Taylors College, Sydney Campus

4 Unit Mathematics

Trial Higher School Certificate Examinations 1998 - 2002

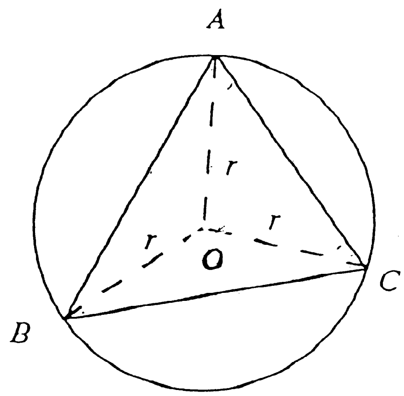
1998

1. **(a)** Find $\int \frac{dx}{x \log_e x}$ **(b)** Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$ **(c)** Find $\int \frac{dx}{(x+1)(x^2+4)}$
(d) Using the substitution $t = \tan \frac{x}{2}$, calculate $\int \frac{15\ dx}{17+8\cos x}$, leaving your answer in terms of t.
(e) **(i)** Differentiate $\frac{x}{\sqrt{x-3}}$ **(ii)** Hence evaluate $\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}\ dx$

2. **(a)** The complex number z is given by $z = 1 - 2i$. Find in simplest form the values of **(i)** $iz + \overline{z}$ **(ii)** $\frac{1}{z}$
(b) The equation $x^2 - (p+iq)x + 3i = 0$, where p and q are real, has roots α and β. The sum of the squares of the roots is 8.
(i) Write down expressions for the sum of the roots and the product of the roots.
(ii) Hence find the possible values of p and q.
(c) **(i)** Express each of the complex numbers $z = 2i$ and $w = 1 + \sqrt{3}i$ in modulus/argument form. Represent the vectors z, w and $z + w$ on an Argand diagram.
(ii) Find the exact values of $\arg\left(\frac{z}{w}\right)$ and $\arg(z+w)$.
(d) **(i)** On an Argand diagram shade in the region satisfy both the conditions $|z - 2i| \leq 1$ and $0 \leq \arg(z - i) \leq \frac{\pi}{6}$
(ii) Find the exact perimeter and the exact area of the shaded region.

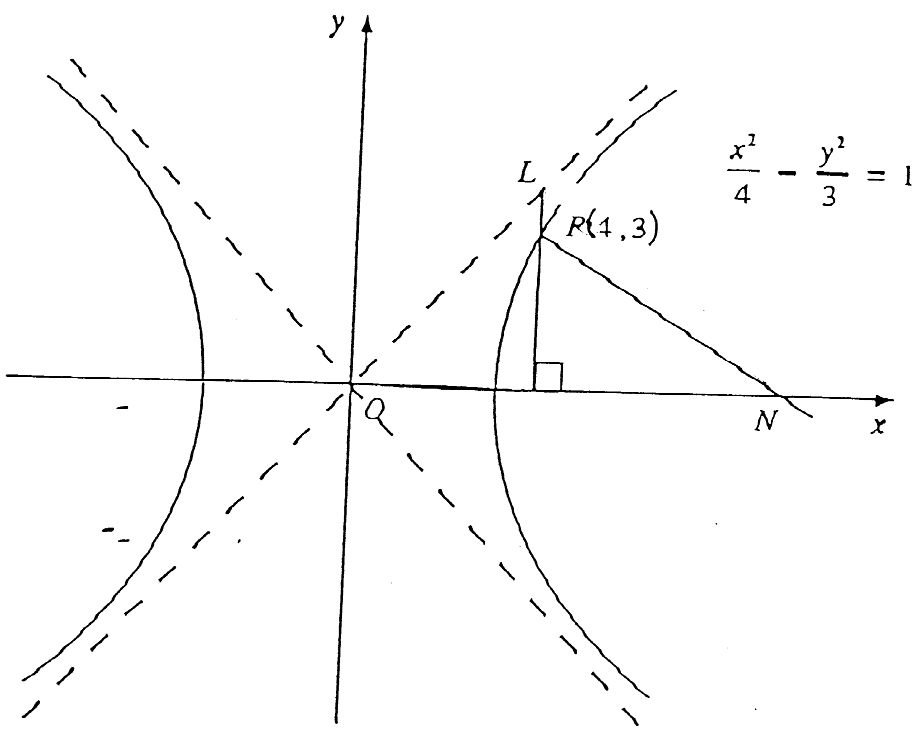
3. **(a)** Show that $\int_0^{\frac{\pi}{4}} x \sin x\ dx = \frac{\sqrt{2}}{8}(4 - \pi)$.
(b) The shape of a particular cake can be represented by rotating the region between the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{4}$, about the line $x = \frac{\pi}{4}$. Using the method of cylindrical shells, find the volume of the cake.

