

$\therefore \angle TCD = \angle TDC$

But $\angle TDC = \angle TAD$ (Alternate segment theorem) (1)

$\angle TCD = \angle TAC$ (Alternate segment theorem)

$\Rightarrow \angle TAD = \angle TAC$ (1)

(iii) $\angle TCD = \angle TAD$ (using c(ii)) (1)

$\therefore$ TCAD is a cyclic quadrilateral as the angles in the same segment at C and A subtended from TD are equal. (1)

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Question 7

a) (i)

$|z^2-2z| = 2|z|$

$|z(z-2)| = 2|z|$

$|z||z-2| = 2|z|$ ①

$|z| \times 2 = 2|z|$

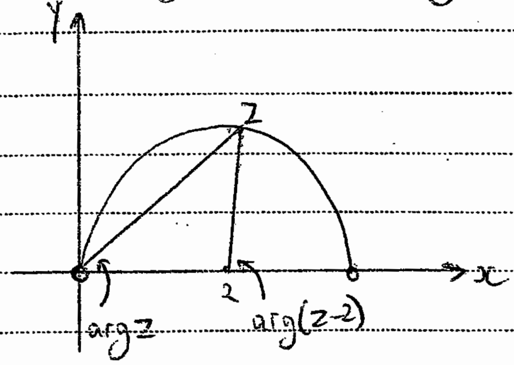
since $|z-2| = 2$ ①

(ii) $\arg(z-2) = k\arg(z^2-2z)$

$\arg(z-2) = k\arg(z(z-2))$

$\arg(z-2) = k\arg z + k\arg(z-$

$(1-k)\arg(z-2) = k\arg z$ ①



$\Rightarrow \arg(z-2) = 2\arg z$

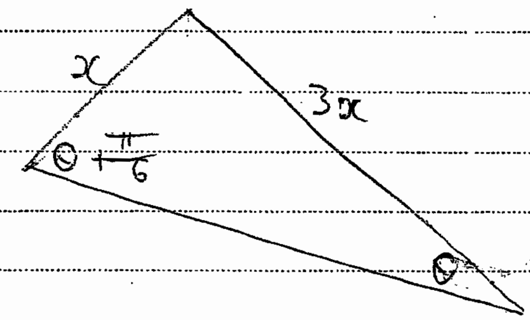
$\therefore 2(1-k)\arg z = k\arg z$ (

$2-2k = k$

$2 = 3k$

$k = \frac{2}{3}$ ①

b)



Using sine rule

$\frac{\sin\theta}{x} = \frac{\sin(\theta+\frac{\pi}{6})}{3x}$ ①

$3\sin\theta = \sin\theta\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos$

$3\sin\theta = \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$

$\sin\theta(3-\frac{\sqrt{3}}{2}) = \frac{1}{2}\cos\theta$ (

$\tan\theta = \frac{\frac{1}{2}}{\frac{6-\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{6-\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{6-\sqrt{3}}\right)$ (

