Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 2

HSC Course

Trial HSC

August 2015

Time allowed: 180 minutes+5 minutes reading time

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice
Questions 1-10
10 Marks

Section II Questions 11-16
90 Marks

SECTION I – Multiple Choice

10 Marks
Attempt questions 1-10.
Allow about 15 minutes for this section.
Fill in your choice on the Multiple Choice answer sheet stapled to the front page of your answer booklet.

Question 1

If $\omega = 2 - 3i$ and $z = 3 + 4i$ then $\omega\bar{z} =$

A. $-6 - 17i$

B. $18 - i$

C. $-6 + 17i$

D. $18 + i$

Question 2

If $z = \sqrt{48} - 4i$, the value of $\arg(z^7)$ is

A. $-\frac{2\pi}{3}$

B. $\frac{2\pi}{3}$

C. $-\frac{5\pi}{6}$

D. $\frac{5\pi}{6}$

Question 3

If $1 + 2i$ is a root of $2x^3 + bx^2 + 2x + 20 = 0$ and b is real, then $b =$

A. -3

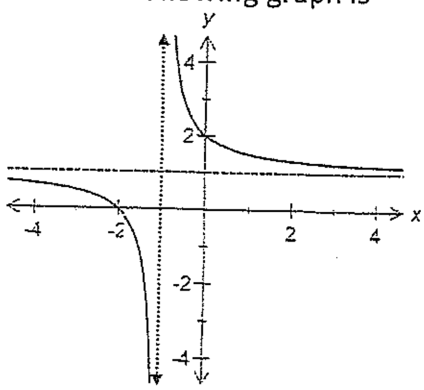
B. -2

C. 0

D. 2

Question 4

The equation of the following graph is



A. $(x-1)(y+1) = 1$

B. $(x+1)(y-1) = 1$

C. $y = \frac{x+2}{x}$

D. $y = \frac{x}{x+1}$

Question 5

The eccentricity of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is equal to

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. $\frac{\sqrt{5}}{3}$

D. $\frac{3}{\sqrt{5}}$

Question 6

With a suitable substitution $\int_0^{\frac{\pi}{6}} cos^3 2x dx =$

A. $\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}}(1-u^2)du$

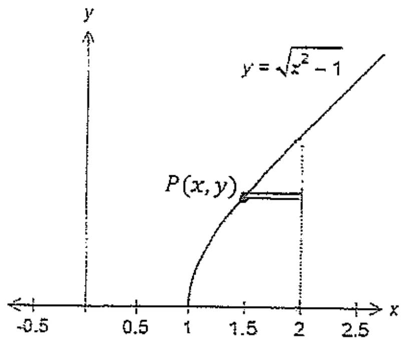
C. $2\int_0^{\frac{\sqrt{3}}{2}}(1-u^2)du$

B. $\frac{1}{2}\int_0^{\frac{1}{2}}(1-u^2)du$

D. $2\int_0^{\frac{1}{2}}(1-u^2)du$

Question 7

The area enclosed by $y=\sqrt{x^2-1}$ and the line $x=2$ and the x axis is rotated about the y axis



The slice at $P(x,y)$ on the curve is perpendicular to the axis of rotation.

The volume δV on the slice of the annulus is

A. $\pi(3-y^2)\delta y$

C. $\pi(4-x^2)\delta x$

B. $\pi(4-(y^2+1))\delta y$

D. $\pi(2-x^2)\delta x$

Question 8

If $e^x+e^y=1$, $\frac{dy}{dx}=$

A. $-e^{x-y}$

C. e^{y-x}

B. e^{x-y}

D. $-e^{y-x}$

Question 9

$\int_0^1 x(1-x)^{99}dx =$

A. $\frac{1}{10010}$

C. $\frac{11}{10010}$

B. $\frac{1}{10100}$

D. $\frac{11}{10100}$

Question 10

If a particle moves in a straight line so that its velocity at any time is given by $v = sin^{-1}x$, then its acceleration will be given by

A. $-cos^{-1}x$

B. $cos^{-1}x$

C. $\frac{-sin^{-1}x}{\sqrt{1-x^2}}$

D. $\frac{sin^{-1}x}{\sqrt{1-x^2}}$

SECTION II

90 Marks
Attempt questions 11-16.
Allow about 2hours 45minutes for this section.
Start each question on a new page.

Question 11

a) Let $z = 5 - 6i$ and $w = 3 + 4i$. Express the following in the form $a + ib$

where a and b are real numbers

i) z^2 1

ii) $\frac{z}{\overline{w}}$ 2

b) i) Express $w = 4 + 4i$ in modulus –argument form 1

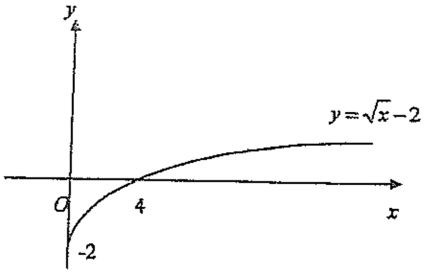
ii) Hence or otherwise find all numbers z such that $z^5 = 4 + 4i$ giving your answers in modulus – argument form. 3

