Name: ....

Sydney Technical High School

Maths Class: ....

Year 12

Mathematics Extension 2

HSC Course

Assessment 4

TRIAL PAPER

August, 2019

Time allowed: *Working time : 3 hours*

Reading time : 5 minutes

Total Marks: *100*

General Instructions:

- Marks for each question are indicated on the question.
- NESA Approved calculators may be used
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided. Refer to it as required

Section I Multiple Choice
Questions 1–10
10 marks

Allow about 15 minutes for this section

Answer on the multiple choice answer sheet supplied in your answer booklet

Section II 90 marks

Attempt Questions 11–16

Allow about 2 hour and 45 minutes for this section

Section 1

10 Marks

Attempt Questions 1 – 10

Allow 15 minutes for this section.

Use the multiple choice answer sheet in your Answer Booklet for Questions 1 – 10.

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1. The value of a, where a is a real number in

$$\cot 2x + \frac{1}{2}\tan x = a\cot x$$ is;

A 2

B -2

C $\frac{1}{2}$

D $-\frac{1}{2}$

2. The equations of the asymptotes of the function $f(x) = \dfrac{x+1}{x^2-4}$ are;

A $x=2, x=-2, x=-1, y=0$

B $x=2, x=-2, y=0$

C $x=2, x=-2$

D $x=2, x=-2, x=-1, y=x$

3. The polynomial $P(x) = x^3 - 2x^2 - 11x + 52$ has a complex root at $x = 3-2i$.

Which expression shows $P(x)$ expressed as a product of real factors?

A $P(x) = (x-4)(x+1)(x+13)$

B $P(x) = (x-4)(x^2-6x+13)$

C $P(x) = (x+4)(x^2+6x+13)$

D $P(x) = (x+4)(x^2-6x+13)$

4. Given the complex numbers $z=1+2i$ and $w=2-i$ the value of $\frac{\bar{z}}{1+w}$ is:

A $\frac{1}{10}(1+7i)$

B $\frac{1}{2}(1-i)$

C $\frac{5}{8}(1+i)$

D $\frac{1}{8}(1-7i)$

5. $\int(x^3 \ln x)dx$ is equivalent to;

A $\frac{1}{4}\left(x^4 \ln x - x^4\right)+C$

B $\frac{1}{4}\left(x^4 \ln x - \frac{1}{4}x^4\right)+C$

C $\frac{1}{4}x^4 \ln x + \frac{x^4}{16}+C$

D $\frac{1}{4}x^4 \ln x + \frac{x^4}{4}+C$

6. A hyperbola has equation $x^2 - 4y^2 = 4$

The distance between its two directrices is:

A $\frac{4\sqrt{5}}{5}$

B $\sqrt{5}$

C $2\sqrt{5}$

D $\frac{8\sqrt{5}}{5}$

7. If $Arg(-1+ai)=-\frac{2\pi}{3}$, the real number *a* is:

A $-\sqrt{3}$

B $-\frac{1}{\sqrt{3}}$

C $\frac{1}{\sqrt{3}}$

D $\sqrt{3}$

8. The locus of a point , P , representing the complex number z on the Argand diagram, described by,

$|z-i|=\text{Im}(z)$ can be expressed as;

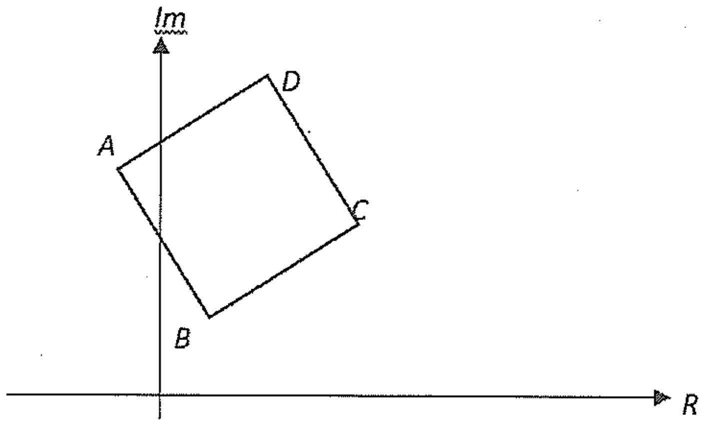
A $x^2=2y-1$

B $x^2+(y-1)^2=y$

C $x^2+y^2=3y-1$

D $x^2+(y-1)^2=1$

9. In the Argand diagram, ABCD is a square and the vertices ***A*** and ***B*** correspond to the complex numbers z_1 and z_2 respectively. Which complex number refers to the diagonal ***BD***?



A $(z_1-z_2)(1-i)$

B $(z_1-z_2)(1+i)$

C $(z_2-z_1)(1+i)$

D $(z_1+z_2)(1-i)$

10. A particle of mass m is moving in a straight line under the action of a force, F, where,

$$F = \frac{m}{x^3}(6 - 10x)$$

Which of the following is an expression for its velocity at any position, if the particle starts from rest at $x = 1$?

A $V = \pm \frac{1}{x}\sqrt{-3 + 10x - 7x^2}$

B $V = \pm x\sqrt{-3 + 10x - 7x^2}$

C $V = \pm \frac{\sqrt{2}}{x}\sqrt{-3 + 10x - 7x^2}$

D $V = \pm x\sqrt{2(-3 + 10x - 7x^2)}$

End of Section 1

Section II

90 Marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Answer each question in the answer booklet – **Starting EACH new question at the TOP of a NEW page.**

In **Questions 11– 16**, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. Given $z = 2 - 3i$ and $w = 4 + i$,

express $z^2 w^{-1}$ in the form of $x + iy$, where x and y are real. 2

b. Express $\dfrac{\left(1-\sqrt{3}i\right)^6}{\left(1+i\right)^4}$ in the form of $x + iy$, where x and y are real numbers 3

c. Find,

i. $\displaystyle\int \frac{4x-12}{x^2-6x+13}\,dx$ 2

ii. $\displaystyle\int \frac{1}{x^2-6x+13}\,dx$ 2

d. Evaluate $\displaystyle\int_0^{\pi} e^{2x} \sin x\,dx$ 3

e. Using a suitable substitution, show that $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx = \sec x + \cos x + C$ 3

Question 12 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. Find $\sqrt{i}$ in the form of $a+ib$, where a and b are real numbers. 2

b.

i. Find all real and complex roots, in modulus argument form of , $z^3 = 1$ 1

ii. Show these roots on the Argand diagram, clearly showing their relative positions. 2

