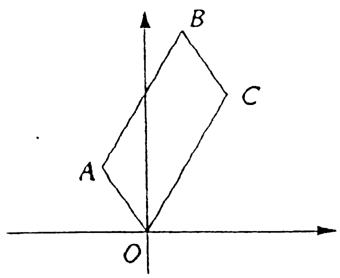
2001

1. **(a)** Find $\int \frac{e^x}{e^x+1}\,dx$ **(b)** Find $\int \frac{e^x}{(e^x+1)^2}\,dx$ **(c)** Find $\int \frac{4x\,dx}{\sqrt{x^4+4}}$ **(d)** Find $\int \frac{dx}{x^2+6x+13}$
(e) Evaluate $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.
(f) **(i)** Use the substitution $x = u^2, u > 0$, to show that $\int_4^{16} \frac{\sqrt{x}}{x-1}\,dx = 4+2\ln 3-\ln 5$.
(ii) Hence use integration by parts to evaluate $\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\,dx$ in simplest exact form.

2. **(a)** Find x and y if $x + iy = \frac{1}{(-1-i\sqrt{3})^{10}}$.
(b)

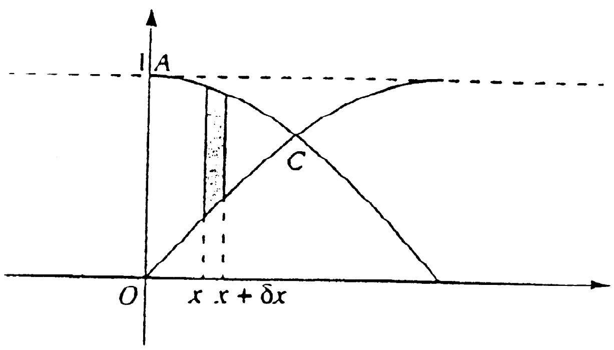


In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$. The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $\angle AOC = 60°$, what complex number does C represent?
(c) In an Argand diagram, the complex number α and $i\alpha$ are represented by the points A and B. z is a variable complex number represented by the point P. $0 < \arg\alpha < \frac{\pi}{2}$.
(i) Draw a diagram showing A, B and the locus of P if $|z-\alpha| = |z-i\alpha|$.
(ii) Draw a diagram showing A, B and the locus of P if $\arg(z-\alpha) = \arg(i\alpha)$.
(iii) Find, in terms of α, the complex number represented by the point of intersection of the two loci in **(i)** and **(ii)**.
(d) It is given that $z = \cos\theta + i\sin\theta$ where $0 < \arg z < \frac{\pi}{2}$.
(i) Show that $z+1 = 2\cos\frac{\theta}{2}\big(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\big)$ and express $z-1$ in modulus/argument form.
(ii) Hence show that $\Re\big(\frac{z-1}{z+1}\big) = 0$.

3. **(a)** Consider the functions $f(x) = |x| + 1$ and $g(x) = \frac{6}{|x|}$.
(i) Solve the equation $f(x) = g(x)$.
(ii) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same set of axes.
(iii) Solve the inequality $g(x) > f(x)$.
(b) $f(x) = 1 - 2\cos x$ for $-\pi \le x \le \pi$. Sketch the graphs of
(i) $y = f(x)$ **(ii)** $y = f(x)^2$ **(iii)** $yO^2 = f(x)$
(c) Consider the function $y = \sin^{-1}(e^x)$
(i) Find the domain and range of the function.

(ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes.

4. **(a)** **(i)** Write down the cartesian equation of the curve with the parametric equations $x = 3\cos\theta$ and $y = 4\sin\theta$.
(ii) Sketch the graph of the curve.
(iii) Find the coordinates of the foci, S_1 and S_2.
(iv) Find the equations of the directrices.
(v) Prove $PS_1 + PS_2 = 8$ where P is any point on the curve.
(b) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y-axis at A, and the C is the first point of intersection of the two graphs to the right of the y-axis.

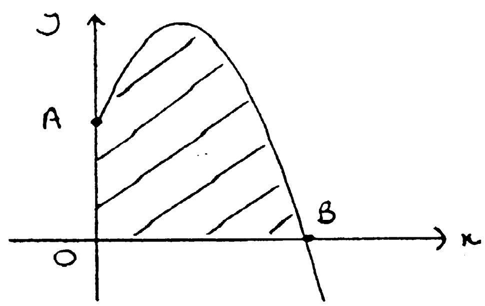


The region OAC is to be rotated about the line $y = 1$.
(i) Write down the coordinates of the point C.
(ii) The shaded stip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by $\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x$.
(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$.