c) Sketch the region in the Argand diagram defined by $|z - 2 + i| < 3$ and $\frac{-\pi}{3} \leq \arg(z - 2 + i) \leq \frac{\pi}{3}$. Indicate whether the points of intersection are included or excluded. You do not need to find the coordinates of points of intersection. 3

d) If $1 + i$ is a zero of $x^3 + ax + b$, find the values of a and b. 2

e) Evaluate $\int_0^1 \frac{5dt}{(2t+1)(2-t)}$ 3

Question 12



a) The diagram shows the graph of the function $f(x) = \sqrt{x} - 2$. On separate diagrams of approximately one third of a page, sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

i) $y = f(|x|)$ 1

ii) $y = [f(x)]^2$ 1

iii) $y = \frac{1}{f(x)}$ 2

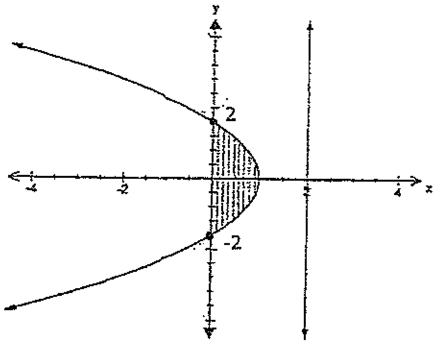
iv) $y = l_n f(x)$ 2

b) If $\propto, \beta, \gamma$ are the roots of the equation $x^3 - 4x^2 + 2x + 5 = 0$, evaluate

i) $\propto^2 + \beta^2 + \gamma^2$ 1

ii) $\propto^3 + \beta^3 + \gamma^3$ 2

c)



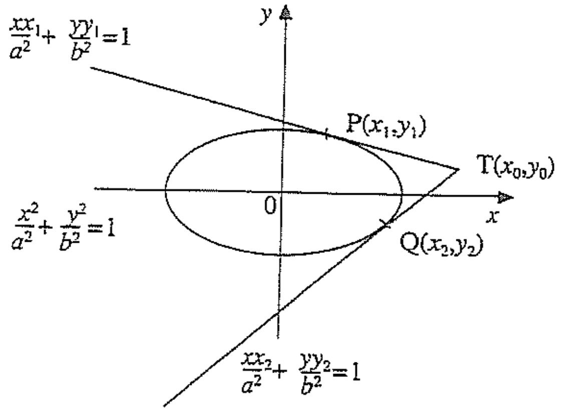
A solid S is formed by rotating the region bounded by the parabola $y^2 = 4(1 - x)$ and the $y - axis$ around the line $x = 2$. By using the method of slicing, find the exact volume of S. 3

d) Find $\int_0^{\frac{\pi}{2}} e^x \cos x dx$ and leave your answer in exact form. 3

Question 13

a)

i) Using the diagram below, show that the equation of the chord of contact PQ of the ellipse is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$ 2



ii) Find the equation of the chord of contact to the ellipse $3x^2 + 4y^2 = 48$ constructed from the external point (6,4) 2

b) Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over the complex field, given that $2 - i$ is a zero. 3

c) Find $\int \frac{1}{x l_n x} dx$ 1

d) Find the volume of the solid of revolution formed when $y = sin^2 \frac{x}{2}$ is rotated about the x axis between $x = 0$ and $x = 2\pi$. 3

e) i) Show that $x = 2$ is a root of multiplicity 3 for $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ 3

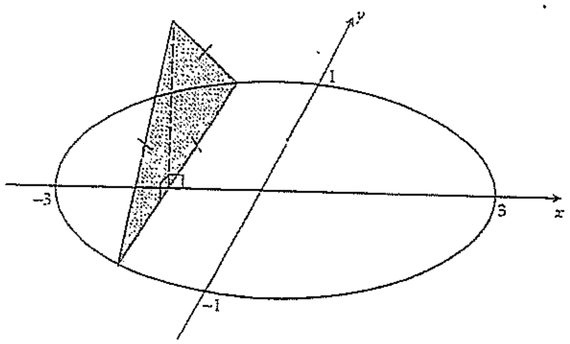
ii) Solve $P(x) = 0$ 1

Question 14

a) i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ 1

ii) Hence find $\int \sin 5x \cos 4x dx$ 2

b)



A solid shape has an elliptical base in the $xy-plane$ as shown below. Sections of the solid taken perpendicular to the $x-axis$ are equilateral triangles. The major and minor axes of the ellipse are of lengths 6 metres and 2 metres respectively.

