SYDNEY TECHNICAL HIGH SCHOOL

(Established 1911)

TRIAL HIGHER SCHOOL CERTIFICATE

2014

Mathematics Extension 2

General Instuctions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Diagrams are not drawn to scale
- Start each question on a new page

Total marks - 100

Section 1 - 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section 2 - 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Name :____________________________

Teacher :____________________________

Section 1

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

---

1 The graph of $y = g(x)$ is a reflection of the graph of $y = f(x)$ about the x axis. Which of the following must be true ?

(A) $g(x) = |\,f(x)|$

(B) $g(x) = -f(x)$

(C) $g(x) = f(-x)$

(D) $g(x) = f^{-1}(x)$

2 The slope of the curve $2x^3 - y^2 = 7$ at the point where $y = -3$ is

(A) 4

(B) -1

(C) -2

(D) -4

3 If $z = -2i$, then $|z^2|$ and $Arg(z^2)$ are respectively

(A) 4 and 0

(B) -4 and π

(C) 4 and π

(D) -4 and $-\pi$

4 If $1 + 2i$ is a root of the polynomial equation $2x^3 + bx^2 + 2x + 20 = 0$,

where b is real, then the value of b is

(A) -3

(B) -2

(C) 0

(D) 2

5 The region bounded by the lines $x = -1$, $y = 1$, $y = -1$ and the curve $x = y^2$

is rotated about the line $x = -2$ to form a solid of revolution.

Which is the correct expression for the volume of this solid ?

(A) $\pi \int_{-1}^{1} y^4 - 4y^2 + 3\, dy$

(B) $\pi \int_{-1}^{1} y^4 + 4y^2 + 3\, dy$

(C) $\pi \int_{-1}^{1} y^4 - 4y^2 + 4\, dy$

(D) $\pi \int_{-1}^{1} y^4 + 4y^2 + 4\, dy$

6 The x axis is tangent to an ellipse at the point $(1, 0)$ and the y axis is tangent to the same ellipse at the point $(0, -2)$.

Which one of the following could be the equation of this ellipse ?

(A) $\frac{(x-1)^2}{4} + (y+2)^2 = 1$

(B) $\frac{(x+1)^2}{4} + (y-2)^2 = 1$

(C) $(x+1)^2 + \frac{(y-2)^2}{4} = 1$

(D) $(x-1)^2 + \frac{(y+2)^2}{4} = 1$

7 Which one of the following relations does **not** have a graph that is a straight line passing through the origin?

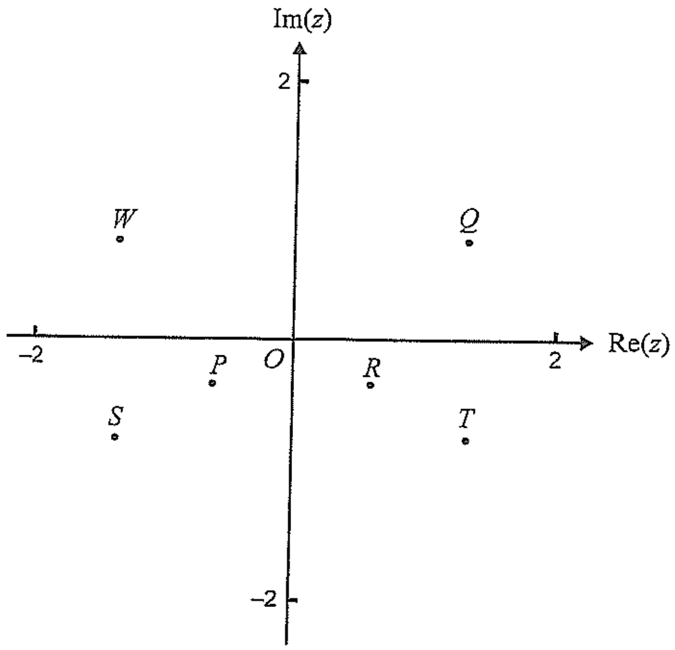
(A) $z + \bar{z} = 0$

(B) $z = i\,\bar{z}$

(C) $3\,Re(z) = Im(z)$

(D) $Re(z) + Im(z) = 1$

8



The point W on the Argand diagram above represents a complex number w where $|w| = 1.5$. The complex number w^{-1} is best represented by the point

(A) P

(B) R

(C) S

(D) T

9 With a suitable substitution, $\int_0^{\frac{\pi}{6}} \cos^3(2x)\,dx$ can be expressed as

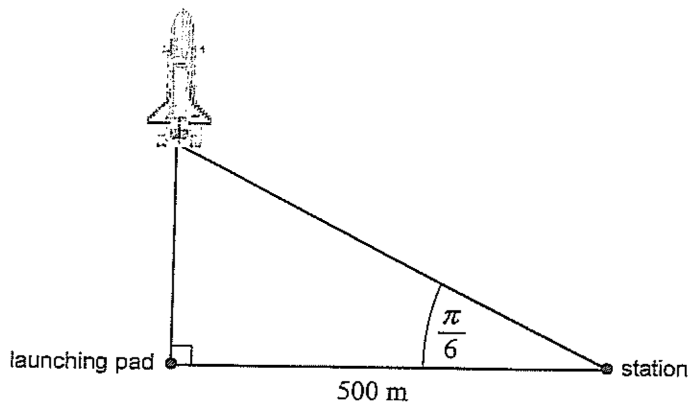
(A) $\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} 1-u^2\,du$