(c) The circle through the vertices of triangle ABC has centre O and radius r.



(i) Show that $BC = 2r \sin A$.
(ii) Use the fact that Area($\triangle OBC$)+Area($\triangle OCA$)+Area($\triangle OAB$) = Area($\triangle ABC$) to show that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

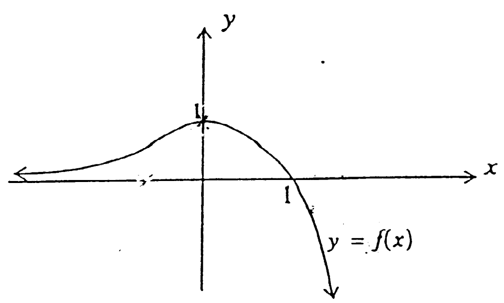
4. (a) The diagram shows the graph of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



The point $P(4, 3)$ lies on the hyperbola. The normal at P to the hyperbola meets the x axis at N. The vertical line through P meets the asymptote in the first quadrant at L.
(i) Show that the normal at P to the hyperbola has equation $x + y = 7$.
(ii) Show that LN is perpendicular to OL.
(b) The points $P\left(2p, \frac{2}{p}\right)$ and $Q\left(2q, \frac{2}{q}\right)$ lie on the rectangular hyperbola $xy = 4$. M is the midpoint of the chord PQ. P and Q move on the rectangular hyperbola so

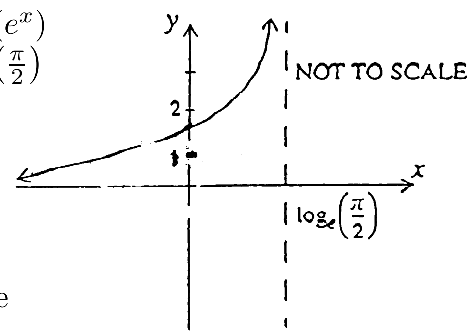
that the chord PQ always passes through the point $R(4,2)$.
(i) Show that the chord PQ has equation $x + pqy = 2(p+q)$
(ii) Show that $pq = p + q - 2$
(iii) Hence show that the locus of M has equation $y = \frac{x}{x-2}$.
(iv) On the same axes sketch the rectangular hyperbola $xy = 4$ and the locus of M, showing clearly the equations of any asymptotes and the point R.

5. (a) The graph of $y = f(x)$ is sketched below. There is a stationary point at $(0, 1)$.



Use this graph to sketch the following without using calculus, showing essential features.
(i) $y = f\left(\frac{x}{2}\right)$ **(ii)** $y = x + f(x)$ **(iii)** $y = \frac{1}{f(x)}$ **(iv)** $y = f\left(\frac{1}{x}\right)$.

