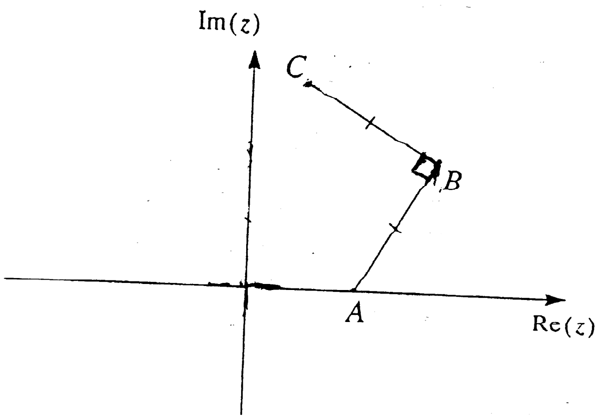
# 1999

**1.** **(a)** Find $\int \sqrt{e^x}\, dx$. **(b)** Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(1+\tan^2 x\right)\tan x\, dx$.
**(c)** Use the substitution $u = \frac{1}{x}$ to show that $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\, dx = 0$.
**(d)** Find $\int \frac{x+7}{x^2+16}\, dx$ **(e)** Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$.

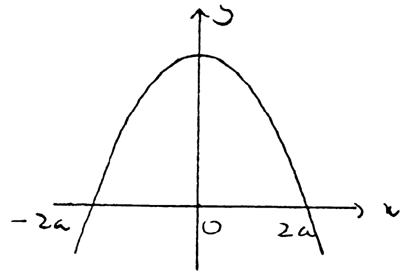
**2.** **(a)** Given $z = \sqrt{6} - \sqrt{2}i$, find: **(i)** $\Re(z^2)$; **(ii)** $|z|$; **(iii)** $\arg z$; **(iv)** $z^4$ in the form $x + iy$; **(v)** $\frac{1}{z^3}$ in modulus-argument form.
**(b)** **(i)** Find that Cartesian equation of the locus represented by $2|z| = 3(z + \overline{z})$.
**(ii)** Sketch the locus on an Argand diagram.
**(c)**



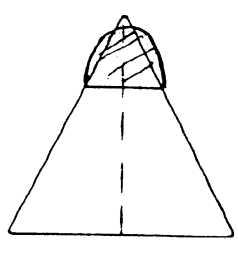

The diagram above shows the fixed points $A, B$ and $C$ in the Argand plane, where $AB = BC$, $\angle ABC = \frac{\pi}{2}$, and $A, B$ and $C$ are in anticlockwise order. The point $A$ represents the complex number $z_1 = 2$ and the point $B$ represents the complex number $z_2 = 3 + \sqrt{5}i$.
**(i)** Find the complex number $z_3$ represented by the point $C$.
**(ii)** $D$ is the point on the Argand plane such that $ABCD$ is a square. Find the complex number $z_4$ represented by $D$.

**3.** **(a)** Consider the function $f(x) = \frac{x-1}{x}$.
**(i)** Sketch the graph $y = f(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.
**(ii)** Use the graph $y = f(x)$ to sketch on separate axes the graphs
**($\alpha$)** $y = |f(x)|$ **($\beta$)** $y = f(|x|)$ **($\gamma$)** $y = \{f(x)\}^2$ **($\delta$)** $y = f^{-1}(x)$ **($\varepsilon$)** $y = \sin^{-1} f(x)$
**(b)** **(i)** Show $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of the equation $z^5 + z - 1 = 0$.
**(ii)** Hence show $z^2 - z + 1$ is a factor of $z^5 + z - 1$. **(iii)** Simplify $(1-\alpha)^{20}$.

**4. (a) (i)** Simpson's rule gives the exact answer to the area under a papabola. Use Simpson's rule to find the area enclosed by the $x$-axis and the parabola shown in the diagram.