(B) $\frac{1}{2}\int_0^{\frac{1}{2}} 1-u^2\,du$

(C) $2\int_0^{\frac{\sqrt{3}}{2}} u^2-1\,du$

(D) $2\int_0^{\frac{1}{2}} 1-u^2\,du$

10



An ascending space shuttle rises vertically from a launching pad. As it rises the shuttle is tracked from a station at ground level 500 metres away. When the angle of elevation of the shuttle is $\frac{\pi}{6}$ radians from the horizontal, and is increasing at a rate of 0.5 radians per second, the speed of the shuttle is closest to

(A) 144 metres per second

(B) 289 metres per second

(C) 333 metres per second

(D) 577 metres per second

Section 2

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Start each question on a new page.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

a) Let $z = -1 + i\sqrt{3}$ and $w = \sqrt{3} + i$

i) Find $z \div \overline{w}$, in simplest $a + ib$ form. 2

ii) Find $Arg(z)$. 1

iii) Find $|z|$. 1

iv) Find the smallest value of n, given that $n > 1$, 1

such that $\mathrm{Arg(z)} = \mathrm{Arg(z^n)}$

b) Find $\int \frac{1}{2x^2 - x}\,dx$ 3

c) Find $\int x \sec^2 x\,dx$ 3

d) Consider the polynomial $P(z) = z^3 + 9z^2 + 28z + 20$. 2

Given that $P(-1) = 0$, fully factorise $P(z)$ over the complex field.

e) Without the use of calculus, sketch the curve $9y^2 = x(3 - x)^2$. 2

Question 12 (15 marks) Start a new page

a) Find the equation of the hyperbola with foci at the points $(\pm 8, 0)$ and vertices at the points $(\pm 5, 0)$ **2**

b) Solve the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ given that it has a triple root. **3**

c) If α, β and γ are the roots of the equation $x^3 - 5x + 3 = 0$ find the polynomial equation with roots $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$. **2**

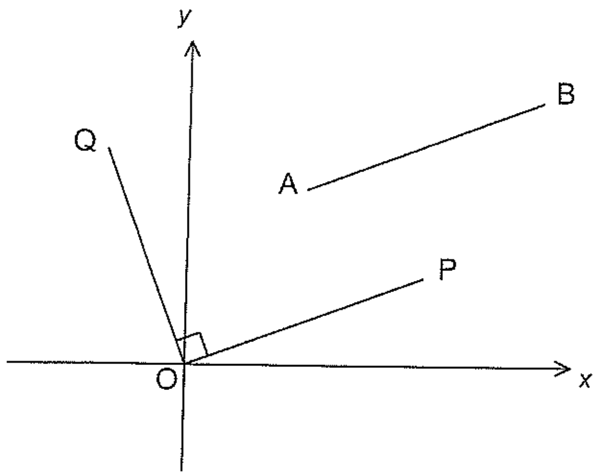
d) i) Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$ **4**

show that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$

ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$. **2**

Question 12 (continued)

e) 2



In the Argand diagram above, intervals AB , OP and OQ are equal in length.

OP is parallel to AB and angle POQ = 90°.

If A and B represent the complex numbers $3 + 5i$ and $9 + 8i$ respectively,

find the complex number which is represented by Q.

Question 13 (15 marks) Start a new page

a) Solve for x, $tan^{-1}3x \; - \; tan^{-1}2x \; = \; tan^{-1}\frac{1}{5}$ 2

b) Use the substitution $x = 10\sin\theta$ to evaluate $\int_0^5 \sqrt{100 - x^2}\, dx$ 4

c) Find the volume of a solid whose base is a triangle with vertices $(0\,,0)$, 4
$(2\,,0)$ and $(0,2)$ and whose cross sections perpendicular to the base and parallel to the y axis are semi-circles.

d) i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 2
at $P(a\sec\theta\,, b\tan\theta)$ is given by $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

ii) If the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta\,, b\tan\theta)$ 3
cuts the x axis at A and the y axis at B, show that $\frac{PA}{PB} = sin^2\theta$

Question 14 (15 marks) Start a new page

a) Consider the function $f(x) = \frac{x^3+4}{x^2}$.

i) Find the coordinates of any stationary points on $y = f(x)$ and determine their nature. 2

ii) For what values of x is the curve $y = f(x)$ is concave up? 2

iii) Sketch the curve $y = f(x)$ showing any important features. 2

iv) On a separate diagram, and without the use of further calculus, sketch the curve $y = \frac{1}{f(x)}$ 2

b) A particle P is projected vertically upwards from the surface of the Earth with initial velocity u. The acceleration due to gravity at any point on its path is given by $-\frac{k}{x^2}$, where x is the distance of the particle from the centre of the Earth and k is a constant.

i) Explain why $k = gR^2$, where R is the radius of the Earth and g is the acceleration due to gravity at the Earth's surface. 1

ii) Neglecting air resistance, show that the velocity v of the particle is given by 3

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$$

iii) If the initial velocity of the particle is given by $u = \sqrt{2gR}$, show that the time taken to reach a height $3R$ above the Earth's surface is given by $\frac{7\sqrt{2R}}{3\sqrt{g}}$. 3