i) Write down the equation of the ellipse. 1

ii) Show that the volume ΔV of a slice taken at $x=d$ is given by $\Delta V = \frac{\sqrt{3}(9-d^2)}{9}\Delta x$ 2

iii) Find the volume of this solid. 3

c) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} sin^7\theta d\theta = 0$ Is this True or False and give a brief reason for your answer. 1

d) Find $\int cos^3 x dx$ 2

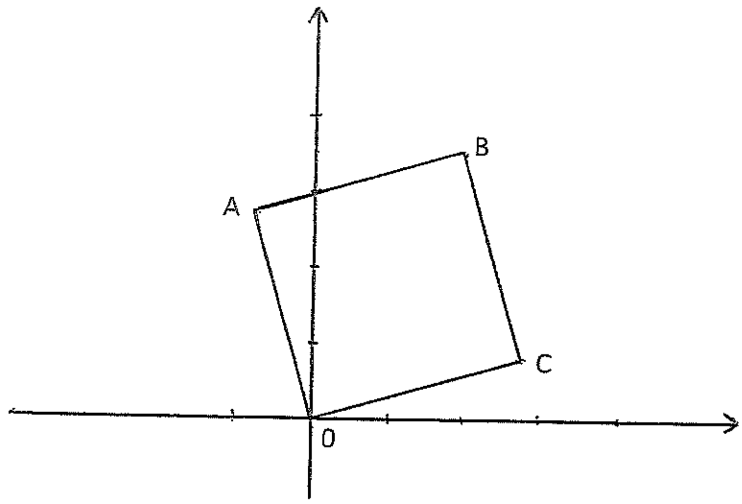
e) Prove by mathematical induction that $2^{n+4} > (n+4)^2$ for all positive integers n. 3

Question 15

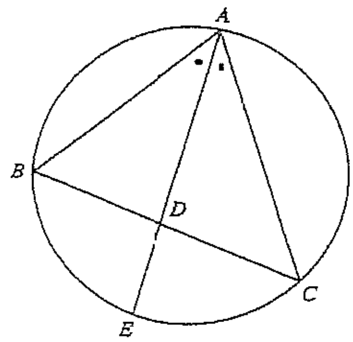
a) $OABC$ is a square on the Argand diagram and is labelled in a clockwise direction. A represents $z = a + ib$ and B represents $4 + 7i$ (see diagram below)

i) Find, in terms of a and b, the complex number represented by C. 2

ii) Hence evaluate a and b. 3



b)



In the diagram, the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E. Copy the diagram into your Writing Booklet.

i) Prove that the triangles ABE and ADC are similar. 2

ii) Show that $AB.AC = AD.AE$ 1

iii) Prove that $AD^2 = AB.AC - BD.DC$ 2

c) i) Given that $\frac{d}{dx}(\cot x) = -cosec^2 x$, show that $\frac{d}{dx}(\cot x)^n = -n cosec^2 x(\cot x)^{n-1}$ 1

ii) If $I_n = \int \cot^n x dx$ for $n \geq 0$ show that $I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$ for $n \geq 2$ 2

iii) Hence evaluate I_5 2

Question 16

a) Find $\int \frac{dx}{\sqrt{4x^2+36}}$ 2

b) The line through O the origin perpendicular to the tangent at $P(cp, \frac{c}{p})$ on the rectangular hyperbola $xy = c^2$ meets the tangent at N. You may assume this tangent has equation $x + p^2y = 2cp$.

i) Show the coordinates of N are $(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4})$ 2

ii) Hence or otherwise find the locus of N as p varies. 2

c) The depth of water in a harbour is 7.2m at low tide and 13.6m at high tide. On Monday, low tide is at 2.05pm and high tide at 8.20pm. The captain of a ship requiring 12m depth of water wants to leave harbour as early as possible on Monday afternoon. Assuming the level of the tides follow simple harmonic motion, and the tide level is represented by the equation $x = -A\cos nt + B$, find

i) Appropriate values of A, B and n and sketch your curve. 2

ii) The earliest time the ship can leave the harbour on Monday afternoon. 2

d) For the curve $x^3 + 3x^2y - 2y^3 = 16$

i) Show that $\frac{dy}{dx} = \frac{x^2+2xy}{2y^2-x^2}$ 1

ii) Find the coordinates of the stationary points on the curve. 2

e) Let P, Q and R represent the complex numbers W_1, W_2, and W_3 respectively. What geometric properties characterise ΔPQR if $W_2 - W_1 = i(W_3 - W_1)$? Give reasons for your answer by sketch or otherwise. 2

END

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Solutions to STHS 2015 Ext. 2 Trial

1. A 2. D 3. C 4. B 5. C

6. A 7. B 8. A 9. B 10. D

11. a. (i) $(5-6i)^2$

$= 25 - 36 - 60i$

$\underline{z^2 = -11 - 60i}$ ①

(ii) $\frac{z}{w} = \frac{5-6i}{3-4i} \times \frac{3+4i}{3+4i}$ ①

$= \frac{15 + 20i - 18i + 24}{25}$

$= \frac{39 + 2i}{25}$ ①

b. (i) $4 + 4i$

$= \sqrt{32}\, \text{cis}\, \frac{\pi}{4}$

$= \underline{4\sqrt{2}\, \text{cis}\, \frac{\pi}{4}}$

(ii) $z^5 = 4\sqrt{2}\, \text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$