**(ii)** Write down the Cartesian equation of the parabola shown in **(i)**.
**(iii)** The latus rectum of a parabola is the focal chord perpendicular to the axis of the parabola. Explain why the equation of the latus rectum is $y = 0$ for this parabola.
**(iv)** The base of a solid is an isosceles triangle with sides 13 cm, 13 cm and 10 cm. The cross-section of the solid is a parabola with its latus rectum lying on this base and perpendicular to the axis of symmetry of the triangle. Find the volume of this solid using the slicing method.



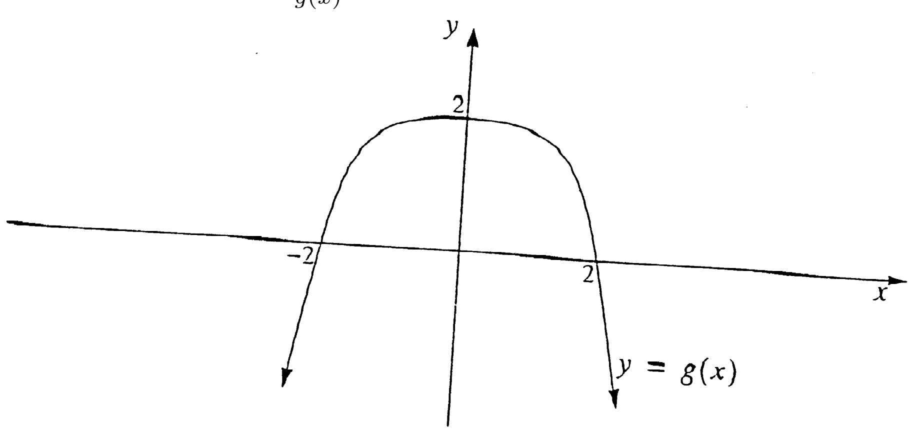
**(b) (i)** Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.
**(ii)** Hence show that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)\,dx = \int_0^{\frac{\pi}{2}} (a\sin^2 x + b\cos^2 x)\,dx$.
**(iii)** Deduce that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)\,dx = \frac{\pi(a+b)}{4}$.

**5. (a) (i)** Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $T\left(2t, \frac{2}{t}\right)$ has equation $x + t^2 y = 4t$.
**(ii)** This tangent cuts the $x$-axis at point $Q$. Show that the line through $Q$ which is perpendicular to the tangent at $T$ has equation $t^2 x - y = 4t^3$.
**(iii)** This line through $Q$ cuts the rectangular hyperbola at the points $R$ and $S$. Show that the midpoint $M$ of $RS$ has coordinates $M(2t, -2t^3)$.
**(iv)** Find the equation of the locus of $M$ as $T$ moves on the rectangular hyperbola.
**(b)** The hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point $Q$.
**(i)** Show this information on a diagram.
**(ii)** Explain why $x^2(x-1)^2 + c^4 = x^2$ has a repeated real root and two complex roots.

**(iii)** Prove that if $k$ is a repeated real root of the polynomial equation $P(x) = 0$ then $k$ is also a root of $P'(x) = 0$.
**(iv)** If $k$ is the repeated real root of $x^2(x-1)^2 + c^4 = x^2$, find the value of $k$ and $c^2$.