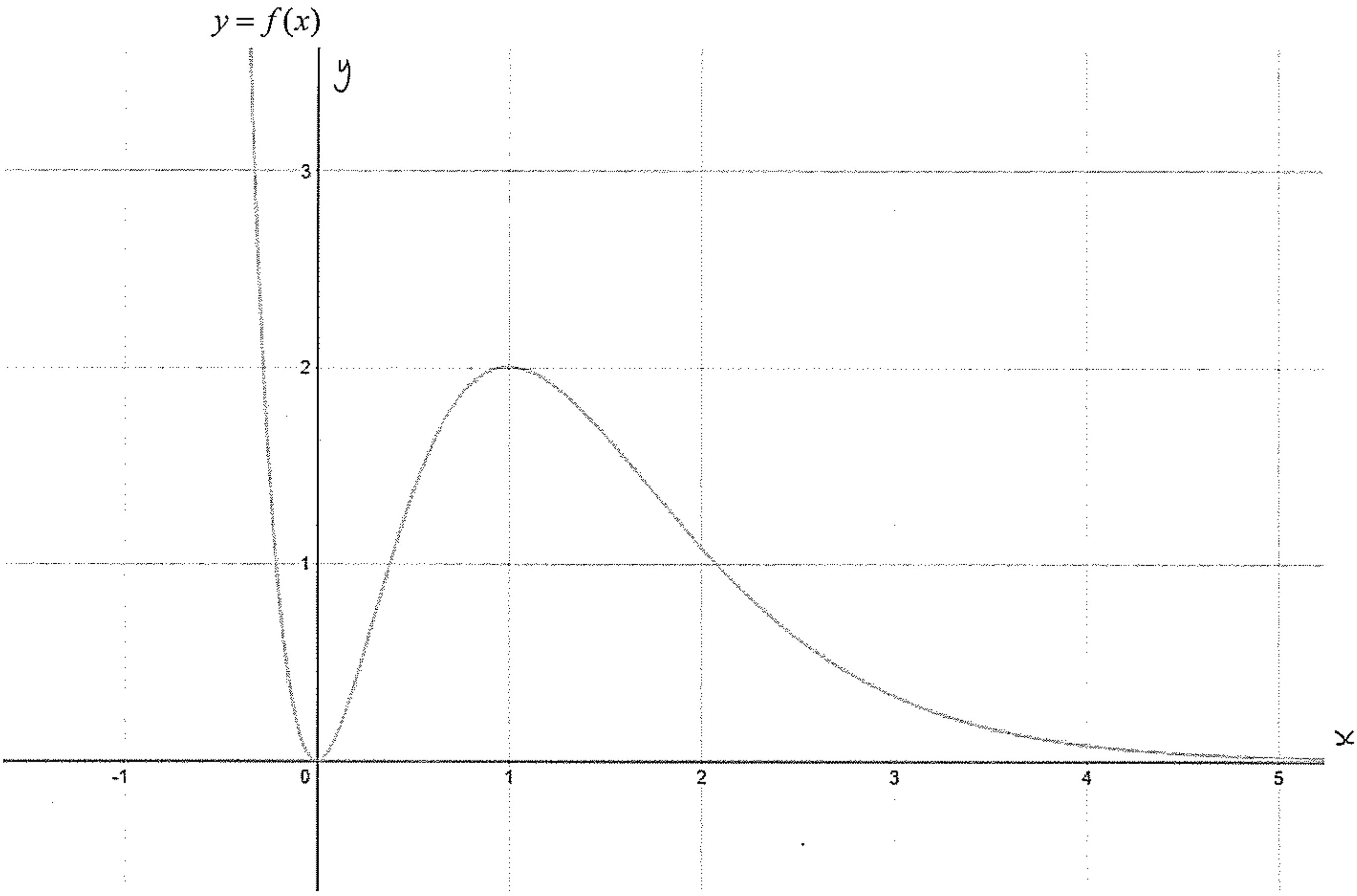
iii. If w is a complex cube root of unity, evaluate, $\left(1-8w+w^2\right)\left(1+w-8w^2\right)$ 2

c. On the Argand diagram, sketch the region held simultaneously by;

$|z-1-3i| \le 2$ and $\frac{\pi}{4} < Arg(z) \le \frac{\pi}{2}$ 3

Question 12 continues on the next page ....

d. Shown below is the sketch of $y = f(x)$



On separate $\frac{1}{3}$ page diagrams draw, showing the location of any turning points and asymptotes,

NEAT sketches of;

i. $y = f|x|$ 1

ii. $y^2 = f(x)$ 2

iii. $y = \ln f(x)$ 2

Question 13 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. Find the equation of the normal to the curve, $2x^2+xy=y^2$, at the point (2, 4). 3

b. i. Find the real numbers A, B and C such that

$$\frac{5-3x}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$ 2

ii. Hence, find $\int \frac{5-3x}{(x-1)(x+1)^2}dx$ 2

c. Find the integers m and n such that,

$(x+1)^2$ is a factor of the polynomial, $P(x)=x^5+2x^2+mx+n$ 3

d. The position at time t of a particle moving in a straight line is given by $x=5-4\cos^2 t$

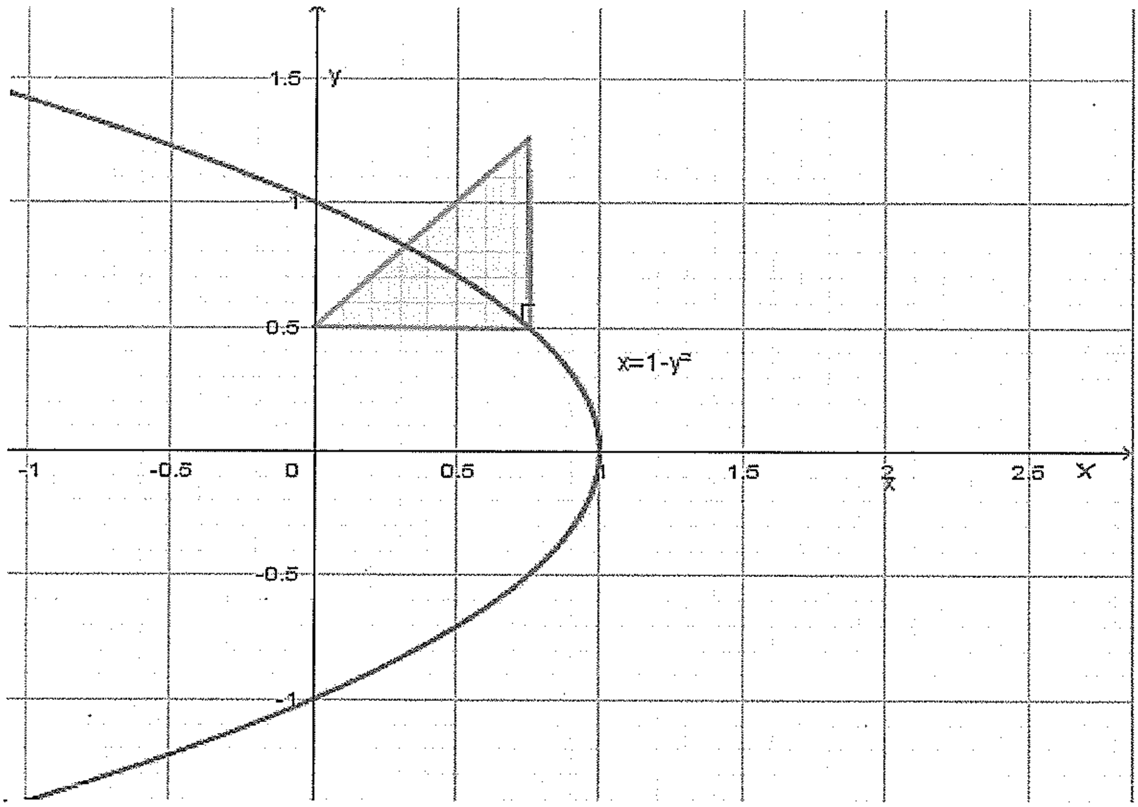
i. Show that the particle is undergoing Simple Harmonic Motion and state the centre of motion. 3

ii. Find the amplitude and period of motion. 2

Question 14 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. The base of a solid is the region enclosed by the parabola $x = 1 - y^2$ and the y - axis. Each cross section perpendicular to the y - axis is a right angle isosceles triangle, as shown in the diagram.



Find the volume of the solid formed. 3

b. The polynomial $P(x) = 2x^3 + 4x^2 - 5x \quad 6$ has roots α, β and γ

i. Find a polynomial with roots α^2, β^2 and γ^2 2

ii. Hence, evaluate $\alpha^2 + \beta^2 + \gamma^2$ 1

iii. Deduce the value of $\alpha^3 + \beta^3 + \gamma^3$ 2

Question 14 continues on the next page.....

c. i. If $I_n = \int_0^1 \left(x^2 - 1\right)^n dx \quad , n = 0, 1, 2, 3, ...$

Show that $I_n = \dfrac{-2n}{2n+1} I_{n-1} \; , \; n = 1, 2, 3, ...$ 3

ii. Hence, use Mathematical Induction to show that;

$$I_n = \frac{\left(-1\right)^n 2^{2n} (n!)^2}{\left(2n+1\right)!} \qquad (n! = 1 \times 2 \times 3 \times 4 \times .... \times (n-1) \times n)$$ 4

