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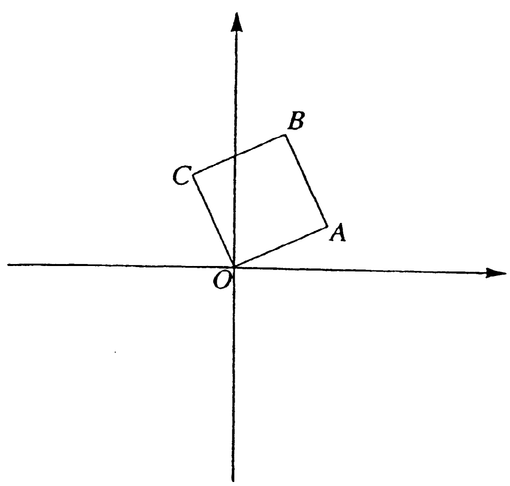
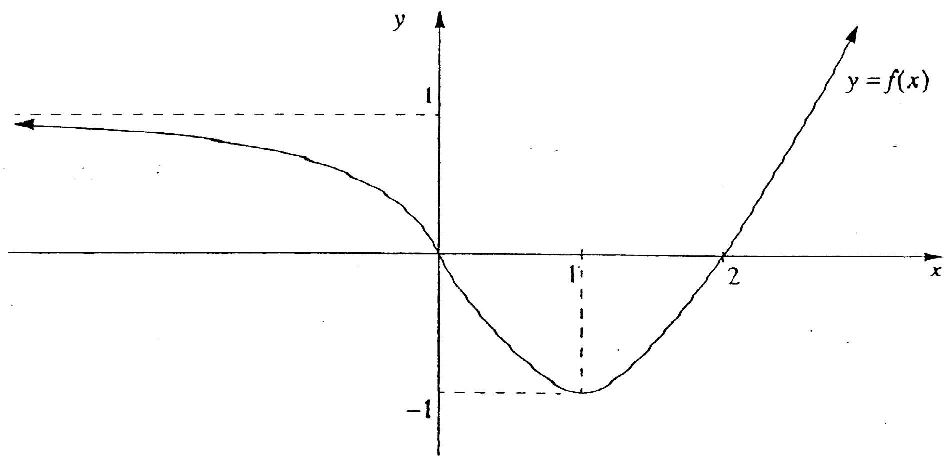
1. **(a)** **(i)** Show that $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$. **(ii)** hence find $\int \frac{dx}{1+e^x}$.
(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+13\sin x}\, dx$ using the substitution $t = \tan\frac{x}{2}$.
(c) **(i)** If $I_n = \int_0^1 (x^2-1)^n\, dx$, $n = 0, 1, 2, \ldots$, show that $I_n = \frac{-2n}{2n+1} I_{n-1}$, $n = 1, 2, 3, \ldots$. **(ii)** Evaluate I_1. **(iii)** Hence use the method of mathematical induction to show that $I_n = \frac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!}$ for all positive integers n.

2. **(a)** Let $z = -1 + i\sqrt{3}$.
(i) Write z in mdulus-argumen form.
(ii) Express in the form $a + ib$, where a and b are real
(α) z^5 **(β)** $z\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
(b)



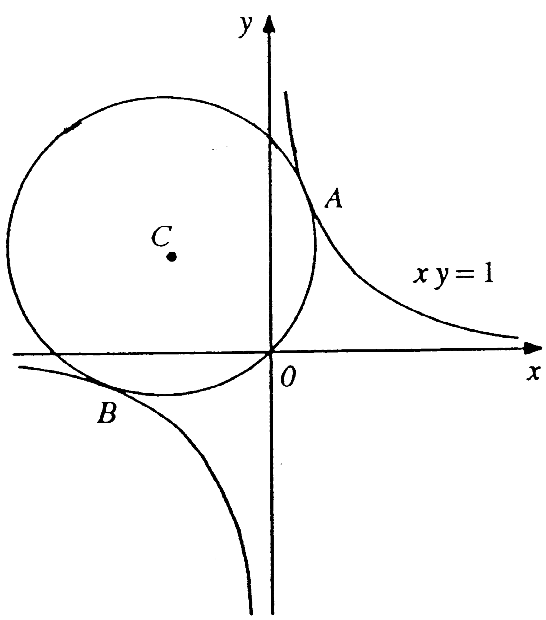
The point A in the Argand diagram sketched above represents the complex number $z = a + ib$, in the first quadrant. The point B represents the complex number $4 + 7i$.
(i) If $OABC$ is a square, find in terms of a and b the complex number represented by the point C. **(ii)** Hence or otherwise evaluate a and b.
(c) **(i)** Sketch and describe the locus of z if $|z - i| = \Im(z)$.
(ii) For what values of m is the line $y = mx$ a tangent to this locus?
(iii) What is the least value of $\arg(z)$ for this locus?
(d) **(i)** Solve $z^3 - 1 = 0$, leaving your answers in modulus-argument form.
(ii) Let ω be one of the non-real roots of $z^3 - 1 = 0$.
(α) Show that $1 + \omega + \omega^2 = 0$. **($\beta$)** Hence simplify $(1+\omega)^8$.