$= 4\sqrt{2}\, \text{cis}\left(\frac{\pi}{4} + \frac{8k\pi}{4}\right)$

$z^5 = 4\sqrt{2}\, \text{cis}\left(\frac{\pi + 8k\pi}{4}\right)$ ①

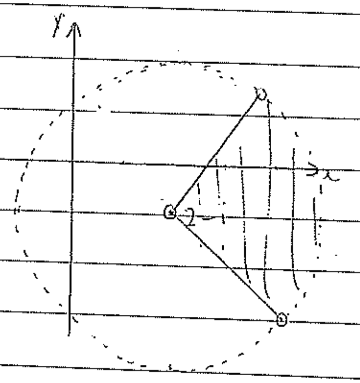
$z = \sqrt{2}\, \text{cis}\left(\frac{\pi + 8k\pi}{20}\right)$ $k = 0, \pm 1, \pm 2$

$z = \sqrt{2}\, \text{cis}\, \frac{\pi}{20},\ \sqrt{2}\, \text{cis}\, \frac{9\pi}{20}$

$\sqrt{2}\, \text{cis}\, \frac{-7\pi}{20},\ \sqrt{2}\, \text{cis}\, \frac{17\pi}{20},$

$\underline{\sqrt{2}\, \text{cis}\, \frac{-15\pi}{20}}$. ②

c)



③

d) If $1+i$ is a zero

$(1+i)^3 + a(1+i) + b = 0$

$1 + 3i + 3i^2 + i^3 + a + ai + b = 0$

$(-2 + a + b) + i(2 + a) = 0$

∴ Equating real/imaginary parts

$-2 + a + b = 0$ and $2 + a = 0$

$a = -2$ and $-2 - 2 + b = 0$

$\underline{a = -2}$ ① and $\underline{b = 4}$ ①

(Can't assume another zero is $1-i$)

e) $\int_0^1 \frac{5\,dt}{(2t+1)(2-t)}$

$\frac{5}{(2t+1)(2-t)} = \frac{a}{2t+1} + \frac{b}{2-t}$

$5 = a(2-t) + b(2t+1)$

$5 = 2a - at + 2bt + b$

$5 = (2b-a)t + (2a+b)$

$2b - a = 0$ and $2a + b = 5$

$2b = a$ ∴ $4b + b = 5$

$b = 1,\ a = 2$ ①

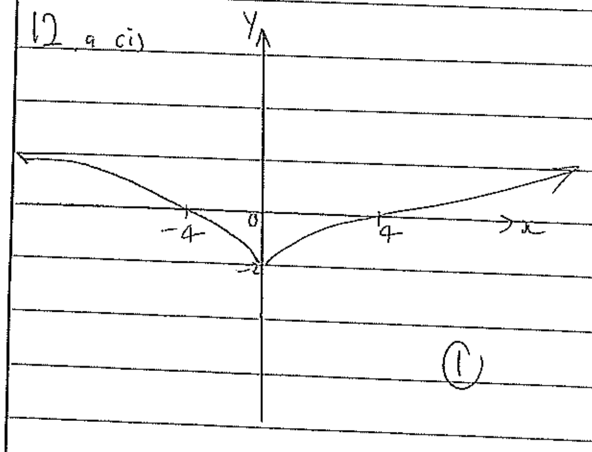
$\int_0^1 \frac{2}{2t+1} + \frac{1}{2-t}\, dt$

$\left[\log_e(2t+1) - \log_e(2-t)\right]_0^1$ ①

$\left[\log_e \frac{2t+1}{2-t}\right]_0^1 = \log_e 3 - \log_e \frac{1}{2}$

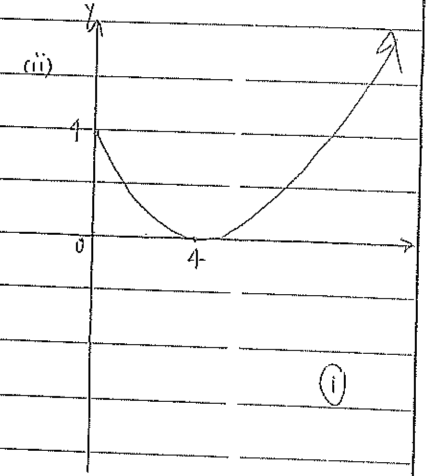
$= \underline{\log_e 6}$ ①

12. a (i)



①

(ii)