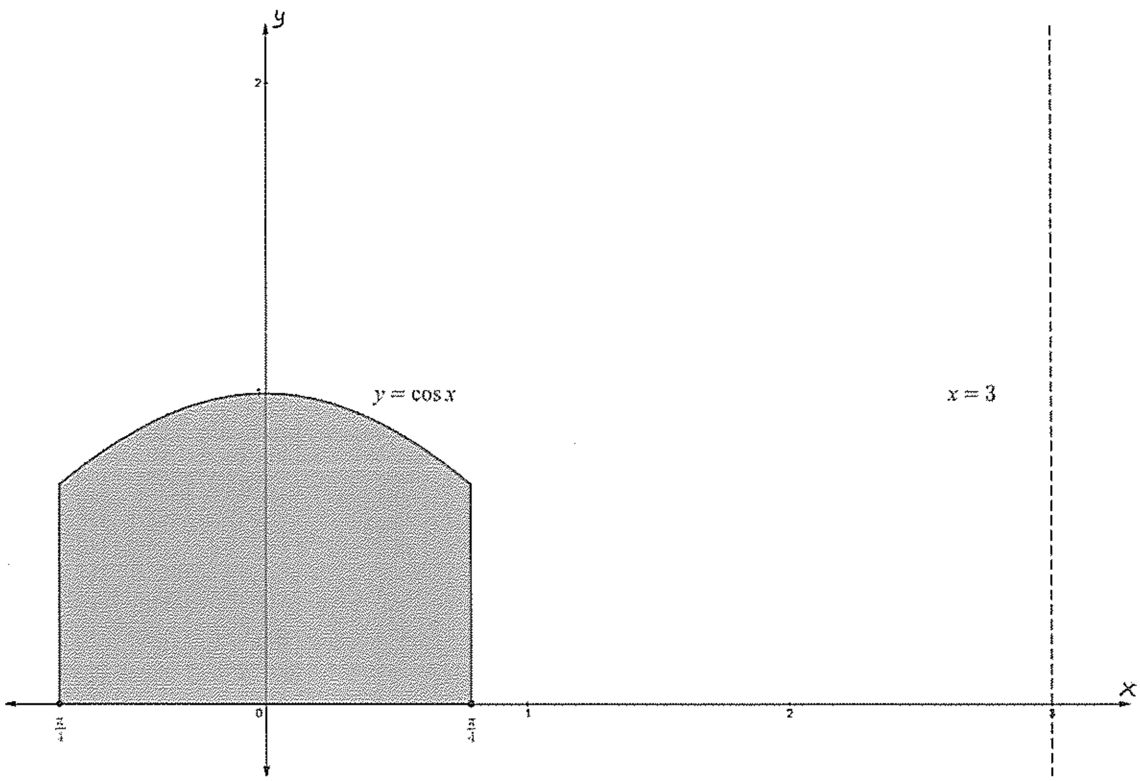
Question 15 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. Using the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, or otherwise,

Find the exact value of $\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}dx$ 3

b. A solid is formed by rotating the region bounded by the curve $y = \cos x$, the $x - axis$ and the lines $x = \frac{\pi}{4}$ and $x = -\frac{\pi}{4}$ about the line $x = 3$ for one revolution.

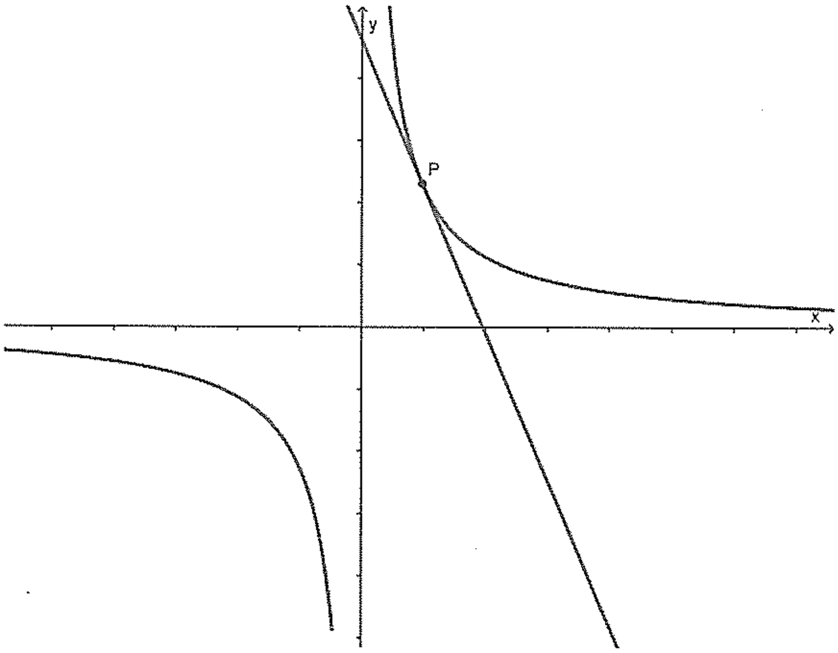


Use the method of cylindrical shells to find the volume of the solid formed. 4

Question 15 continues on the next page........

c. The rectangular hyperbola with equation $xy = 9$ is shown on the Cartesian plane below.

The tangent at the point $P(3p, \frac{3}{p})$ is shown.



i. Show that the equation of the tangent to the rectangular hyperbola $xy = 9$ at the point

$P(3p, \frac{3}{p})$ is $x + p^2 y = 6p$ 2

ii. The tangent at another point $Q(3q, \frac{3}{q})$ intersects the tangent at P, at the point $R(x_0, y_0)$

Prove that $p + q = \frac{6}{y_0}$ and also that $pq = \frac{x_0}{y_0}$ 2

iii. If the length of the chord PQ is d units, find an expression for d^2 in terms of p and q in factorised form. 2

iv. If $d = 6$, deduce that the point R lies on the curve whose equation is;

$\left(x^2 + y^2\right)\left(9 - xy\right) = x^2 y^2$ 2

Question 16 (15 MARKS)

Start this question at the TOP of a NEW page in your Answer Booklet.

a. A particle of mass m kg is projected vertically upwards from the ground with a velocity μ ms^{-1} in a medium whose resistance is given by mkv^2 Newtons, where v is the speed at that instant (in ms^{-1}) and k is a positive constant and g is acceleration due to gravity.

i. Prove that the time taken to reach the highest point is $\dfrac{1}{\sqrt{kg}}\tan^{-1}\left(\mu\sqrt{\dfrac{k}{g}}\right)$ seconds. 3

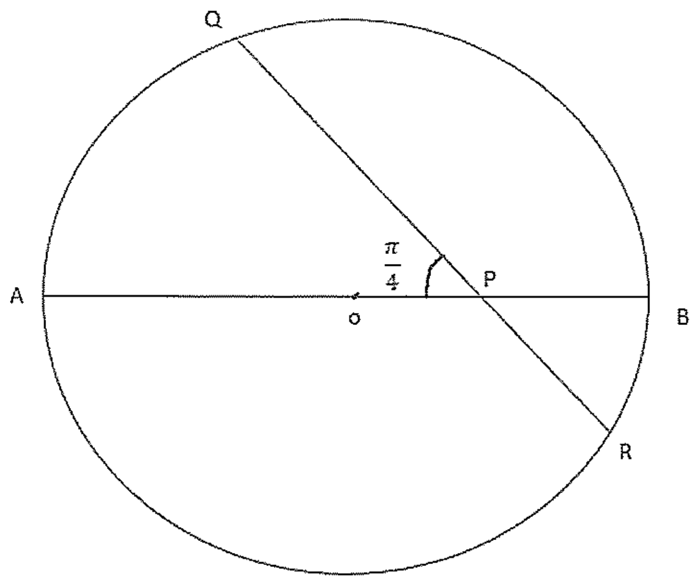
ii. Prove that the greatest height reached is $\dfrac{1}{2k}ln\left(1+\dfrac{k\mu^2}{g}\right)$ metres. 3

iii. How fast is the particle going when it reaches the ground again? 3

b. Two sides of a triangle are in the ratio of $3:2$ and the angles opposite these sides differ by 45°. If the smaller of the two angles is α, find the exact value of $\tan\alpha$ 2

Questions 16 continues on the next page ....

c. Circle centre O has diameter AB intersecting the chord QR at P, at an angle of $\frac{\pi}{4}$ as shown.