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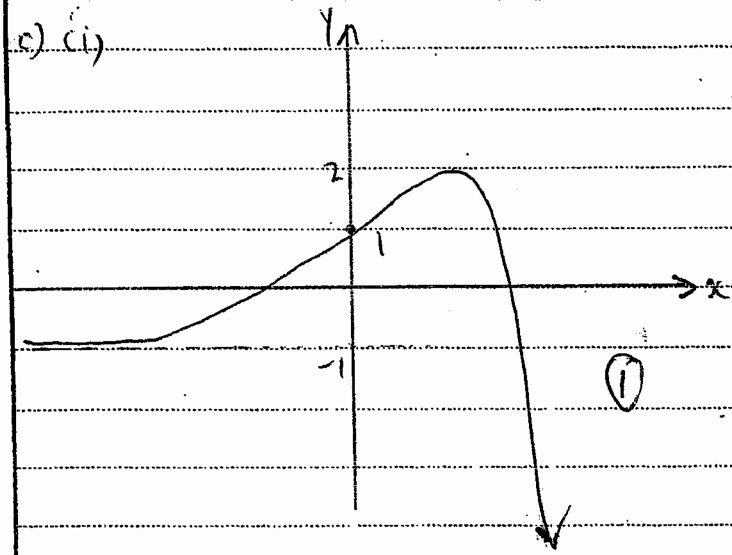
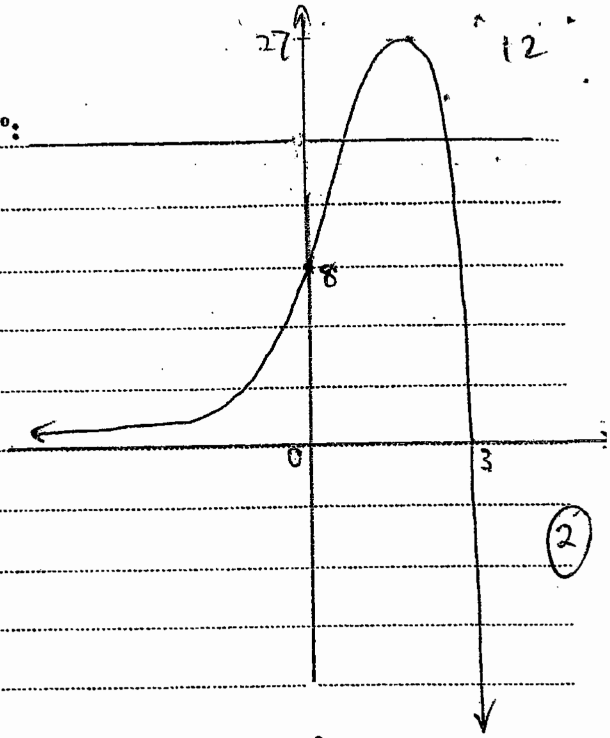
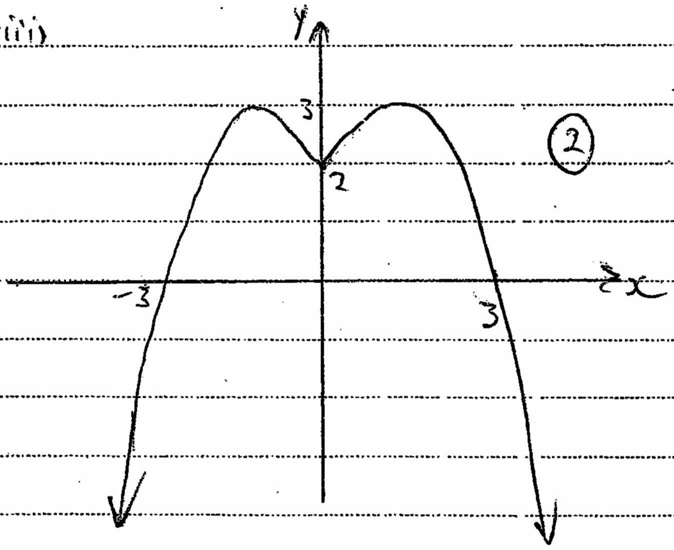
c) (i)

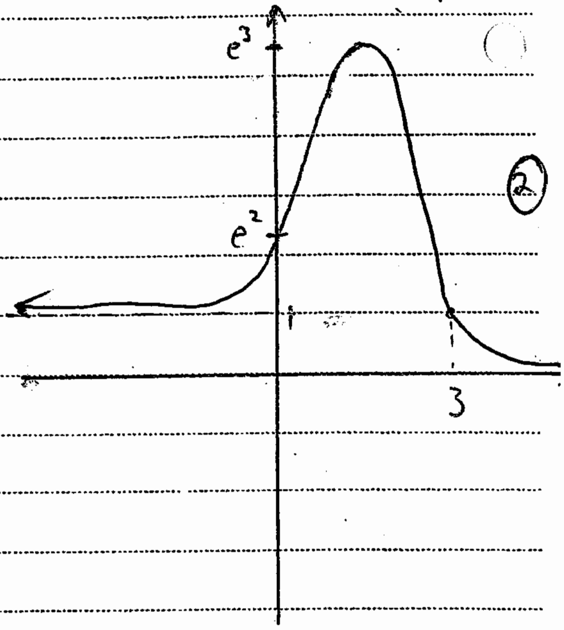


(ii)



c(iii)