①

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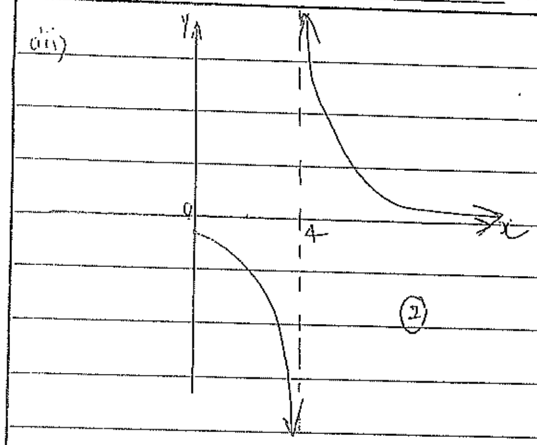
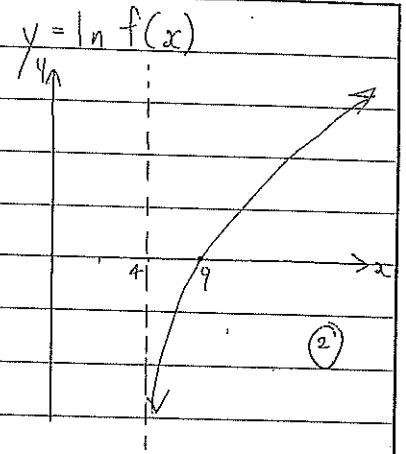
(iii)



(2)

(iv) $y = \ln f(x)$

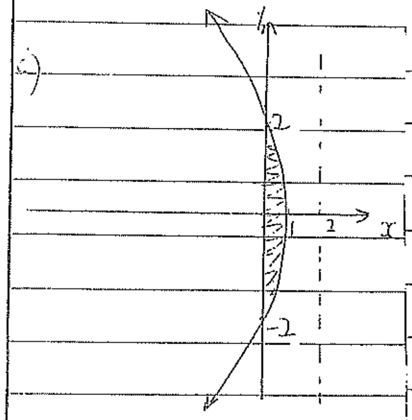


(2)

b)(i) $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$= 16 - 2 \times 2$
$= 12$ (1)

(ii) $\alpha^3 - 4\alpha^2 + 2\alpha + 5 = 0$
$\beta^3 - 4\beta^2 + 2\beta + 5 = 0$
$\gamma^3 - 4\gamma^3 + 2\gamma + 5 = 0$
$\alpha^3 + \beta^3 + \gamma^1 = 4(\alpha^2+\beta^2+\gamma^2)$ (1)
$- 2(\alpha+\beta+\gamma) - 15$
$= 4 \times 12 - 2 \times 4 - 15$
$= 25$ (1)

c)



$\delta V = [\pi(2)^2 - \pi(2-x)^2]\delta y$
$= [4\pi - \pi(4 - 4x + x^2)]\delta y$
$= [4\pi - 4\pi + 4\pi x - \pi x^2]\delta y$
$= \pi[4x - x^2]\delta y$ (1)

Since $y^2 = 4(1-x)$
$y = 2(1-x)^{\frac{1}{2}}$
$\frac{y}{2} = (1-x)^{\frac{1}{2}}$
$\frac{y^2}{4} = 1 - x$
$x = 1 - \frac{y^2}{4}$ (1)

$\therefore V = 2\pi\int_0^2 4(1-\frac{y^2}{4}) - (1-\frac{y^2}{4})^2\, dy$

Student Name: ______ Teacher Name: ______

$V = 2\pi\int_0^2 4 - y^2 - 1 + \frac{y^2}{2} - \frac{y^4}{16}\, dy$
$= 2\pi\int_0^2 3 - \frac{y^2}{2} - \frac{y^4}{16}\, dy$
$= 2\pi\left[3y - \frac{y^3}{6} - \frac{y^5}{80}\right]_0^2$
$= 2\pi\left[6 - \frac{8}{6} - \frac{32}{80}\right]$
$= 2\pi\left[\frac{64}{15}\right]$
$= \frac{128\pi}{15}$ (1)

d) $\int_0^{\frac{\pi}{2}} e^x \cos x\, dx$
$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \times -\sin x\, dx$ (1)
$= e^{\frac{\pi}{2}} \times 0 - 1 + \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\, dx$
$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x\, dx = -1 + e^{\frac{\pi}{2}} - 0$ (1)
$\Rightarrow \int_0^{\frac{\pi}{2}} e^x \cos x\, dx = \frac{1}{2}(e^{\frac{\pi}{2}} - 1)$ (1)

13d)(i) T lies on tangents at P and Q
$\therefore$ satisfies both tangent equations.
ie: $\frac{x_0 x_1}{a^2} + \frac{y_0 y_1}{b^2} = 1$ and
$\frac{x_0 x_2}{a^2} + \frac{y_0 y_2}{b^2} = 1$ (1)

It follows that since both $P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation of the form
$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$
then this must be the equation of the chord of contact joining P and Q. (1)