i. Copy the diagram, into your answer booklet and construct $TR \perp AB$, where T lies on the circumference

ii. Use triangle QOT to find an expression for QT in terms of AB, with reasons. 2

iii. Prove that $AB^2 = 2(PQ^2 + PR^2)$ 2

END OF THE EXAMINATION

Suggested Solutions

1	C	2	B	3	D	4	B	5	B
6	D	7	A	8	A	9	A	10	C

(1) $\cot 2x + \frac{1}{2}\tan x = \dfrac{1}{\tan 2x} + \dfrac{\tan x}{2}$

$= \dfrac{1-\tan^2 x}{2\tan x} + \dfrac{\tan^2 x}{2\tan x}$

$= \dfrac{1}{2\tan x}$

$= \frac{1}{2}\cot x \qquad a = \frac{1}{2}$ (C)

(2) $f(x) = \dfrac{x+1}{(x+2)(x-2)}$ horiz $x = \pm 2$ $\quad x \to \infty$, $y \to 0$ (B)

(3) $x_1 = 3-2i \quad x_2 = 3+2i \quad \therefore x^2-6x+13$

$\alpha\beta\gamma = -52$ $\quad p(x) = (x+\gamma)(x^2-6x+13)$

$\gamma = -4$ $\quad (x+4)(x^2-6x+13)$ (D)

(4) $\dfrac{z}{1+\omega} = \dfrac{1-2i}{1+(2-i)} = \dfrac{1-2i}{3-i} \cdot \dfrac{3+i}{3+i}$

$= \dfrac{3+i-6i+2}{10}$

$= \dfrac{5-5i}{10} = \dfrac{1-i}{2}$ (B)

(5) $\int x^3 \ln x \, dx = \dfrac{x^4}{4}\ln x - \int \dfrac{x^4}{4} \cdot \dfrac{1}{x} \, dx$

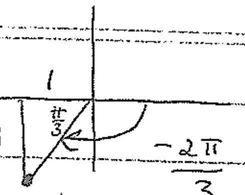
$= \frac{1}{4}\left[x^4 \ln x - \dfrac{x^4}{4}\right] + C$ (B)

(6) $x^2 - 4y^2 = 4$ $\quad a=2$ $\quad$ distance $= 2 \times \dfrac{a}{e}$

$\dfrac{x^2}{4} - y^2 = 1$ $\quad b=1$

$e = \dfrac{\sqrt{5}}{2}$ $\quad = \dfrac{8}{\sqrt{5}}$ (D)

7. $\arg(-1 + ai) = \dfrac{-2\pi}{3}$



$a = -\sqrt{3}$ (A)

$\tan\dfrac{\pi}{3} = \dfrac{y}{1}$

$\sqrt{3} = y$ But 3rd quad.

8. $|z - i| = Im(z)$

$|(x+iy) - i| = Im(x+iy)$

$|x + (y-1)i| = y$

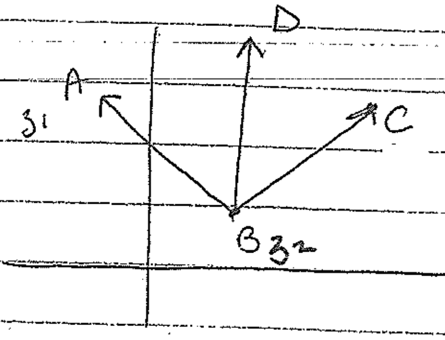
$\therefore \sqrt{x^2 + (y-1)^2} = y$ (A)

$x^2 + (y-1)^2 = y^2$

$x^2 + y^2 - 2y + 1 = y^2$

$x^2 = 2y - 1$

9.



$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BC}$

$= \overrightarrow{BA} + (-i)\overrightarrow{BA} \Rightarrow \overrightarrow{BA}(1-i) = (1-i)(z_1 - z_2)$

OR $= (z_1 - z_2) + -i(z_1 - z_2)$

$= z_1(1-i) - z_2(1-i)$

$= (1-i)(z_1 - z_2)$ (A)

10. $F = \dfrac{m}{x^3}(6 - 10x)$ $\qquad F = m\ddot{x}$

$\therefore \ddot{x} = \dfrac{6-10x}{x^3}$

$\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{6-10x}{x^3}$

$\displaystyle\int_0^v \frac{d}{dx}v^2 = \int_1^x \frac{12-20x}{x^3}\,dx$

$v^2 = \left[\dfrac{-6}{x^2} + \dfrac{20}{x}\right]_1^x$

$= -6/x^2 + 20/x - [-6+20]$

$v^2 = -6/x^2 + 20x/x^2 - \dfrac{14x^2}{x^2}$ (C)

$V = \pm\sqrt{\dfrac{2}{x^2}\left(-3 + 10x - 7x^2\right)}$

Question 11

a) $z = 2-3i$ and $\omega = 4+i$

$$z^2\omega^{-1} = (2-3i)^2 \cdot \frac{1}{4+i}$$

$$= \frac{(-5-12i)}{4+i} \cdot \frac{4-i}{4-i} \quad — 1$$

$$= \frac{-20+5i-48i-12}{16+1}$$

$$= \frac{-32-43i}{17} \quad — 1$$

b) $\dfrac{(1-\sqrt{3}i)^6}{(1+i)^4}$

$$\left.\begin{aligned} 1-\sqrt{3}i &= 2\text{cis}(-\pi/3) \\ 1+i &= \sqrt{2}\,\text{cis}(\pi/4) \end{aligned}\right] 1$$

$$= \left.\frac{2^6 \text{ cis}(6 \times -\pi/3)}{\sqrt{2}^4 \text{ cis}(4 \times \pi/4)}\right] 1$$

$$= 2^4 \text{ cis}(-2\pi - \pi) \qquad (-3\pi \approx -\pi = \pi)$$

principal arg

$$= 2^4 \text{ cis}\,\pi$$

$$= 2^4[\cos\pi + i\sin\pi]$$

$$= 2^4[-1+0]$$

$$= -16 \longleftarrow 1$$

c) (i) $\displaystyle\int \frac{4x-12}{x^2-6x+13}\,dx$

(Do not penalise $+C$)

$$= 2\int \frac{2x-6}{x^2-6x+13}\,dx$$

$$= 2\ln|x^2-6x+13| + C$$

(ii) $\displaystyle\int \frac{1}{x^2-6x+13}\,dx$

$$= \int \frac{1}{x^2-6x+9+4}\,dx$$

$$= \int \frac{1}{(x-3)^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\frac{(x-3)}{2} + C$$

d) $\int_0^{\pi} e^{2x}\sin x\,dx$

let $I = \int_0^{\pi} e^{2x}\sin x\,dx$

$= e^{2x}\cdot -\cos x\Big]_0^{\pi} - \int 2e^{2x}\cdot -\cos x\,dx$

$= -\cos x\, e^{2x}\Big]_0^{\pi} + 2\int e^{2x}\cos x\,dx$

$= -(-1)e^{2\pi} - (-1e^0) + 2\left[e^{2x}\sin x\Big]_0^{\pi} - \int 2e^{2x}\sin x\,dx \right]$

$= e^{2\pi} + 1 + 2\left[0 - 0\right] - 4I$

$5I = e^{2\pi} + 1$

$I = \dfrac{e^{2\pi}+1}{5}$

e) $\int \dfrac{\sin^3 x}{\cos^2 x}\,dx$

$= \int \dfrac{\sin x \cdot \sin^2 x}{\cos^2 x}\,dx$

$= \int \dfrac{1-\cos^2 x}{\cos^2 x}\,\sin x\,dx$ let $u = \cos x$, $du = \dfrac{dx}{-\sin x}$

$= \int \dfrac{1-u^2}{u^2} - du$

$= -\left(\dfrac{u^{-1}}{-1} - u\right) + c$

$= 1/u + u + c$ $(u = \cos x)$

$= \sec x + \cos x + c.$

Question 12

a) $\sqrt{i} = x + iy$

$i = (x+iy)^2$

$= x^2 - y^2 + 2xyi$

equating $x^2 - y^2 = 0$ — ①

$2xy = 1$

$xy = 1/2$ — ②

By inspection

$x = \pm 1/\sqrt{2} \quad y = \pm 1/\sqrt{2}$

$\therefore \sqrt{i} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$ or $\dfrac{-1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}} i$ ②

b) I. $z^3 = 1 \qquad z^3 - 1 = 0$

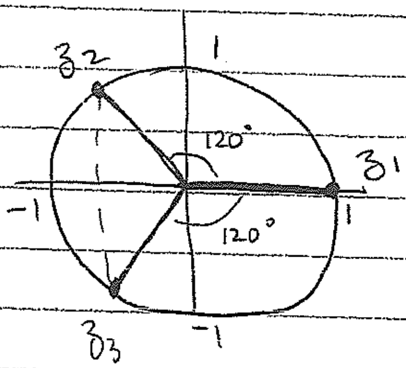
$(z-1)(z^2+z+1) = 0$

$z = 1 \qquad z = \dfrac{-1 \pm \sqrt{3}i}{2}$

$z = 1\text{cis} \pm \dfrac{2\pi}{3}$

$\therefore z = 1\text{cis}0, \quad 1\text{cis} \pm \dfrac{2\pi}{3}$ ①

II.