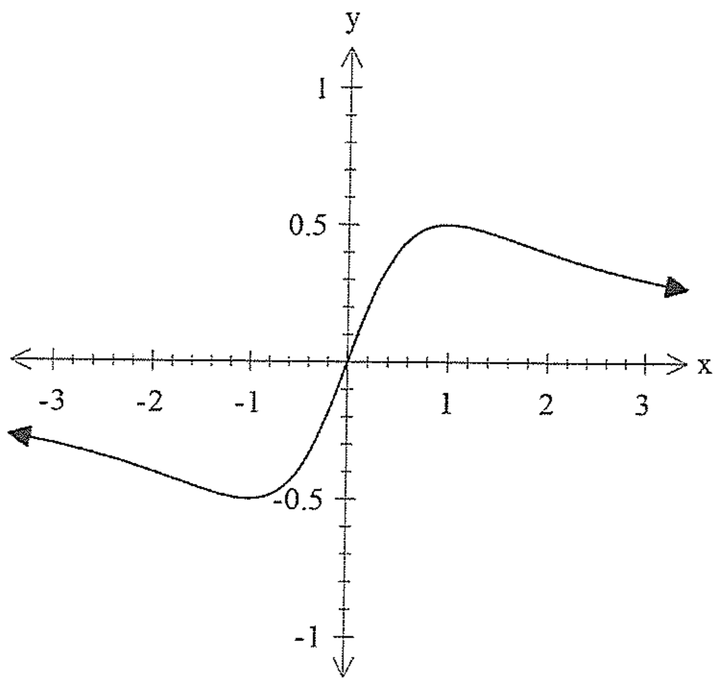
Question 15 (15 marks) Start a new page

a) For the relation $x^3 + y^3 = 6xy$, find $\frac{dy}{dx}$ in terms of x and y . 2

b) Sketch the region on the Argand diagram defined by 2

$$(z - 3 + i)(\overline{z} - 3 - i) \leq 9$$

c) The following is a graph of $y = \frac{x}{x^2+1}$

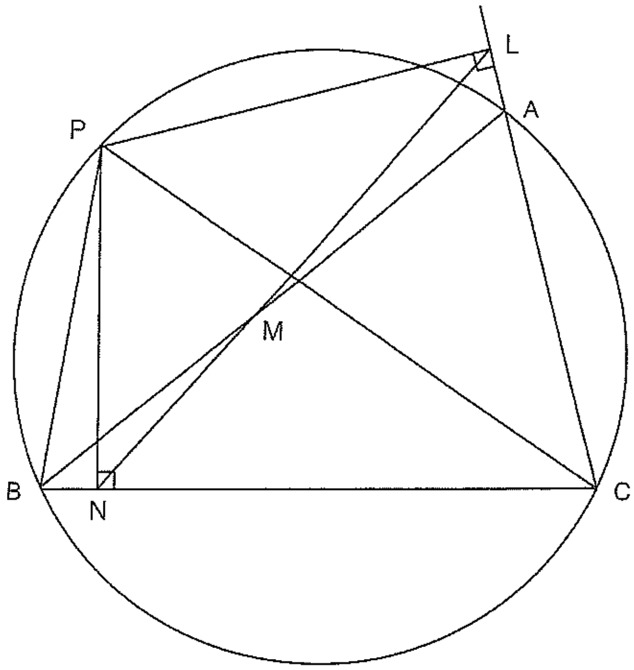


The area bounded by the curve $y = \frac{x}{x^2+1}$, the line $x = 1$ and the x axis 4

is rotated about the y axis. Using the method of cylindrical shells, or otherwise, find the volume of the solid of revolution formed.

Question 15 (continued)

d)



ABC is an acute angled triangle inscribed in a circle. P is a point on the minor arc AB such that PL is perpendicular to CA (produced) and PN is perpendicular to BC.

LN cuts AB at M.

Copy or trace the diagram into your Writing Booklet.

i) Explain why PNCL is a cyclic quadrilateral. 1

ii) Show that $\angle PBM = \angle PNM$. 3

iii) Show that PM is perpendicular to AB. 3

Question 16 (15 marks) Start a new page

a) If w is a complex root of the equation $x^3 = 1$.

i) Show that the other complex root is w^2. 1

ii) Show that $1 + w + w^2 = 0$. 1

iii) Find in its simplest form, the cubic equation whose roots are 3

3, $2w + w^2$ and $2w^2 + w$.

b) i) Given that for $k > 0$, $2k + 3 > 2\sqrt{(k+1)(k+2)}$, 4

Use mathematical induction to show that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$

for all positive integers n.

ii) Use the graph of $y = \frac{1}{\sqrt{x}}$ to show that 3

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_{1}^{n} \frac{1}{\sqrt{x}}\,dx$$

iii) Hence show that $198 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 199$ 3

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXT 2 SOLUTIONS 2014 TRIAL HSC

1. B
2. D
3. C
4. C
5. B
6. D
7. D
8. A
9. A
10. C

11. a) i) $\dfrac{-1+i\sqrt{3}}{\sqrt{3}-i} \times \dfrac{\sqrt{3}+i}{\sqrt{3}+i}$