(ii) $3x^2 + 4y^2 = 48 \equiv \frac{x^2}{16} + \frac{y^2}{12} = 1$
ie: $a^2 = 16$ (1) $b^2 = 12$ and using
$(6, 4) \equiv (x_0, y_0)$, chord of contact is
$\frac{6x}{16} + \frac{4y}{12} = 1$ or $\frac{3x}{8} + \frac{y}{3} = 1$
(1)

b) If $2-i$ is a zero, so is $2+i$ (real coefficients)

$\therefore$ quadratic factor is $x^2-(2-i+2+i)x+(2-i)(2+i)$

x^2-4x+5 (1)

By inspection

$x^4-5x^3+7x^2+3x-10=(x^2-4x+5)(x^2-x-2)$

$=(x^2-4x+5)(x+1)(x-2)$ (1)

$\therefore$ Zeros are $x=2\pm i, -1, 2$ (1)

c) $\int \frac{1}{x\ln x}\,dx = \log_e(\log_e x)+c$ (1)

d) $V=\pi\int y^2\,dx$

$=\pi\int_0^{2\pi}\sin^4\frac{x}{2}\,dx$

$=\pi\int_0^{2\pi}\left[\sin^2\frac{x}{2}\right]^2dx$

$=\pi\int_0^{2\pi}\left[\frac{(1-\cos x)}{2}\right]^2dx$ (1)

$=\frac{\pi}{4}\int_0^{2\pi}1-2\cos x+\cos^2 x\,dx$

$=\frac{\pi}{4}\int_0^{2\pi}1-2\cos x+\frac{\cos 2x+1}{2}\,dx$

$=\frac{\pi}{4}\left[\frac{3x}{2}-2\sin x+\frac{\sin 2x}{4}\right]_0^{2\pi}$ (1)

$=\frac{\pi}{4}\left[3\pi\right]$

$=\frac{3\pi^2}{4}$ units^3 (1)

Student Name:

Teacher Name:

e)(i) $P(x)=x^4-3x^3-6x^2+28x-24$

$P'(x)=4x^3-9x^2-12x+28$

$P''(x)=12x^2-18x-12=0$

$2x^2-3x-2=0$

$(2x+1)(x-2)=0$

$x=\frac{-1}{2}$ or 2 (1)

could be roots of multiplicity 3.

$P(2)=2^4-3\times2^3-6\times2^2+28\times2-24$

$=16-24-24+56-24$

$=0$ (1)

$\therefore$ $x=2$ is root of multiplicity 3 (1)

(ii) $P(x)=(x-2)^3(x+3)$ by inspection

$\therefore$ $x=2$ or -3 (1)

Student Name: ______________ Teacher Name: ______________

14 a) (i) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

LHS $= \sin A\cos B + \sin B\cos A + \sin A\cos B - \sin B\cos A$

$= 2\sin A\cos B$

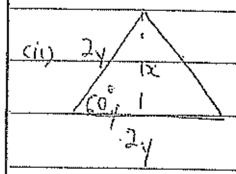
$= RHS$ (1)

(ii) $\int \sin 5x\cos 4x\,dx$

$= \frac{1}{2}\int \sin 9x + \sin x\,dx$ (1)

$= -\frac{1}{18}\cos 9x - \frac{1}{2}\cos x + C$ (1)

b) (i) $\frac{x^2}{9} + \frac{y^2}{1} = 1$ (1)

(ii) 

$\sin 60 = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2} = \frac{x}{2y}$

$\therefore x = \frac{y\sqrt{3}}{1}$

$\delta V = \frac{1}{2}\times 2y\times y\sqrt{3}\times\delta x$

$\delta V = y^2\sqrt{3}\times\delta x$ (1)

Using $\frac{x^2}{9} + \frac{y^2}{1} = 1$

When $x = d$

$y^2 = 1 - \frac{d^2}{9}$

$\therefore \delta V = \left(1 - \frac{d^2}{9}\right)\sqrt{3}\,\delta x$

$\delta V = \frac{\sqrt{3}(9-d^2)}{9}\,\delta x$ (1)

(iii) $V = 2\int_0^3 \frac{\sqrt{3}(9-x^2)}{9}\,dx$ (1) by symmetry across the y axis

$= \frac{2\sqrt{3}}{9}\left[9x - \frac{x^3}{3}\right]_0^3$ (1)

$= \frac{2\sqrt{3}}{9}[27-9]$

$= 4\sqrt{3}$ units^3 (1)

Student Name: ______________ Teacher Name: ______________

c) (i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta\,d\theta = 0$ is true as since

$\sin\theta$ is odd, $\sin^7\theta$ will be odd (1)

so $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{odd} = 0$, property of odd functions.

d) $\int \cos^3 x\,dx$

$\int \cos^2 x\times\cos x\,dx$

$\int (1-\sin^2 x)\cos x\,dx$ (1)

$\int \cos x - \cos x\sin^2 x\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + C$ (1)

e) Show true for $n=1$

$2^5 > 5^2$

$32 > 25$ $\therefore$ true for $n=1$

Assume true for $n=k$

ie: $2^{k+4} > (k+4)^2$ (1)