must be on unit circle

must have z_2 & z_2

in allignment & mark 120° ②

III. as ω is a root $\omega^3 = 1$ and $(\omega^2 + \omega + 1) = 0$

Now $(1 - 8\omega + \omega^2)(1 + \omega - 8\omega^2)$

$(1 + \omega^2 - 8\omega)(1 + \omega - 8\omega^2)$ $\qquad 1 + \omega^2 = -\omega$

$(-\omega - 8\omega)(-\omega^2 - 8\omega^2)$ $\qquad 1 + \omega = -\omega^2$

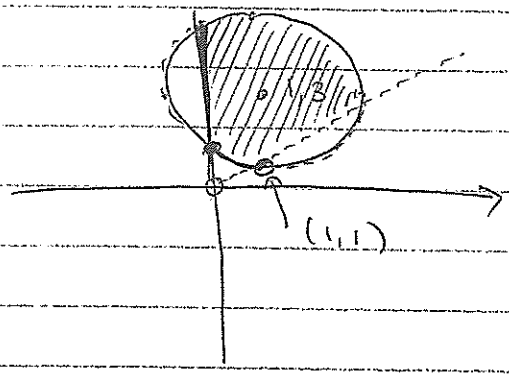
$= (-9\omega)(-9\omega^2)$

$= 81\omega^3$

$= 81 \times 1$

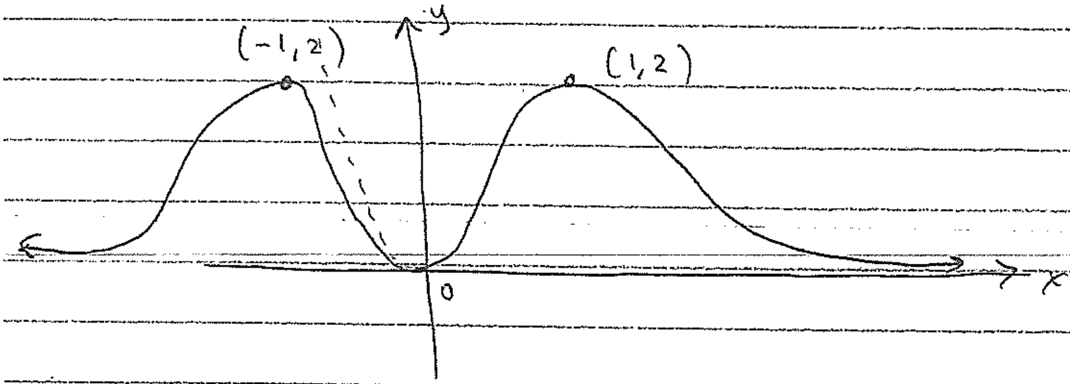
$= 81.$ ②

d. $|z - (1+3i)| \le 2 \;\cap\; \frac{\pi}{4} < \arg(z) \le \frac{\pi}{2}$



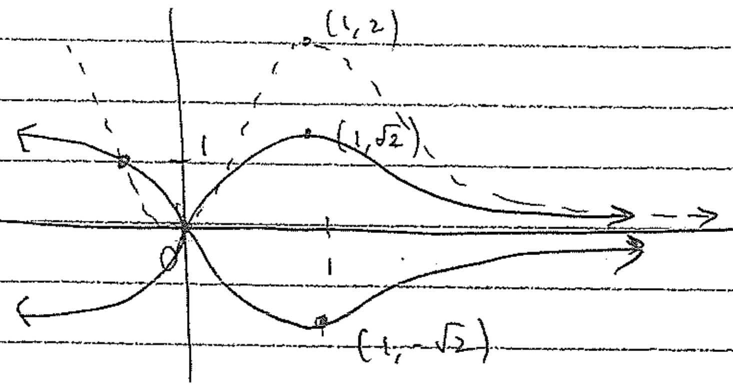
3

e) $y = f|x|$ keeps $x \geq 0$ and reflects this in the y-axis



1

ii) $y^2 = f(x) \longrightarrow y = \pm\sqrt{f(x)}$



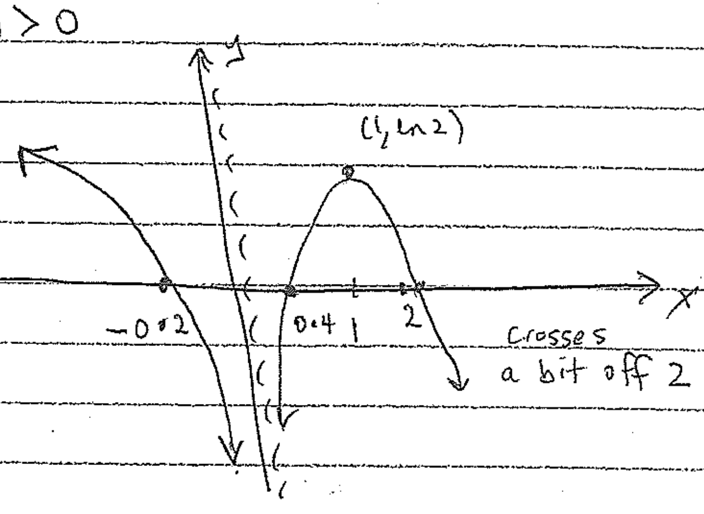
2

iii) $y = \ln f(x)$ $\qquad f(x) > 0$

$\ln 1 = 0$

$\ln(<1) = \text{neg}$

$\ln(>1) = +\text{ve}$



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Question 13

a) $2x^2 + xy = y^2$ at $(2,4)$

$4x + x\frac{dy}{dx} + 1 \cdot y = 2y\frac{dy}{dx}$

$\frac{dy}{dx}(x - 2y) = -y - 4x$

$\frac{dy}{dx} = \frac{-y-4x}{x-2y}$ @ $(2,4)$

$m_T = \frac{-4-8}{2-8}$

$= \frac{-12}{-6}$

$= 2$

$m_N = -1/2$

$\therefore y - 4 = \frac{-1}{2}(x-2)$

$2y - 8 = -x + 2$

$x + 2y - 10 = 0$

b) $5 - 3x = A(x+1)^2 + B(x+1)(x-1) + C(x-1)$

let $x = 1$

$2 = A(2)^2 + 0 + 0$ $\quad A = 1/2$

let $x = -1$

$8 = 0 + 0 + C(-2)$ $\quad C = -4$

let $x = 0$

$5 = 1/2(1)^2 + B(1)(-1) - 4(-1)$

$1 = 1/2 - B$

$B = -1/2$

$B = -1/2$

ii) $\int \frac{1/2}{x-1} + \frac{-1/2}{x+1} + \frac{-4}{(x+1)^2}\, dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \frac{4(x+1)^{-1}}{-1 \times 1} + c$

$= \frac{1}{2}\ln\frac{|x-1|}{|x+1|} + \frac{4}{x+1} + c$.