5. **(a)** **(i)** Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2.
(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3.
(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections.
(b) **(i)** Solve the $\cos 5\theta = -1$ for $0 \leq \theta \leq 2\pi$.
(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^2\theta - 20\cos^3\theta + 5\cos\theta$.
(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$.
(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers.

6. **(a)** **(i)** Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$.

(ii) Let $I_n = \int \frac{t^n}{1+t^n}\ dt$. Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2},\ n > 2$.
(iii) Show that $\int_0^1 \frac{t^6}{1+t^2}\ dt = \frac{13}{15} - \frac{\pi}{4}$.
(b)



The graph shows part of the curve whose parametric equations are $x = 2t + 1$, $y = (5 - 2t)(3 + 2t)$.
(i) Find the values of t corresponding to the points A and B on the curve.
(ii) The volume V of the solid formed by rotating the shaded area about the y axis is to be calculated using cylindrical shells. Express V in the form $V = 2\pi \int_a^b f(t)\ dt$. Specify the limits of integration a and b and the function $f(t)$. You may leave $f(t)$ in unexpanded form. Do NOT evaluate this integral.
(c) $P\big(cp, \frac{c}{p}\big)$ and $Q\big(cq, \frac{c}{q}\big)$ are two points on the hyperbola $xy = c^2$ where $p, q > 0$.
(i) Show the point of intersection, T of the tangents at P and Q is $\big(\frac{2cpq}{p+q}, \frac{2c}{p+q}\big)$. [You may assume the equation of the tangent at P is $x + p^2y = 2cp$.]
(iii) The chord PQ produced, passes through $(0, c)$. Find the equation of the locus of T **precisely**. [You may assume the equation of the chord PQ is $x + pqy = c(p+q)$.]

7. (a) A, B and C are three distinct points on a horizontal straight line that is on the same level as the foot P of a vertical tower PQ of height h. The distances AB and BC are both equal to d and the angles of elevation of the top Q of the tower from the points A, B, C are equal to α, β, γ respectively.
(i) If the line ABC passes through the foot P of the tower so that A, B, C are all on the same side of P, show that $2\cot\beta = \cot\alpha + \cot\gamma$.
(ii) If the line ABC does not pass through the foot P of the tower, use the cosine rule in each of $\triangle ABP, \triangle CBP$ to show that $h^2(\cot^{\alpha} - 2\cot^2\beta + \cot^2\gamma) = 2d^2$.
(b) After t minutes the number N of bacteria in a culture is given by $N = \frac{a}{1+be^{-ct}}$ for some constants $a > 0, b > 0$ and $c > 0$. Initially there are 300 bacteria in the culture and the number of bacteria is initially increasing at a rate of 20 per minute. As t increases indefinitely the number of bacteria in the culture approaches a limiting value of 900.
(i) Show that $\frac{dN}{dt} = \frac{c}{a}(a - N)N$.
(ii) Find the values of a, b and c.
(iii) Show that the maximum rate of increase in the number of bacteria occurs when $N = 450$. Sketch the graph of N against t.

8. (a) Two students were asked to find $\frac{dy}{dx}$ for the curve $\frac{x^2}{y}+y=3$. Student A used the quotient rule and found $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$. Student B first multiplied the equation by y and then differentiated to get a different expression for $\frac{dy}{dx}$. Has one of the students made a mistake or can the two students be reconciled? Justify your answer.
(b) (i) $(\boldsymbol{\alpha})$ Differentiate $y=\log_e(1+x)$, and hence draw $y=x$ and $y=\log_e(1+x)$ on one set of axes.
$(\boldsymbol{\beta})$ Using this graph, explain why $\log -e(1+x) < x$, for all $x>0$.
(ii) $(\boldsymbol{\alpha})$ Differentiate $y=\frac{x}{1+x}$, and hence draw $y=\frac{x}{1+x}$ and $y=\log_e(1+x)$ on one set of axes.
$(\boldsymbol{\beta})$ Using this graph, explain why $\frac{x}{1+x}<\log_e(1+x)$, for all $x>0$.
(iii) Use the inequalities of parts **(i)** and **(ii)** to show that

$$\frac{\pi}{8}-\frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2.$$

(c) You may assume that, for all positive real numbers a and b, $\sqrt{ab}\le\frac{a+b}{2}$.
(i) Show that for all positive integers n, ${}^nC_0+{}^nC_1+\cdots+{}^nC_n=2^n$.
(ii) Prove that for all positive integers n, $\left(\sqrt{{}^nC_1}+\sqrt{{}^nC_2}+\cdots+\sqrt{{}^nC_n}\right)^2\le n(2^n-1)$.
You may use the identity $(x_1+x_2+\cdots+x_n)^2=(x_1^2+x_2^2+\cdots+x_n^2)+\sum_{i<j}2x_ix_j$.