$= \dfrac{-\sqrt{3} - i + 3i - \sqrt{3}}{4}$

$= \dfrac{-\sqrt{3}+i}{2}$

ii) $Arg(z) = \dfrac{2\pi}{3}$

iii) $|z| = 2$

iv) $2m\pi + \dfrac{2\pi}{3} = n \times \dfrac{2\pi}{3}$

$\therefore \quad n = 4$

b) $\displaystyle\int \frac{1}{2x^2 - x}\,dx$

$\dfrac{1}{x(2x-1)} = \dfrac{A}{x} + \dfrac{B}{2x-1}$

$\therefore 1 = A(2x-1) + Bx$

sub $x = 0$

$1 = -A$

$\Rightarrow A = -1$

$\therefore B = 2$

$= \displaystyle\int \frac{2}{2x-1} - \frac{1}{x}\,dx$

$= \ln(2x-1) - \ln x + c$

$= \ln\left(\dfrac{2x-1}{x}\right) + c$

c) $\int x \sec^2 x \, dx$ $\quad u = x \quad v = \tan x$

$\quad u' = 1 \quad v' = \sec^2 x$

$= x\tan x - \int \tan x \, dx$

$= x\tan x + \ln(\cos x) + c$

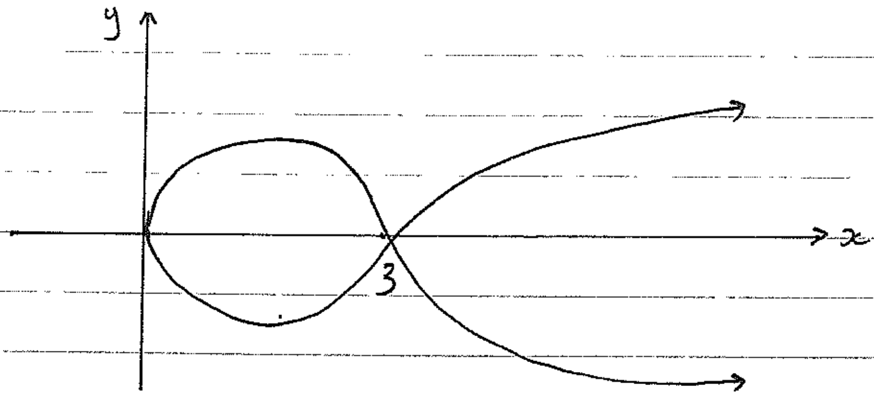
d) $P(z) = z^3 + 9z^2 + 28z + 20$

$= (z+1)(z^2 + 8z + 20)$

$= (z+1)((z+4)^2 + 4)$

$= (z+1)(z+4+2i)(z+4-2i)$

e)



12 a) $S(\pm 8, 0)$ $\quad ae = 8$

$A(\pm 5, 0)$ $\quad a = 5$

$e = \frac{8}{5}$

$b^2 = a^2(e^2 - 1)$

$= 25\left(\frac{64}{25} - 1\right)$

$= 39$

$\therefore \frac{x^2}{25} - \frac{y^2}{39} = 1$

b) $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$

$P'(x) = 4x^3 - 15x^2 - 18x + 81$

$P''(x) = 12x^2 - 30x - 18$

$\therefore \quad 6(2x^2 - 5x - 3) = 0$

$6(2x+1)(x-3) = 0$

$x = -\frac{1}{2}, 3$

$P'(3) = 0 \qquad P'(-\frac{1}{2}) \neq 0$

$\therefore$ roots are $3, 3, 3, \alpha$

sum of roots $\quad 3 + 3 + 3 + \alpha = 5 \quad \left(-\frac{b}{a}\right)$

$\alpha = -4$

$\therefore$ roots $\quad x = 3, 3, 3, -4$

c) let $y = \frac{1}{x+1}$

$\therefore \quad x = \frac{1}{y} - 1$

$\therefore$ required equation $\quad P\left(\frac{1}{y} - 1\right) = 0$

$$\left(\frac{1}{y} - 1\right)^3 - 5\left(\frac{1}{y} - 1\right) + 3 = 0$$

$$\frac{1}{y^3} - \frac{3}{y^2} + \frac{3}{y} - 1 - \frac{5}{y} + 5 + 3 = 0$$

$$\frac{1}{y^3} - \frac{3}{y^2} - \frac{2}{y} + 7 = 0 \qquad (\times y^3)$$

$$1 - 3y - 2y^2 + 7y^3 = 0$$

d) i) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$ $\qquad u = x^n \qquad v = -\cos x$

$\qquad u' = n(x)^{n-1} \qquad v' = \sin x$

$$= -x^n \cos x \Big]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx$$

$u = x^{n-1} \qquad v = \sin x$

$u' = (n-1)x^{n-2} \qquad v' = \cos x$

$$= 0 + n\left[\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx \right.$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

ii) $I_4 = \int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$

$$= 4\left(\frac{\pi}{2}\right)^2 - 4\times 3 \, I_2$$

$$= 4\left(\frac{\pi}{2}\right)^2 - 12\left[2\left(\frac{\pi}{2}\right)^1 - 2\times 1 \, I_0\right]$$

$$= 4\left(\frac{\pi}{2}\right)^2 - 24\left(\frac{\pi}{2}\right) + 24\int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \pi^2 - 12\pi + 24\left(-\cos x\right)_0^{\frac{\pi}{2}}$$