Need to show true for $n=k+1$

$2^{k+1+4} = 2\times 2^{k+4}$

$> 2\times(k+4)^2$ from assumption

$= 2(k^2+8k+16)$

$= 2k^2+16k+32$

$= k^2+10k+25+k^2+6k+7$

$= (k+5)^2+k^2+6k+7$ (1)

so $2^{(k+1)+4} > (k+5)^2+k^2+6k+7$

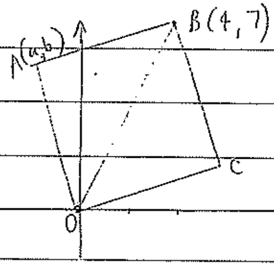
Since k is a positive integer then $k^2+6k+7 >$

$\therefore 2^{k+5} > (k+5)^2$ (1)

Since result is true for $n=1$, by induction it is true for all positive integers

Student Name: ____________ Teacher Name: ____________

15 (i)



$\vec{OA}$ is rotated clockwise by 90° to give $\vec{OC}$

$\therefore$ multiply $\vec{OA}$ by $-i$ ①

$(a+ib) \times -i$

$= b - ai$ ①

(ii) Now $\vec{OA} + \vec{OC} = \vec{OB}$

$a+ib + b - ia = 4 + 7i$

$(a+b) + i(b-a) = 4 + 7i$ ①

$a+b = 4$ ①, $b - a = 7$ ②

$\therefore$ ① + ② $= 2b = 11$

$\therefore b = 5\frac{1}{2}, \quad a = -1\frac{1}{2}$ ①

b)(i) Join BE

In Δ's ABE and ADC,

$\angle BAE = \angle DAC$ (given) ①

$\angle BEA = \angle ACD$ (angles in the same segment)

$\therefore \Delta ABE \ ||| \ \Delta ADC$ (equiangular) ①

(ii) $\frac{AB}{AD} = \frac{AE}{AC}$ corresponding sides of similar triangles in the same ratio

$\therefore AB.AC = AD.AE$ as required ①

(iii) $AD.DE = BD.DC$ (product of intercepts of intersecting chords) ①

Using $AB.AC - AD.AE$

Student Name: ____________ Teacher Name: ____________

To prove that

$AD^2 = AB.AC - BD.DC$

$= AD.AE - BD.DC$ from part (ii)

$= AD.AE - AD.DE$ from above

$= AD(AD + DE) - AD.DE$

$= AD^2$ ①

c)(i) $\frac{d}{dx}(\cot x)^n = n(\cot x)^{n-1} \times -\operatorname{cosec}^2 x$ by function of a function rule. ①

(ii) $I_n = \int \cot^n x \, dx$

$= \int \cot^2 x \cdot \cot^{n-2} x \, dx$

$= \int (\operatorname{cosec}^2 x - 1)\cot^{n-2} x \, dx$ ①

$= \int \operatorname{cosec}^2 x \cdot \cot^{n-2} x \, dx - I_{n-2}$

$I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$ as req'd. ①

(iii) $I_5 = \frac{-\cot^4 x}{4} - I_3$

$= \frac{-\cot^4 x}{4} - \left[\frac{-\cot^2 x}{2} - I_1\right]$ ①

$= \frac{-\cot^4 x}{4} + \frac{\cot^2 x}{2} + \int \cot x \, dx$

$I_5 = \frac{-\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log_e|\sin x| + c$ ①

Student Name: ______ Teacher Name: ______

16 a) $\int \frac{dx}{\sqrt{4x^2+36}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2+9}}$ (1)

$= \frac{1}{2}\log_e\left(x+\sqrt{x^2+9}\right)+c$ (1)

b) (i) Tangent has equation $x + p^2 y = 2cp$.

i.e: $y = \frac{-1}{p^2}x + \frac{2c}{p}$ gradient is $\frac{-1}{p^2}$.

perpendicular gradient is p^2.

$\Rightarrow$ Equation of line through $(0,0)$ is

$y = p^2 x$
$x + p^2 y = 2cp$ } solve simultaneously (1)

$x + p^2 \times p^2 x = 2cp$

$x(1+p^4) = 2cp$

$x = \frac{2cp}{1+p^4} \Rightarrow y = p^2 \times \frac{2cp}{1+p^4}$ (1)

N is $\left(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right)$ as required.