Question 8

a) $\sqrt{\frac{x}{v}} + \sqrt{\frac{y}{v}} = 1$

when $x = 0$, $y = v$

Differentiate implicitly

$$\frac{1}{2}\left(\frac{x}{v}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{y}{v}\right)^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\left(\frac{x}{v}\right)^{-\frac{1}{2}}}{\left(\frac{y}{v}\right)^{-\frac{1}{2}}}$$

$$= -\frac{\sqrt{\frac{y}{v}}}{\sqrt{\frac{x}{v}}} = -k\sqrt{\frac{y}{x}} \quad ①$$

k constant

Now as $x \to 0$

$\frac{dy}{dx} \to -\infty$

$\Rightarrow$ tangent is vertical at $x = 0$.

$\therefore$ the y axis must be a tangent at $(0, v)$ ①

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b) $V = \frac{4}{3}\pi r^3$ $\quad A = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\quad \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{dA}{dt}$

$\therefore \frac{dV}{dr} \times \frac{dr}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ (1)

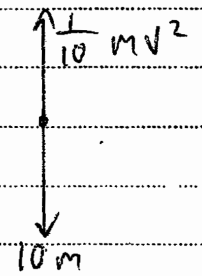
$4\pi r^2 = 8\pi r$

$r^2 - 2r = 0$

$r(r-2) = 0$

when $r = 0$ or 2 (1)

c) (i)



$m \times a = 10m - \frac{1}{10}mv^2$

$\therefore a = 10 - \frac{1}{10}v^2$

$a = \frac{100 - v^2}{10}$ (1)

(ii) $\frac{dv}{dt} = \frac{100 - v^2}{10}$

$\frac{dt}{dv} = \frac{10}{(10-v)(10+v)}$

$\frac{10}{(10-v)(10+v)} = \frac{a}{10-v} + \frac{b}{10+v}$ (1)

$10 = a(10+v) + b(10-v)$

When $v = 10$

$10 = 20a \quad \therefore a = \frac{1}{2}$

When $v = -10$

$10 = 20b \quad \therefore b = \frac{1}{2}$

$\therefore dt = \left[\frac{\frac{1}{2}}{10-v} + \frac{\frac{1}{2}}{10+v}\right] dv$ (

$t = \frac{1}{2}\log_e\left[\frac{10+v}{10-v}\right] + c$

when $t = 0$, $v = 0$

$\therefore 0 = \frac{1}{2}\log_e 1 + c \quad \therefore c = 0$

$t = \frac{1}{2}\log_e\left[\frac{10+v}{10-v}\right]$ (1)

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(iii) $t = \frac{1}{2}\ln\left[\frac{10+v}{10-v}\right]$

$e^{2t} = \frac{10+v}{10-v}$ (1)

$10e^{2t} - ve^{2t} = 10 + v$

$10e^{2t} - 10 = v(1+e^{2t})$

$\underline{v = \frac{10(e^{2t}-1)}{e^{2t}+1}}$ (1)

$\frac{dx}{dt} = \frac{10(e^{2t}-1)}{e^{2t}+1}$

$= \frac{10(e^{t}-e^{-t})}{e^{t}+e^{-t}}$ (1)

$x = 10\log_e(e^t + e^{-t}) + c$

when $t = 0$, $x = 0$

$0 = 10\log_e(2) + c$

$c = -10\log_e 2$

$\therefore \underline{x = 10\log_e\left(\frac{e^t + e^{-t}}{2}\right)}$ (1)

(iv) As $t \to \infty$, v approaches terminal velocity

Now in

$v = \frac{10(e^{2t}-1)}{e^{2t}+1}$, $\frac{e^{2t}-1}{e^{2t}+1} \to 1$ as $t \to \infty$ (1)

$\therefore v \to 10$ $\therefore$ terminal velocity is $\underline{10\,m/s}$.

(v) $v = 8 \Rightarrow \begin{cases} t = \frac{1}{2}\ln\left(\frac{10+8}{10-8}\right) = \frac{1}{2}\ln 9 = \underline{\ln 3} \quad (1) \\ x = 10\ln\left(\frac{e^{\ln 3} + e^{-\ln 3}}{2}\right) \end{cases}$

$= 10\ln\left(\frac{3+\frac{1}{3}}{2}\right) = \underline{10\ln\frac{5}{3}}$ (1)