$$= \pi^2 - 12\pi + 24\left(0 - -1\right)$$

$$= \pi^2 - 12\pi + 24$$

e) P represents $6 + 3i$

$\therefore Q$ represents $i(6 + 3i)$

$= -3 + 6i$

13 a) $\tan^{-1}3x - \tan^{-1}2x = \tan^{-1}\frac{1}{5}$

$$\tan(\tan^{-1}3x - \tan^{-1}2x) = \frac{1}{5}$$

$$\frac{3x - 2x}{1 + 6x^2} = \frac{1}{5}$$

$$6x^2 - 5x + 1 = 0$$

$$(3x-1)(2x-1) = 0$$

$$x = \frac{1}{3}, \frac{1}{2}$$

b) $\int_0^5 \sqrt{100 - x^2}\, dx$ $\quad x = 10\sin\theta$

$dx = 10\cos\theta\, d\theta$

$x = 0 \quad \theta = 0$

$x = 5 \quad \theta = \frac{\pi}{6}$

$$= \int_0^{\frac{\pi}{6}} \sqrt{100 - 100\sin^2\theta}\,.\,10\cos\theta\, d\theta$$

$$= 100\int_0^{\frac{\pi}{6}} \sqrt{\cos^2\theta}\,\cos\theta\, d\theta$$

$$= 100\int_0^{\frac{\pi}{6}} \cos^2\theta\, d\theta$$

$$= \frac{100}{2}\int_0^{\frac{\pi}{6}} 1 + \cos 2\theta\, d\theta$$

$$= 50\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{6}}$$

$$= 50\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - (0)\right]$$

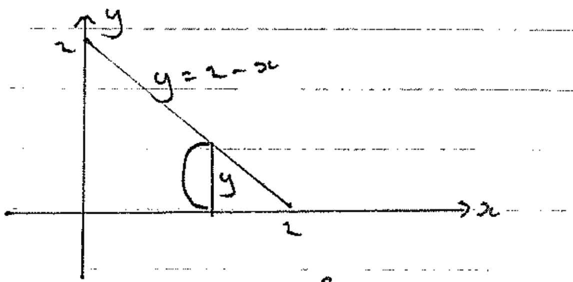
$$= \frac{25\pi}{3} + \frac{25\sqrt{3}}{2}$$

c) $V = \frac{\pi}{8}\int_0^2 y^2\, dx$

$$= \frac{\pi}{8}\int_0^2 (2-x)^2\, dx$$

$$= -\frac{\pi}{8}\cdot\frac{1}{3}\left[(2-x)^3\right]_0^2$$

$$= \frac{\pi}{3} \text{ cu. units}$$



$A(x) = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi\left(\frac{1}{4}y^2\right)$

d) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot y' = 0$$

$$y' = \frac{b^2 x}{a^2 y}$$

$$\therefore \quad m_T = \frac{b^2 a \sec\theta}{a^2 b \tan\theta}$$

$$= \frac{b \sec\theta}{a \tan\theta}$$

$$\therefore \quad y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

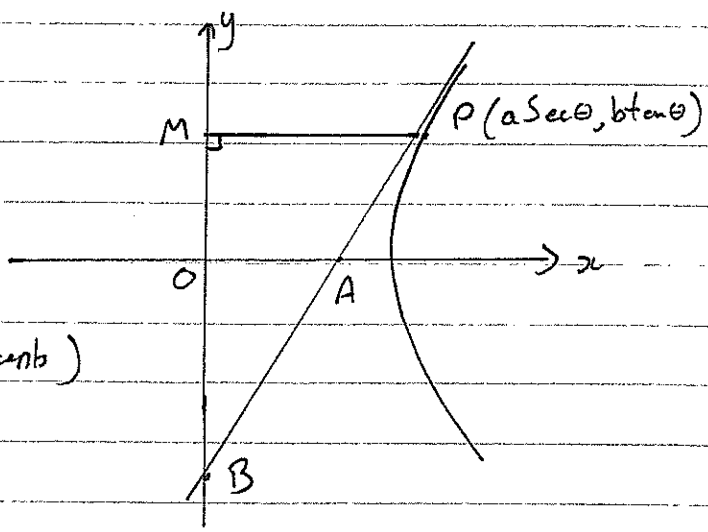
$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$

$$bx\sec\theta - ay\tan\theta = ab$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

ii) $M(0, b\tan\theta)$

$B\left(0, \frac{-b}{\tan\theta}\right)$



$$\frac{PA}{PB} = \frac{MO}{MB}$$ (ratio of intercepts)