3. (a)



Given the function $y = f(x)$ in the diagram above, sketch on separate diagrams, showing all intercepts, turning points and asymptotes:
(i) $y = f(-x)$, **(ii)** $y = f(|x|)$, **(iii)** $y = f(2x)$, **(iv)** $y = e^{f(x)}$, **(v)** $y = \tan^{-1} f(x)$
(b) (i) Graph the function $y = x^3(x-2)$. You need not use calculus, but you must show the behaviour near any x-intercepts.
(ii) Differentiate $y^2 = x^3(x-2)$ implicitly, and hence show that for $y > 0$, $\frac{dy}{dx} = (2x-3)\sqrt{\frac{x}{x-2}}$.
(iii) Sketch $y^2 = x^3(x-2)$, paying particular attention to the behaviour of the curve near its x-intercepts. (You do not need to find the coordinates of any inflections.)

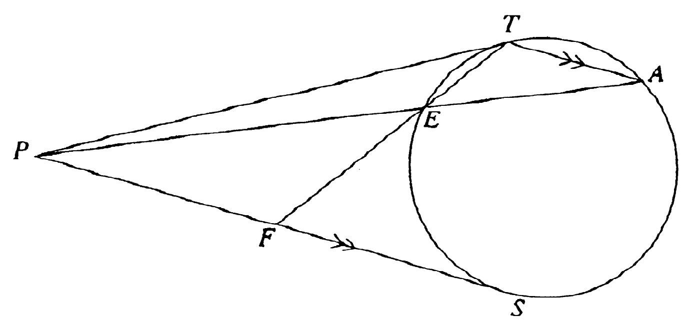
4. (a)



The circle with centre $C(= c, c)$, where $c > 0$, passes through the origin O and

touches the curve $xy = 1$ at the points A and B.
(i) Show that the x coordinates $x = \alpha$ and $x = \beta$ of the points A and B satisfy the equation $x^4 + 2cx^3 - 2cx + 1 = 0$.
(ii) Explain why the equation $x^4 + 2cx^3 - 2cx + 1 = 0$ has real roots $\alpha, \alpha, \beta, \beta$.
(iii) Use the relationships between the roots and the coefficients of this equation to find the exact values of c, α and β.
(b) **(i)** On the same axes sketch the graphs of $y = \sqrt{1 - x^2}$ and $y = \frac{1}{\sqrt{1-x^2}}$.
(ii) The region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$, the coordinate axes and the line $x = \frac{1}{2}$ is rotated through one complete revolution about the ine $x = 6$. Use the method of cylindrical shells to show that the volume V unit^3 of the solid of revolution is given by $V = 2\pi \int_0^{\frac{1}{2}} \frac{6-x}{\sqrt{1-x}}\, dx$.
(iii) Hence find the value of V in simplest exact form.

5. **(a)** The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.
(i) Write down the equations of the two asymptotes of the hyperbola.
(ii) Show that the acute angle α between the two asymptotes satisfies $\tan\alpha = \frac{2ab}{a^2-b^2}$.
(iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP.NP = \frac{a^2b^2}{a^2+b^2}$.
(iv) Hence show that the area of $\triangle PMN$ is $\frac{a^3b^3}{(a^2+b^2)^2}$ square units.
(b)

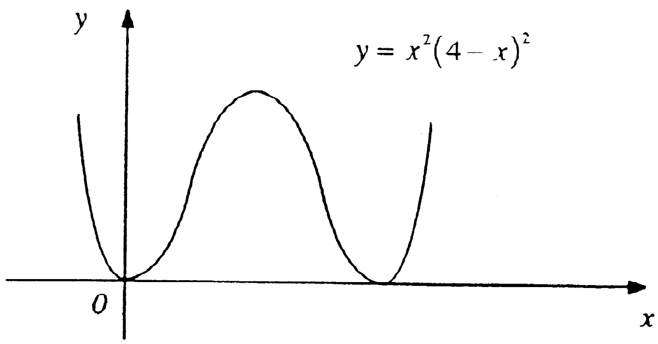


The diagram shows two tangents PT and PS drawn to a circle from a point P outside the circle. Through T, a chord TA is drawn parallel to the other tangent PS. The secant PA meets the circle at E, and TE produced meets PS at F.
(i) Prove that $\triangle EFP|||\triangle PFT$. **(ii)** Hence show that $PF^2 = TF \times EF$.
(iii) Hence or otherwise prove that F is the midpoint of PS.