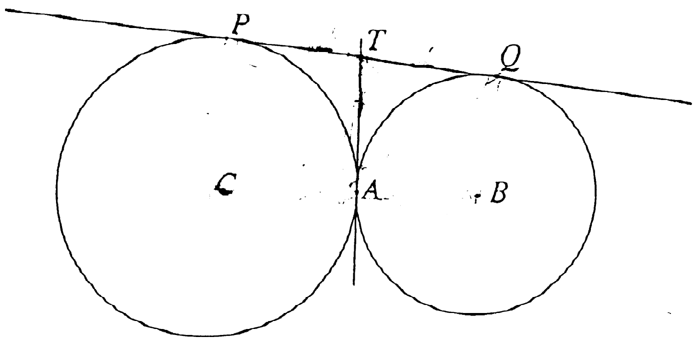
(b) The diagram shows part of the curve $y = \tan(e^x)$ where $x < \log_e\left(\frac{\pi}{2}\right)$. The part to the right of $\log_e\left(\frac{\pi}{2}\right)$ has not yet been drawn.
(i) Find the smallest positive solution to the equation $\tan(e^x) = 0$.
(ii) Show that $y = \tan(e^x)$ is an increasing function.
(iii) Copy the diagram and hence sketch the curve $y = \tan(e^x)$ for $x < \log_e\left(\frac{3\pi}{2}\right)$.



(iv) Find the equation of the inverse function of $y = \tan(e^x)$ for the case when
(α) $x < \log_e\left(\frac{\pi}{2}\right)$ **(β)** $\log_e\left(\frac{\pi}{2}\right) < x < \log_e\left(\frac{3\pi}{2}\right)$.

6. (a) (i) Prove that $\tan^{-1} n - \tan^{-1}(n-1) = \tan^{-1} \frac{1}{n^2-n+1}$, where n is a positive integer.
(ii) Hence evaluate $\tan^{-1} 1 + \tan^{-1} \frac{1}{3} + \cdots + \tan^{-1} \frac{1}{n^2-n+1}$.
(iii) Hence find $\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{n^2-n+1}$.
(b) (i) On the same diagram sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 4$ showing clearly the coordinates of any points of intersection with the x axis or the y axis, and the equations of any asymptotes.
(ii) Shade the region where $(x^2 + y^2 - 9)(x^2 - y^2 - 4) \geq 0$
(c) (i) Use DeMoivre's theorem to show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$, $(\cos\theta \neq 0)$ where n is a positive integer.

(ii) Hence show that the equation $(1+z)^4+(1-z)^4=0$ has roots $\pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.
(iii) Hence show that $\tan^2\frac{\pi}{8}=3-2\sqrt{2}$.
7. (a)



Two circles, centres C and B, touch externally at A. PQ is a direct common tangent touching the circles at P and Q respectively. The common tangent at A meets PQ at T.
(i) Show that the common tangent at A bisects PQ.
(ii) Let M be the midpoint of CB. Prove that $MT||CP$.
(iii) Prove that the circle with BC as diameter touches the line PQ.
(b) A curve is defined by the parametric equations $x=\cos^3\theta, y=\sin^3\theta$ for $0<\theta<\frac{\pi}{4}$.
(i) Show that the equation of the normal to the curve at the point $P(\cos^3\phi, \sin^3\phi)$ is $x\cos\phi - y\sin\phi = \cos 2\phi$
(ii) The normal at P meets the x axis at A and the y axis at B. Show that $AB = 2\cot 2\phi$
(c) **(i)** If $x\geq 0$ show that $\frac{2x}{1+x^2}\leq 1$.
(ii) Show that $e^a\geq 1+a^2$ for $a\geq 0$.
8. (a) A High School Student Representative Council consists of twelve students, one boy and one girl from each of years 7 to 12. At their meetings the twelve students sit around a circular table. Find how many seating arrangements are possible
(i) without restriction
(ii) if all the boys sit next to each other
(iii) if no two boys sit next to each other
(iv) if the boy and girl from each year group sit opposite each other.
Suppose now that a committee of six students is chosen at random from the members of the Student Representative Council.
(v) Find the probability that the committee contains exactly two students who belong to the same year group.
(b) A sequence is defined by $u_1=2, u_2=8$ and $u_n=4u_{n-1}-4u_{n-2}$, where $n\geq 3$ is a positive integer.
(i) Use the method of Mathematical Induction to show that $u_n=n2^n$ for all positive integers $n\geq 1$.
(ii) Without using the method of Mathematical Induction again, show that
$\sum_{r=1}^{n} u_r = 2+(n-1)2^{n+1}$.