$$= \frac{b\tan\theta}{b\tan\theta + \frac{b}{\tan\theta}}$$

$$= \frac{b\tan^2\theta}{b(\tan^2\theta + 1)}$$

$$= \frac{\tan^2\theta}{\sec^2\theta}$$

$$= \sin^2\theta$$

14

a) $f(x) = \dfrac{x^3 + 4}{x^2}$, $x \neq 0$

1) $f'(x) = \dfrac{x^2(3x^2) - (x^3+4)(2x)}{x^4}$

$= \dfrac{3x^4 - 2x^4 - 8x}{x^4}$

$= \dfrac{x^4 - 8x}{x^3}$

$= \dfrac{x^3 - 8}{x^3}$

st. pts when $y' = 0$

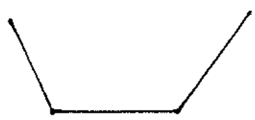
$x^3 - 8 = 0$

$x = 2$

$\therefore y = 3$

test

x	2^-	2	2^+
y'	-ve	0	+ve



$\therefore$ min at $(2, 3)$

ii) $$f''(x) = \frac{x^3(3x^2) - (x^3 - 8)3x^2}{x^6}$$

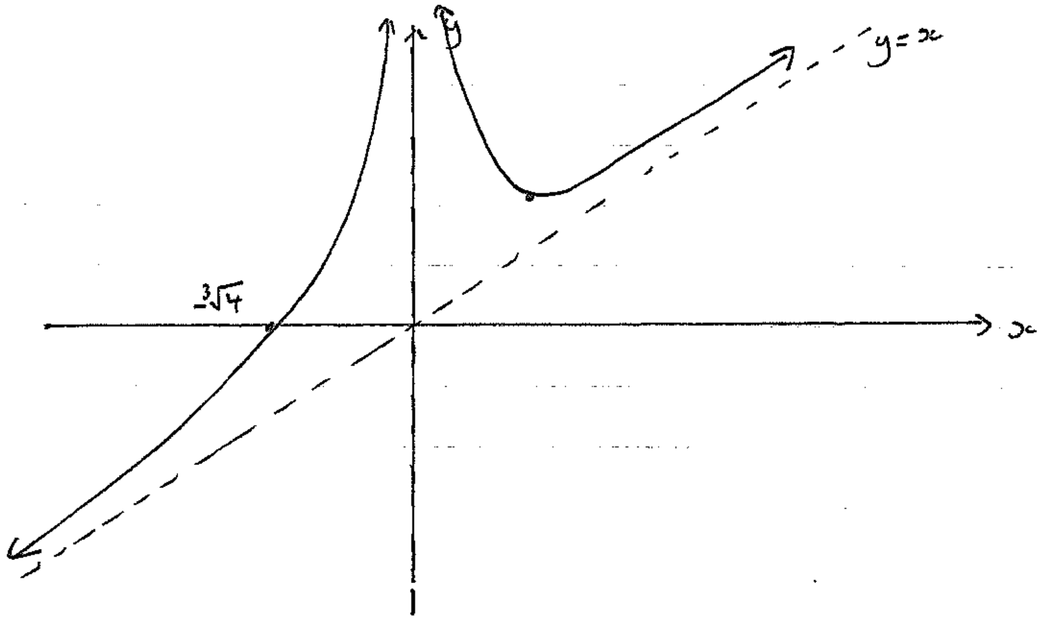
$$= \frac{3x^5 - 3x^5 + 24x^2}{x^6}$$

$$= \frac{24}{x^4}$$

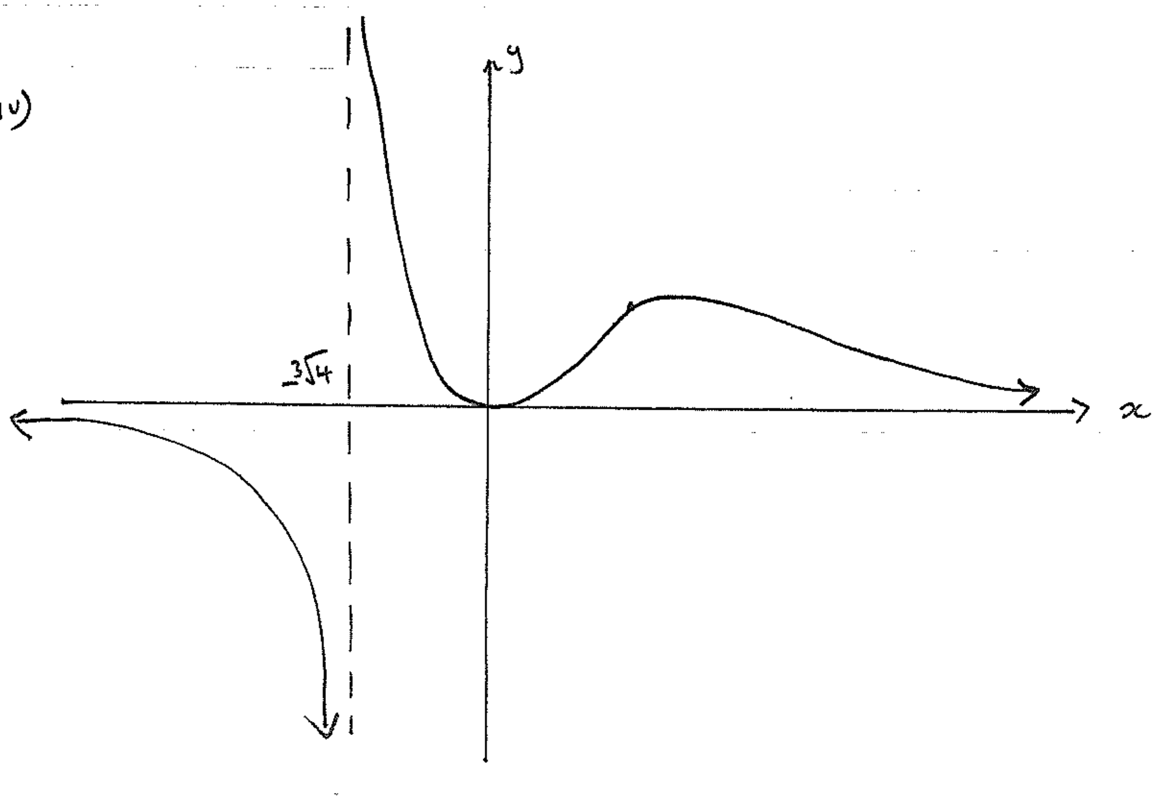
which is positive for all x, except $x = 0$