c) $P(x) = x^5 + 2x^2 + mx + n$ $\quad (x+1)^2 \Longrightarrow$ double root

$\therefore P(-1) = P'(-1) = 0$

$P(-1) = -1 + 2 - m + n = 0$

$-m + n = -1$

$m - n = 1 \quad —①$

$P'(x) = 5x^4 + 4x + m$

$P'(-1) = 5 - 4 + m = 0$

$m = -1$

$\therefore n = -2$

d) $x = 5 - 4\cos^2 t$

$= 5 - 4\left[\tfrac{1}{2}(1 + \cos 2t)\right]$

$= 5 - 2(1 + \cos 2t)$

$x = 3 - 2\cos 2t \qquad \therefore \quad 2\cos 2t = 3 - x$

Now

$\dot{x} = 4\sin 2t$

$\ddot{x} = +8\cos 2t$

$= +4(2\cos 2t)$

$= 4(3 - x)$

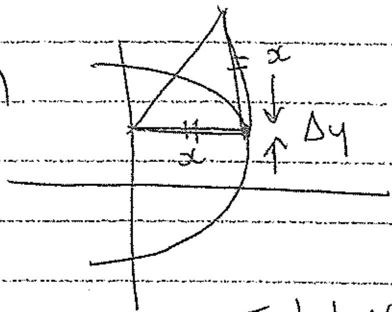
$\ddot{x} = -4(x - 3) \qquad \therefore$ SHM as $\ddot{x} = -n^2 x$

Centre is at $x = 3$ $(\ddot{x} = 0)$

ii) amp $= 2$ $\quad$ period $= \dfrac{2\pi}{n}$ $(n = 2)$

$= \pi$

Question 14

a) 

$$\Delta A = \frac{1}{2}x^2$$

$$\Delta V = \frac{1}{2}x^2 \Delta y \qquad x = 1 - y^2$$

$$x^2 = 1 - 2y^2 + y^4$$

$$\text{Total Volume} = \lim_{\Delta y \to 0} \sum_{-1}^{1} \frac{1}{2}(1 - 2y^2 + y^4)\Delta y$$

$$V = \frac{1}{2}\int_{-1}^{1}(1 - 2y^2 + y^4)\,dy$$

$$= \frac{1}{2}\left[y - \frac{2}{3}y^3 + \frac{y^5}{5}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

$$= \frac{1}{2}\left[2 - 4/3 + 2/5\right]$$

$$= \frac{8}{15} \qquad u^3$$

b) $P(x) = 2x^3 + 4x^2 - 5x - 6$

i. replace x with $\sqrt{x}$ into $P(x) = 0$

$$2(\sqrt{x})^3 + 4(\sqrt{x})^2 - 5(\sqrt{x}) - 6 = 0$$

$$2x\sqrt{x} + 4x - 5\sqrt{x} - 6 = 0$$

$$\sqrt{x}(2x - 5) = 6 - 4x$$

$$x(2x-5)^2 = (6-4x)^2$$

$$x\left[4x^2 - 20x + 25\right] = 36 - 48x + 16x^2$$

$$4x^3 - 36x^2 + 73x - 36 = 0$$

ii) $\therefore \alpha^2 + \beta^2 + \gamma^2 = \frac{36}{4} = 9$

iii) Now
$$\left.\begin{aligned} 2\alpha^3 &= -4\alpha^2 + 5\alpha + 6 \\ 2\beta^3 &= -4\beta^2 + 5\beta + 6 \\ 2\gamma^3 &= -4\gamma^2 + 5\gamma + 6 \end{aligned}\right] +$$

sub α, β, γ into $P(x)$

$$2(\alpha^3+\beta^3+\gamma^3) = -4(\alpha^2+\beta^2+\gamma^2)+5(\alpha+\beta+\gamma)+3(6)$$

$$= -4(9) + 5\left(\frac{-4}{2}\right) + 18$$

$$= -28$$

$$\therefore \alpha^3+\beta^3+\gamma^3 = \underline{-14}$$

(1) $$I_n = \int_0^1 (x^2-1)^n \, dx$$

$$= \int_0^1 (x^2-1)^1 . (x^2-1)^{n-1} \, dx$$

$$= \int_0^1 x^2(x^2-1)^{n-1} - \int_0^1 (x^2-1)^{n-1} \, dx$$

$$I_n = \int_0^1 x^2(x^2-1)^{n-1} - I_{n-1} \longleftarrow ①$$

$$\therefore I_n + I_{n-1} = \int_0^1 x^2(x^2-1)^{n-1} \, dx$$

$$= \int_0^1 x . x(x^2-1)^{n-1} \, dx$$

(Now Parts) } ①

$$= x . \frac{(x^2-1)^n}{n . 2}\Big]_0^1 - \int_0^1 \frac{1 . (x^2-1)^n}{2n} \, dx$$

$$= \frac{1(0)}{2n} - \frac{0}{2n} - \frac{1}{2n} I_n$$

So $$I_n + I_{n-1} = \frac{-1}{2n} I_n$$

$$2n I_n + 2n I_{n-1} = -I_n$$ } ← ① around here

$$I_n(2n+1) = -2n I_{n-1}$$

$$I_n = \frac{-2n}{2n+1} . I_{n-1}$$ QED

c)

ii) Test $n=1$

$$I_1 = \frac{-2(1)}{2(1)+1} I_0 \qquad \text{and} \qquad I_1 = \frac{(-1)^1 \cdot 2^{2(1)} \cdot (1!)^2}{(2\times 1+1)!}$$

$$= \frac{-2}{3}\int_0^1 1\,dx \qquad = \frac{-1 \cdot 4 \cdot 1}{1\times 2\times 3}$$

$$= \frac{-2}{3}\left[x\right]_0^1 \qquad = \frac{-2}{3}$$

$$= -2/3$$

$\therefore$ true for $n=1$ 1

Assume true for $n=K$ ie $I_K = \dfrac{(-1)^K 2^{2K} (K!)^2}{(2K+1)!}$

Prove true for $n=K+1$

ie aim to prove $I_{K+1} = \dfrac{(-1)^{K+1} 2^{2(K+1)} ((K+1)!)^2}{\cancel{(2(K+1)+1)!}}$

all for

Now $I_{K+1} = \dfrac{-2(K+1)}{2(K+1)+1} \cancel{I_K}$ from part (i) 1

$$= \frac{-2(K+1)}{2K+3} \cdot \frac{(-1)^K 2^{2K} (K!)^2}{(2K+1)!} \quad \text{by assumption.}$$

$$= \frac{-1^1 \times 2(K+1) \cdot (-1)^K \cdot 2^{2K} (K!)^2}{(2K+1)!\,(2K+3)} \times \frac{2K+2}{2K+2}$$ 2

$$= \frac{(-1)^{K+1} \times 2(K+1)\, 2^{2K} (K!)^2 \cdot 2(K+1)}{(2K+1)!\,(2K+2)(2K+3)}$$

$$= \frac{(-1)^{K+1} \cdot 2^{2K+2} (K!)^2 (K+1)^2}{(2K+3)!}$$

$$= \frac{(-1)^{K+1} \cdot 2^{2(K+1)} ((K+1)!)^2}{(2K+3)!}$$ 1

QED

$\therefore$ If true for $n=K$ also true for $n=K+1$. As true $n=1$ also true $n=2,3,4$
Hence By M.I true all positive integer n.

Question 15.

a) let $I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx$

Now $I = \int_0^{\pi/2} \frac{\cos^n(\pi/2 - x)}{\cos^n(\pi/2-x) + \sin^n(\pi/2-x)}\,dx$

$= \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx \longleftarrow 1$

$\therefore\ 2I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} + \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx \longleftarrow 1$

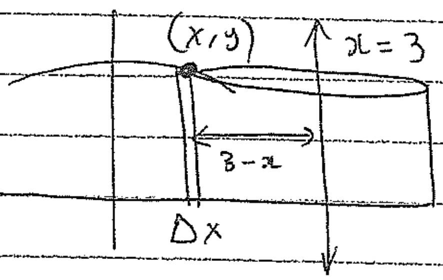
$2I = \int_0^{\pi/2} 1\,dx$

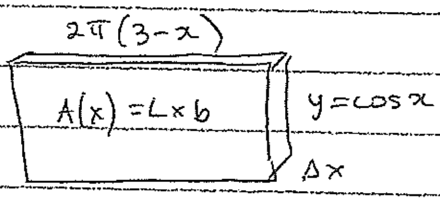
$2I = \left[x\right]_0^{\pi/2}$

$2I = \pi/2$

$\therefore\ I = \pi/4.$ 1

b)





1 $\Delta A(x)$ OR similar

$\Delta V = 2\pi(3-x)\cos x\,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi(3-x)\cos x)\,\Delta x$