(ii) $x = \frac{2cp}{1+p^4}$ $y = \frac{2cp^3}{1+p^4}$ for the locus of N. Square both sides of the tangent at P:

$x^2 + 2xyp^2 + p^4y^2 = 4c^2p^2$. Now solve simultaneously with $y = p^2x$ by substituting $p^2 = \frac{y}{x}$ (1)

$\Rightarrow x^2 + 2xy \times \frac{y}{x} + \frac{y^2}{x^2}y^2 = 4c^2\frac{y}{x}$

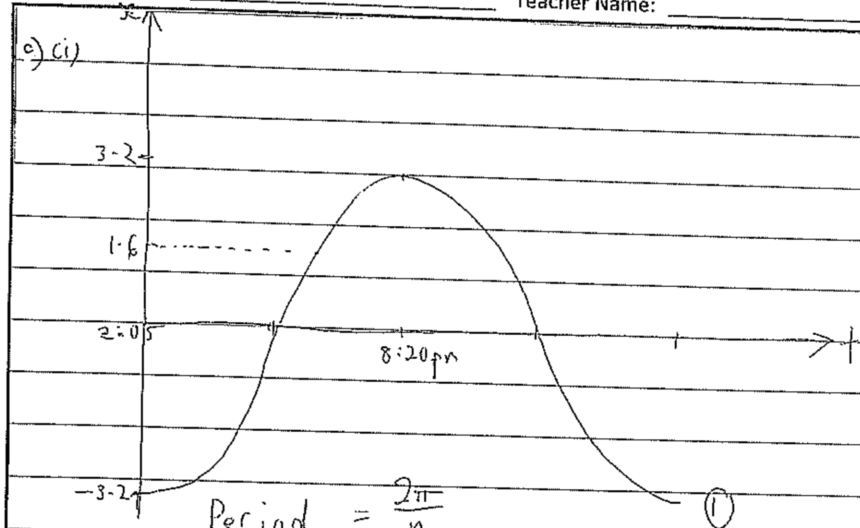
$\times x^2$ $\times x^2$

$x^4 + 2x^2y^2 + y^4 = 4c^2xy$

$(x^2+y^2)^2 = 4c^2xy$ must be the locus of N. (1)

Student Name: ______ Teacher Name: ______

c) (i)



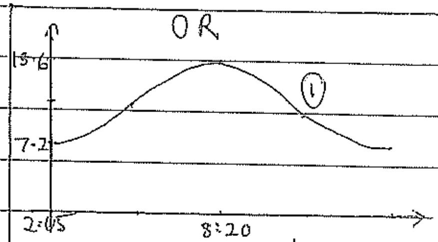
Period $= \frac{2\pi}{n}$

$750 = \frac{2\pi}{n}$

$n = \frac{2\pi}{750}$

$n = \frac{\pi}{375}$ (1)

OR



$x = -A\cos nt$

$x = -3{\cdot}2\cos\frac{\pi t}{375}$ (1)

$x = -3{\cdot}2\cos\frac{\pi t}{375} + 10{\cdot}4$ (1)

(ii) Need to solve for $x = 1{\cdot}6$ (or $x = 12$)

$1{\cdot}6 = -3{\cdot}2\cos\frac{\pi t}{375}$ (1)

$-\frac{1}{2} = \cos\frac{\pi t}{375}$

$\frac{\pi t}{375} = \frac{2\pi}{3}$ (first)

$t = \frac{2}{3} \times 375 = 250$ minutes

earliest departure $= 2{:}05 + 4$ hrs $10 = 6{:}15$ pm (1)

d) (i) $x^3 + 3x^2y - 2y^3 = 16$ differentiating implicitly,

$3x^2 + 6xy + 3x^2\frac{dy}{dx} - 6y^2\frac{dy}{dx}$

$-6y^2\frac{dy}{dx} + 3x^2\frac{dy}{dx} = -3x^2 - 6xy$

$\frac{dy}{dx}(3x^2 - 6y^2) = -3x^2 - 6xy$

$$\frac{dy}{dx} = \frac{-3x^2 - 6xy}{3x^2 - 6y^2}$$

$$= \frac{3(x^2 + 2xy)}{3(2y^2 - x^2)}$$

$$\frac{dy}{dx} = \frac{x^2 + 2xy}{2y^2 - x^2}$$ as required ①

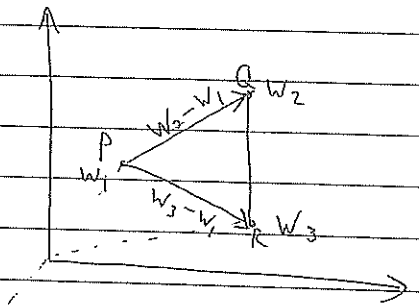
(ii) Stationary points where $\frac{dy}{dx} = 0$

ie. $x^2 + 2xy = 0$

$x(x + 2y) = 0$

$x = 0$ or $x = -2y$ $\therefore (0, -2)$ ① and $(-4, 2)$ ①

e)



Multiplying by i rotates vector $W_3 - W_1$ anticlockwise by $90°$. ①

Also if $W_2 - W_1 = i(W_3 - W_1)$

$|W_2 - W_1| = |i(W_3 - W_1)|$

$= |i| \times |W_3 - W_1|$

$= |W_3 - W_1|$ ①

$\therefore \triangle PQR$ is a right angle isosceles $\triangle$.