$\therefore$ concave up for all x, $x \neq 0$.

iii)



iv)



b) i) $\ddot{x} = \dfrac{-k}{x^2}$ when $x = R$ $\ddot{x} = -g$

$\therefore k = R^2 g$

ii) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = -kx^{-2}$

$\frac{1}{2}v^2 = kx^{-1} + c$

when $x = R$ $v = u$

$\therefore \frac{1}{2}u^2 - \frac{k}{R} = c$

$\frac{1}{2}v^2 = \frac{1}{2}u^2 - \frac{k}{R} + \frac{k}{x}$ but $k = R^2 g$

$v^2 = u^2 - \dfrac{2R^2g}{R} + \dfrac{2R^2g}{x}$

$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

iii) If $u = \sqrt{2gR}$

$v^2 = 2gR - 2gR + \dfrac{2gR^2}{x}$

$v = \dfrac{\sqrt{2gR^2}}{\sqrt{x}}$

$\dfrac{dx}{dt} = \dfrac{\sqrt{2gR^2}}{\sqrt{x}}$

$\dfrac{dt}{dx} = \dfrac{x^{\frac{1}{2}}}{\sqrt{2gR^2}}$

$t = \dfrac{2x^{\frac{3}{2}}}{3\sqrt{2gR^2}} + c$

when $x = R$, $t = 0$

$\therefore c = \dfrac{-2R^{\frac{3}{2}}}{3\sqrt{2gR^2}}$

$\therefore t = \dfrac{2x^{\frac{3}{2}}}{3\sqrt{2gR^2}} - \dfrac{2R^{\frac{3}{2}}}{3\sqrt{2gR^2}}$ when $x = 4R$

$t = \dfrac{2(4R)^{\frac{3}{2}}}{3\sqrt{2gR^2}} - \dfrac{2R^{\frac{3}{2}}}{3\sqrt{2gR^2}}$

$$t = \frac{16\sqrt{R}}{3\sqrt{2g}} - \frac{2\sqrt{R}}{3\sqrt{2g}}$$

$$= \frac{14\sqrt{R}}{3\sqrt{2g}}$$

$$= \frac{7\sqrt{2R}}{3\sqrt{g}}$$

15 a) $3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$

$$3x^2 - 6y = \frac{dy}{dx}(6x - 3y^2)$$

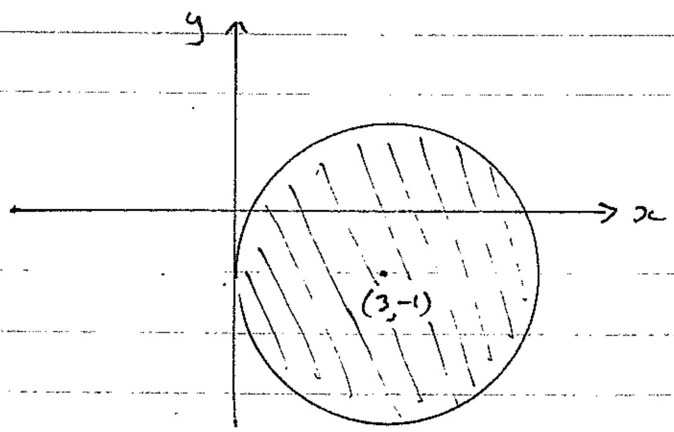
$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}$$

$$= \frac{x^2 - 2y}{2x - y^2}$$

b) $(x + iy - 3 + i)(x - iy - 3 - i) \leq 9$

$$((x-3) + i(y+1))((x-3) - i(y+1)) \leq 9$$

$$(x-3)^2 + (y+1)^2 \leq 9$$



c) $V = \int_0^1 2\pi x \left(\frac{x}{x^2+1}\right) dx$

$= 2\pi \int_0^1 \frac{x^2}{x^2+1} dx$

$= 2\pi \int_0^1 \frac{x^2+1-1}{x^2+1} dx$

$= 2\pi \int_0^1 1 - \frac{1}{x^2+1} dx$

$= 2\pi \left[x - \tan^{-1} x \right]_0^1$

$= 2\pi \left[(1 - \tan^{-1} 1) - (0 - \tan^{-1} 0) \right]$

$= 2\pi \left(1 - \frac{\pi}{4}\right)$ cu units

d) i) $\angle PNC = \angle PLC = 90^\circ$ (given)

$\therefore$ PNCL is cyclic (opposite angles supplementary)

ii) let $\angle PBM = x$

$\therefore \angle PCA = x$ (angles at circumference equal)

$\therefore \angle PNL = x$ (angles at circumference equal, PNCL cyclic)

$\therefore \angle PBM = \angle PNM$

iii) PMNB is cyclic (PM subtends equal angles at B and N)

$\therefore \angle PMB = \angle PNB$ (PB subtends equal angles at M and N)

but $\angle PNB = 90^\circ$ (given)