$= 2\pi \int_{-\pi/4}^{\pi/4} 3\cos x - x\cos x\,dx$ = integr 1

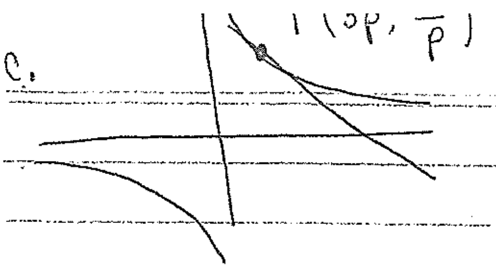
$= 2\pi \int_{-\pi/4}^{\pi/4} 3\cos x - \underbrace{2\pi \int_{-\pi/4}^{\pi/4} x\cos x\,dx}_{\text{odd}\times\text{even}=\text{odd}}$

$\therefore$ zero as symmetrical Limits

$= 2\pi \times 2\int_0^{\pi/4} 3\cos x\,dx$

$= 4\pi \times \left[3\sin x\right]_0^{\pi/4}$

$= 12\pi\left[1/\sqrt{2} - 0\right] = \boxed{6\sqrt{2}.\pi\ u^3}$ 1 c.



$$xy = 9 \longrightarrow y = 9x^{-1}$$

$$\frac{dy}{dx} = \frac{-9}{x^2} \qquad x = 3p$$

$$m_T = \frac{-9}{9p^2}$$

$$= -1/p^2 \quad (1)$$

$$\therefore\ y - 3/p = -1/p^2 (x - 3p)$$

$$p^2 y - 3p = -x + 3p$$

$$x + p^2 y = 6p \quad — (1)$$

ii. tangent at Q $\quad x + q^2 y = 6q \quad$ meets TP in $R(x_0\ y_0)$

$$x + p^2 y = 6p$$

$\therefore$ R satisfies Both tangents

ie $\quad x_0 + q^2 y_0 = 6q \quad — (1)$

and $\quad x_0 + p^2 y_0 = 6p \quad — (2)$

(2) − (1)

$$p^2 y_0 - q^2 y_0 = 6p - 6q$$

$$y_0 (p^2 - q^2) = 6(p - q)$$

$$y_0 (p+q)(p-q) = 6(p-q)$$

$$(p+q) y_0 = 6$$

$$p + q = \frac{6}{y_0}$$

and (1) × p $\quad p x_0 + q^2 p y_0 = 6pq$

(2) × q $\quad q x_0 + p^2 q y_0 = 6pq \quad$] equate these

$\therefore\ p x_0 + q^2 p y_0 = q x_0 + p^2 q y_0 \qquad$ Need $\dfrac{x_0}{y_0}$

$$p x_0 - q x_0 = pq (p y_0 - q y_0)$$

$$x_0 (p - q) = pq (p - q) y_0 \qquad \text{cancel } (p - q)$$

$$x_0 = pq\, y_0$$

OR $\quad pq = \dfrac{x_0}{y_0}$

III. $P = (3p, 3/p)$ $\quad Q(3q, 3/q)$

$$d^2 = PQ^2 = (3p-3q)^2 + \left(\frac{3}{p} - \frac{3}{q}\right)^2$$

$$d^2 = 9(p-q)^2 + 9(1/p - 1/q)^2 \quad — (1)$$

$$d^2 = 9\left[(p-q)^2 + \frac{(q-p)^2}{(pq)^2}\right]$$

$$= 9\left[(p-q)^2 + \frac{(p-q)^2}{(pq)^2}\right]$$

$$= 9(p-q)^2\left[1 + \frac{1}{(pq)^2}\right] \quad — (1)$$

IV.

Now $d = 6$ and Locus of x means $x_0 \to x$

$y_0 \to y$

$$\therefore \; 36 = 9\left[(p+q)^2 - 4pq\right]\left[1 + \frac{1}{(pq)^2}\right]$$

sub in (1) from part (II)

$$4 = \left[\left(\frac{6}{y}\right)^2 - 4\left(\frac{x}{y}\right)\right]\left[1 + \frac{y^2}{x^2}\right]$$

$$4 = \left[\frac{36}{y^2} - \frac{4x}{y}\frac{y}{y}\right]\left[1 + \frac{y^2}{x^2}\right]$$

est
cess

$$4 = \frac{4}{y^2}\left[9 - xy\right]\left[\frac{1}{x^2}\left(x^2+y^2\right)\right]$$

factorise out $\frac{4}{y^2}$ & $\frac{1}{x^2}$

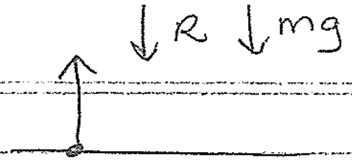
$$1 = \frac{1}{x^2y^2}(9-xy)(x^2+y^2)$$

$$x^2y^2 = (9-xy)(x^2+y^2)$$

$\therefore$ R lies on $(x^2+y^2)(9-xy) = x^2y^2$

Question 16

Resolving forces



$x=0,\ t=0,\ v=\mu$

$$F = m\ddot{x} = -R - mg \qquad R = mkv^2$$

$$m\ddot{x} = -mkv^2 - mg$$

$$\ddot{x} = -kv^2 - g$$

i) highest point $v=0$ $\therefore$ from $t=0$ to $t=T$

and $v=\mu$ to $v=0$

use $\ddot{x} = dv/dt = -kv^2 - g$

$$\frac{dt}{dv} = \frac{-1}{kv^2+g}$$

$$\int_0^T dt = -\frac{1}{k}\int_\mu^0 \frac{1}{v^2+g/k}\,dv$$

$$t\Big]_0^T = +\frac{1}{k}\int_0^u \frac{1}{v^2+(\sqrt{g}/\sqrt{k})^2}\,dv$$

$$T = \frac{1}{k}\cdot\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,v\Big]_0^u$$

$$T = \frac{1}{\sqrt{kg}}\left[\tan^{-1}\sqrt{\frac{k}{g}}\,u - \tan^{-1}\sqrt{\frac{k}{g}}\times 0\right]$$

$$= \frac{1}{\sqrt{kg}}\tan^{-1}\mu\sqrt{\frac{k}{g}}$$

QED

ii) Now $\ddot{X} = v\dfrac{dv}{dx}$ and from $x=0$ to $x=H$

and $v=\mu$ to $v=0$

$$v\frac{dv}{dx} = -kv^2 - g$$

$$\frac{dv}{dx} = \frac{-kv^2-g}{v}$$

$$\text{Now } \frac{dx}{dv} = \frac{-v}{kv^2+g}$$

$$\int_0^H dx = \int_\mu^0 \frac{-v}{kv^2+g}\,dv$$

$$x\Big]_0^H = +\int_0^u \frac{v}{kv^2+g}\,dv$$

$$H = \frac{1}{k}\cdot\frac{1}{2}\ln(kv^2+g)\Big]_0^\mu$$

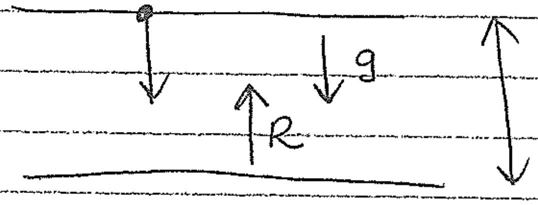
$$\therefore\ H = \frac{1}{2k}\ln\left(k\mu^2+g\right) - \frac{1}{2k}\ln(g)$$

$$= \frac{1}{2k}\ln\left(\frac{k\mu^2+g}{g}\right)$$

$$= \frac{1}{2k}\ln\left(1+\frac{k\mu^2}{g}\right)$$

iii) Speed at return to the ground

$x=0 \quad v=0 \quad t=0$



$$H = \frac{1}{2k}\ln\left(\frac{k\mu^2+g}{g}\right)$$

$$\ddot{x} = -kv^2 + g$$

$$v\frac{dv}{dx} = -kv^2 + g$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

from $x=0$ to $x=H$
and $v=0$ to $v=V$

$$\int_0^H dx = \int_0^V \frac{v}{g-kv^2}$$

$$x\Big]_0^H = \frac{-1}{2k}\ln\left(g-kv^2\right)\Big]_0^V$$

$$H = \frac{-1}{2k}\ln(g-kV^2) - \left[\frac{-1}{2k}\ln(g)\right]$$

$$\frac{1}{2k}\ln\left(\frac{k\mu^2+g}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g-kV^2}\right)$$