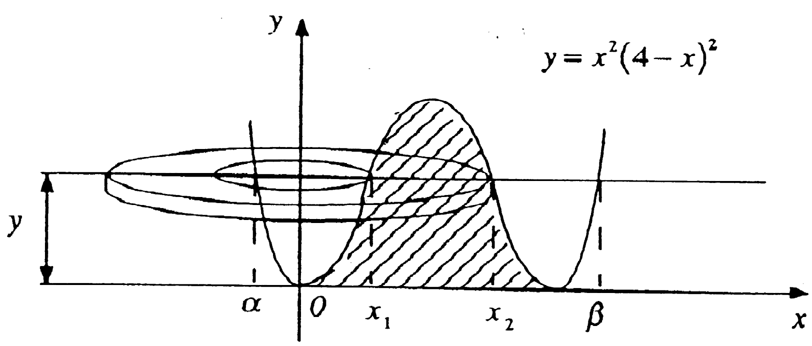
6. **(a)** Let α, β and χ be the roots of $x^3 - x^2 + 2x - 1 = 0$. Write down an equation with roots
(i) $\alpha + \beta, \beta + \chi$ and $\alpha + \chi$
(ii) $\frac{\alpha}{\beta\chi}, \frac{\beta}{\alpha\chi}$ and $\frac{\chi}{\alpha\beta}$.
(b) Conside the hyperbola $xy = c^2$ and the distinct points $P\big(c_1, \frac{c}{t_1}\big)$ and $Q\big(ct_2, \frac{c}{t_2}\big)$ on it.
(i) Show that the equation of the tangent at $\big(ct, \frac{c}{t}\big)$, where $t \neq 0$, is $x + t^2y = 2ct$.

(ii) Show that the tangents at P and Q intersect at $M\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.
(iii) Show that if $t_1t_2 = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin.

7. (a) Show that the stationary points of $y = \{f(x)\}^2$ are exactly those points on the curve that have x coordinates which are zeros of either $f(x)$ or $f'(x)$.
(b)



Use the graph of $y = x(4 - x)$ to justify the features shown on the graph above. Copy the graph of $y = x^2(4 - x)^2$ and mark on the coordinate axes the values of x and y at the stationary points.
(c)



The shaded region is rotated through one revolution about the y axis. The volume of the solid formed is found by taking slices perpendicular to the y axis. The typical slice shown in the diagram is at a height y above the x axis.
(i) Deduce that α, x_1, x_2, β, as shown in the diagram, are roots of $x^4 - 8x^3 + a6x^2 - y = 0$.
(ii) Use the symmetry in the graph to explain why $\frac{x_1+x_2}{2} = 2$ and $\frac{\alpha+\beta}{2} = 2$. Hence, by considering the coefficients of the equation in **(i)**, show that $\alpha\beta = -x_1x_2$ and deduce that $x_1x_2 = \sqrt{y}$ and $x_2 - x_1 = 2\sqrt{4 - \sqrt{y}}$.
(iii) Show that the volume of the solid of revolution is given by $V = 8\pi \int_0^{16} \sqrt{4 - \sqrt{y}}\, dy$. Use the substitution $y = (4 - u)^2$ to evaluate this integral and find the exact volume.

8. (a) (i) If $p > 0$ and $q > 0$ are positive real numbers, show that $p + q \geq 2\sqrt{pq}$.
(ii) Hence show that
(α) $\left(\sqrt[3]{p} + \sqrt[3]{p^2}\right)\left(\sqrt[3]{q^4} + \sqrt[3]{p^5}\right) \geq 4pq$
(β) $\sqrt[3]{\frac{p}{q}} + \sqrt[3]{\frac{q^2}{p^2}} + \sqrt[3]{\frac{q^4}{p^4}} + \sqrt[3]{\frac{p^5}{q^5}} \geq 4$
(b) Amy and Zoe both applied for Olympic Games tickets to the finals in Athletics, Basketball, Hockey, Swimming and Tennis. When Amy applied, the probability of getting a ticket to any one of these finals was one in five. When Zoe applied, the probability of getting a ticket to any one of the Athletics, Hockey or Tennis finals was still one in five, but the probability of getting a ticket to either one of the Basketball or Swimming finals was one in ten. Find the probability, correct to 4 decimal places, that
(i) Amy gets tickets to exactly four finals.
(ii) Zoe gets a ticket to exactly one final.
(iii) Amy gets tickets to four more finals than Zoe.
(iv) Only one of Amy and Zoe gets tickets to both the Basketball and Swimming finals.