$\therefore \angle PMB = 90^\circ$

$\therefore PM \perp AB$

16 a) i) substitute w^2 into x^3 gives

$$(w^2)^3 = (w^3)^2$$
$$= 1$$

$\therefore$ w^2 is a solution of $x^3 = 1$ if w is a solution

ii) 3 roots of $x^3 = 1$ are 1, w and w^2

$\therefore$ sum of roots $= -\frac{b}{a}$

$\therefore$ $1 + w + w^2 = 0$

iii)

polynomial in form $(x-3)(x^2 - S_1 x + S_2) = 0$

where $S_1 = 2w + w^2 + 2w^2 + w$
$= 3w + 3w^2$
$= 3(w + w^2)$
$= -3$

$S_2 = (2w + w^2)(2w^2 + w)$
$= 4w^3 + 2w^2 + 2w^4 + w^3$
$= 5w^3 + 2w^2 + 2w^4$ $\quad (w^4 = w^3 \times w)$
$= 5 + 2w^2 + 2w$
$= 3 + 2(1 + w^2 + w)$
$= 3$

$\therefore$ required polynomial is $(x-3)(x^2 + 3x + 3) = 0$

$x^3 + 3x^2 + 3x - 3x^2 - 9x - 9 = 0$

$$x^3 - 6x - 9 = 0$$

b) i) Step 1 : test $n=1$ LHS $= \frac{1}{\sqrt{1}}$ RHS $= 2(\sqrt{2}-1)$

$= 1$ $= 2\sqrt{2}-2$

≈ 0.828

$\therefore$ true for $n=1$

Step 2 : assume true for $n=k$

i.e. $\sum_{r=1}^{k} \frac{1}{\sqrt{r}} > 2(\sqrt{k+1}-1)$

show for $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} = \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}}$$

$$> 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}}$$

$$= 2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}}$$

$$= \frac{2(k+1)}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} - 2$$

$$= \frac{2k+3}{\sqrt{k+1}} - 2$$

$$> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2$$

$$= 2(\sqrt{k+2}-1)$$

which is the required result

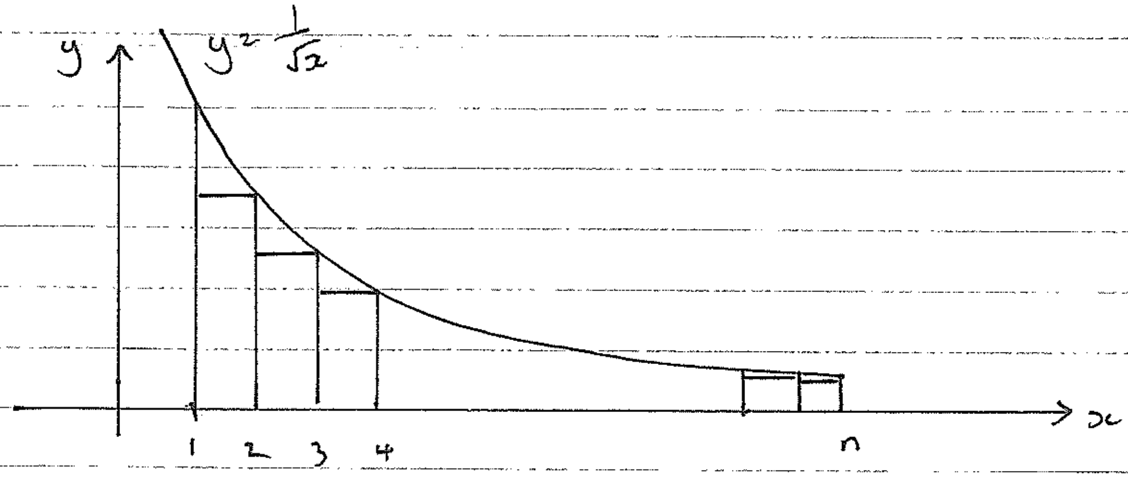
$\therefore$ true for $n=k+1$ if true for $n=k$

Step 3 : As true for $n=1$, result is true for $n=1+1$, ie $n=2$

As true for $n=2$, result is true for $n=2+1$, ie $n=3$

and so on for all positive integers n.

ii)



Area under curve from $x=1$ to n is given by

$$\int_1^n \frac{1}{\sqrt{x}}\,dx$$

Area of rectangles is given by

$$1\times\frac{1}{\sqrt{2}} + 1\times\frac{1}{\sqrt{3}} + 1\times\frac{1}{\sqrt{4}} + \cdots + 1\times\frac{1}{\sqrt{n}}$$

but area of rectangles < area under curve

$$\therefore \quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < \int_1^n \frac{1}{\sqrt{x}}\,dx$$

$$\therefore \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 1 + \int_1^n \frac{1}{\sqrt{x}}\,dx$$

$$\therefore \quad \sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_1^n \frac{1}{\sqrt{x}}\,dx$$

iii) $$2\left(\sqrt{n+1} - 1\right) < \sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_1^n \frac{1}{\sqrt{x}}\,dx$$ from (i) and (ii)

let $n = 10000$

$$2\left(\sqrt{10001} - 1\right) < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 1 + \int_1^{10000} \frac{1}{\sqrt{x}}\,dx$$

$$198.0099 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 1 + \left[2\sqrt{x}\right]_1^{10000}$$

$$198 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 1 + 2\left(\sqrt{10000} - 1\right)$$

$$198 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 199$$