Now $\dfrac{k\mu^2+g}{g} = \dfrac{g}{g-kV^2}$

$$(g-kV^2)(k\mu^2+g) = g^2$$

Looking for $V^2 \rightarrow V$

$$kV^2 = g - \frac{g^2}{k\mu^2+g}$$

$$V^2 = \frac{g}{k} - \frac{g^2}{k^2\mu^2+gk}$$

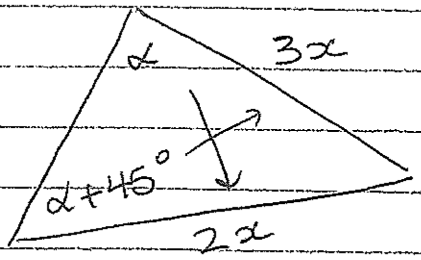
$$V^2 = \frac{kg\mu^2}{k^2\mu^2 + gk}$$

$$= \frac{g\mu^2}{k\mu^2 + g}$$

positive motion

$$V = \mu\sqrt{\frac{g}{k\mu^2+g}} \quad m/s.$$

b)



$$\frac{\sin\alpha}{2x} = \frac{\sin(\alpha+45^\circ)}{3x}$$

$$3\sin\alpha = 2\sin(\alpha+45^\circ)$$

$$3\sin\alpha = 2\sin\alpha.\cos 45^\circ + 2\sin 45^\circ\cos\alpha$$

$$3\sin\alpha = \frac{2}{\sqrt{2}}(\sin\alpha + \cos\alpha)$$

$$3\sin\alpha = \sqrt{2}\sin\alpha + \sqrt{2}\cos\alpha$$

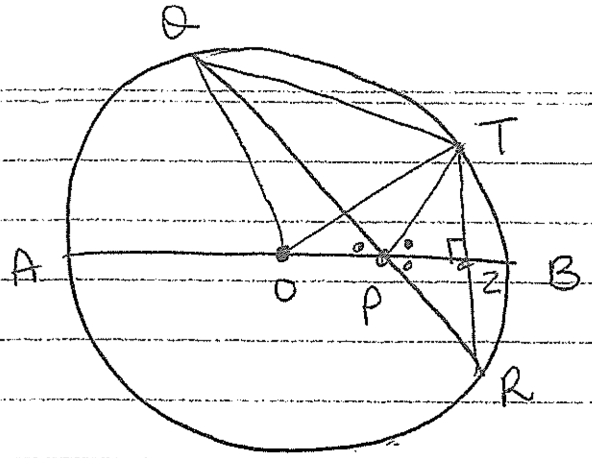
$$\sin\alpha(3-\sqrt{2}) = \sqrt{2}\cos\alpha$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{2}}{3-\sqrt{2}}$$

$$\tan\alpha = \frac{\sqrt{2}}{3-\sqrt{2}} \cdot \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{3\sqrt{2}+2}{7}.$$

c)



let TR cross AB at Z

1. As $TR \perp OB$ $ZT = RZ$ a perpendicular to a chord from the centre bisects the chord
and $\angle QPO = \angle BPR = \pi/4$ vertically opposite
and Now $\Delta TPZ \equiv \Delta RPZ$ (SAS, PZ common)

$\angle QOT = 2\angle QRT$ (angle at the centre is twice the angle at the circumference, arc QT)
$= \pi/2$

Now ΔQOT $OQ = OT = \frac{1}{2}AB$ radii

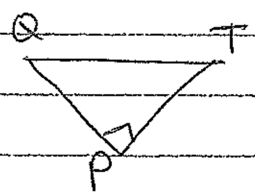
Now $QT^2 = (\frac{1}{2}AB)^2 + (\frac{1}{2}AB)^2$ (Pythagoras theorem)

$QT^2 = \frac{AB^2}{4} + \frac{AB^2}{4}$

$QT^2 = \frac{1}{2}AB^2$ ← ~~except~~ this $(QT = \frac{1}{\sqrt{2}}AB)$

ii. $\angle TPZ = \pi/4$ corresponding angle in congruent triangles
$\therefore \angle QPT = \pi/2$ straight line

Now ΔQPT



$QT^2 = PQ^2 + PT^2$ (Pyth)

$\frac{AB^2}{2} = PQ^2 + PR^2$

But $PT = PR$ corresponding sides in congruent triangles.

$\therefore AB^2 = 2(PQ^2 + PR^2)$.

QED.

2019 EXTENSION 2 TRIAL MARKER FEEDBACK

QUESTION 11

a. Algebraic errors - especially in expanding the bracket when realising the denominator
b. Mostly well done but again algebraic errors were made
c. Well done
d. Integration by parts needs care, especially in algebra.
e. See the suggested solutions for the simplest substitution.

QUESTION 12

a) Poor algebra led to students not obtaining full marks.

bi) Students able to find real and complex root, well answered

bii) When drawing an Argand diagram, use a ruler and a compass to assist. The question required that the solutions were placed in a unit circle and z_1, z_2 and z_3 in alignment and $\frac{2\pi}{3}$ marked for the angles.

biii) Algebraic errors caused the loss of marks

c) Need to draw accurate sketches. Use appropriate equipment. The circle and the straight line were graphed correctly however many errors when shading the region.

di) The graph of $y = f|x|$ was well drawn

dii) Students had difficulties when drawing the graph of $y^2 = f(x)$. Many students did not graph on the left of the y-axis. Please see solutions!!!

diii) Students had difficulties when drawing the graph of $y = lnf(x)$. Many students did not graph on the left of the y-axis. Many were unaware that ln1=0, ln(<1)=negative and ln(>1)=posistive.

QUESTION 13

d) To prove SHM students must show and state that acc = $-n^2X$ where X=x-b in this question – it was easier to change the format of x at the start and then differentiate, this also meant that students could easily find the correct centre, amplitude and period – the amp is not the coefficient of $\cos^2 x$

QUESTION 14

a) You need to show the marker how you built the equation formed. YOU must state and show

$$\lim_{\Delta y \to 0} \sum_{-1}^{1} \frac{1}{2}(1 - y^2)^2 \Delta y \text{ initially}$$

Students made silly errors of changing limits resulting in incorrect answers.

b) Algebraic errors caused the loss of marks

biii) The results from bi were required to solve the question.

ci) Students who correctly used integration by parts were able to obtain full marks.

Poor algebra led to the loss of marks.

cii) NESA does not abbreviate. MI also stands for mission impossible. Please do not abbreviate.

Poor setting out for Mathematical induction proofs.

Many students miraculously went from $(2k+3)(2k+1)! = (2k+3)!$ marks were not awarded.

QUESTION 15

a. Many tried to expand the trig ratios instead of using complementary angles and ratios which made the problem much harder

b. Again, care needed in the working, especially with negative signs. Some students failed to recognise that the integral between $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ for $x\cos x$ is 0 because odd function and incorrectly doubled an integral between $\frac{\pi}{4}$ and 0. Some students also tried to find the volume by doing an inner cylinder minus a large cylinder and students who used this method were largely unsuccessful due to the copious amounts of difficult algebra.

c i. Well done.

c ii. Proving $p+q=\frac{6}{y}$ was well done but some students struggled to prove $pq=\frac{x}{y}$ as they did not realise that they needed to use $p+q=\frac{6}{y}$ to do so.

c iii. Students struggled to factorise the algebra correctly

c iv. Again, algebra manipulation and substitution skills let some students down when trying to deduce the equation. When completing conics, you MUST check and double check your algebra.

QUESTION 16

aiii. Question needed to be started again with new origin from point of fall and answer to part ii needed to be used to get correct velocity

b. SINE RULE!!! Smaller angle is opposite smaller side. You were told alpha was the smaller angle so other angle needed to be alpha +45.

ci. Needed to hop scotch your way around to prove < QOT at centre is 90, then use Pythagoras.

cii. Either congruence of 2 triangles or correct use of intersecting chords theorem.