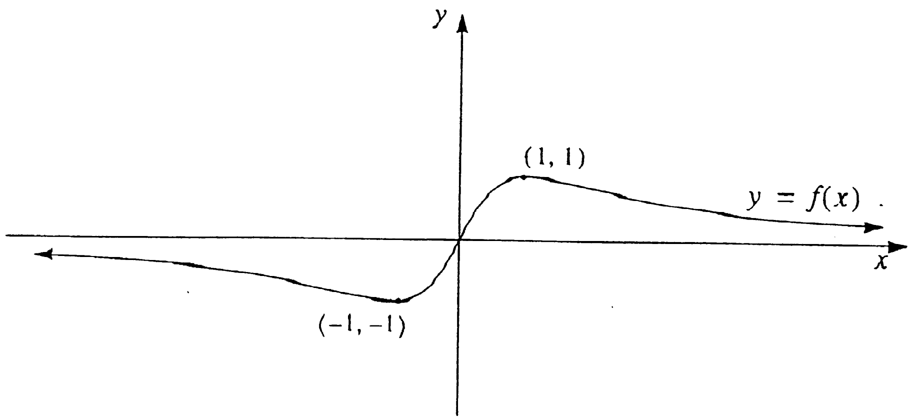
**6. (a) (i)** Show that $\int x \tan^{-1} x\, dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c,$ $c$ constant.
**(ii)** $I_n = \int_0^1 x^n \tan^{-1} x\, dx,\ n = 0, 1, 2, \ldots$. Show that $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2,\ I_1 = \frac{\pi}{4} - \frac{1}{2}$ and $I_n = \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}\cdot I_{n-2},\ n = 2, 3, 4, \ldots$.
**(b)** Two tangents are drawn from the external point $T(x_0, y_0)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, meeting it at $P$ and $Q$.
**(i)** Write down the equation of the chord $PQ$.
**(ii)** If the chord $PQ$ touches the circle $x^2 + y^2 = 9$, then by considering the distance of the chord from the origin, or otherwise, show that the point $T(x_0, y_0)$ statisfies $\frac{9x_0^2}{256} + \frac{y_0^2}{9} = 1$.
**(iii)** Give a geometrical description of the locus of $T$.

**7. (a)** The equation $x^3 - x^2 - 3x + 2 = 0$ has roots $\alpha, \beta$ and $\chi$.
**(i)** Use the value of $\alpha + \beta + \chi$ to find the monic polynomial equation with roots $2\alpha + \beta + \chi, \alpha + 2\beta + \chi$ and $\alpha + \beta + 2\chi$.
**(ii)** Find the monic polynomial equation with roots $\alpha^2, \beta^2$ and $\chi^2$.
**(b) (i)** The diagram shows the graph of $y = g(x)$ where $g(x) = 2 - \frac{x^4}{8}$. Use it to sketch the graph of $y = \frac{1}{g(x)}$.




**(ii)** Find the area enclosed by $y = \frac{1}{g(x)}$, the $x$-axis and the lines $x = 1$ and $x = \sqrt{2}$.
**(c) (i)** Consider the following statements.
**($\alpha$)** If $P(x)$ is an odd function and $Q(x)$ is an even function then $P(Q(x))$ is odd.
**($\beta$)** If $P(x)$ is an odd function and $Q(x)$ is an even function then $Q(P(x))$ is even.
Indicate whether each of these statements is true or false. Give a reason for your answer.

**(ii)** The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{2x}{x^2+1}$.




Sketch the graph of $y = g(f(x))$ where $g(x) = 2 - \frac{x^4}{8}$.

**8. (a) (i)** Show that for all values of $A$ and $B$ $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$.
**(ii)** Use the method of mathematical induction to show that for all positive integers $n$, $\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$.
**(iii)** Hence show that $\cos 2x + \cos 4x + \cos 6x + \cdots + \cos 16x = 8\cos 9x \cos 4x \cos 2x \cos x$.
**(b) (i)** If $x > 0, y > 0$ show that $x + y \geq 2\sqrt{xy}$.
**(ii)** Hence show that if $x > 0, y > 0, z > 0$ then $(x+y)(y+z)(z+x) \geq 8xyz$.
**(iii)** If $a, b, c$ are the sides of a triangle with semi-perimeter $S = \frac{1}{2}(a+b+c)$ then Heron's formula states that the area $\Delta$ of the triangle is given by $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$. By choosing suitable values for $x, y, z$ show that $\Delta^2 \leq \frac{(a+b+c)abc}{16}$.

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