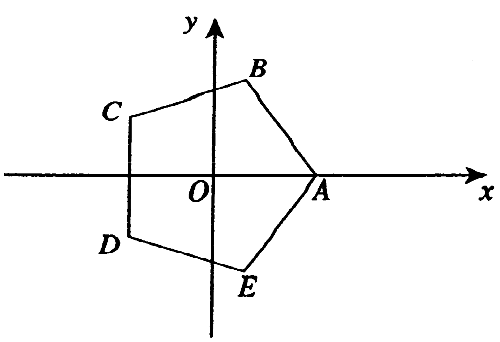
2002

1. **(a)** Find $\int \frac{dx}{\sqrt{9+4x^2}}$.
(b) **(i)** Find real constants A, B and C such that $\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$.
(ii) Hence find $\int \frac{x^2+5x+2}{(x^2+1)(x+1)}\,dx$.
(c) Evaluate $\int_1^5 x\sqrt{2x-1}\,dx$.
(d) Evaluate $\int_0^1 x^5 e^{x^3}\,dx$.
(e) **(i)** Simplify $\sin(A-B)+\sin(A+B)$.
(ii) Hence find $\int \sin 5x \cos 3x\,dx$.

2. **(a)** Let $z = \frac{2-4i}{1+i}$.
(i) Find $\overline{z}$, giving your answer in the form $a+bi$, where a and b are real.
(ii) Find iz.
(b) Find a and b if $(a+ib)^2 = 3-4i$, where a and b are real and $a>0$.
(c) Consider the region defined by $|z-4i| \le 3$.
(i) Sketch the region.
(ii) Determine the maximum value of $|z|$.
(iii) Determine the maximum value of $\arg z$, where $-\pi < \arg z \le \pi$.
(d)



In the diagram above, the complex numbers z_0, z_1, z_2, z_3, z_4 are represented by the vertices of a regular polygon with centre O and vertices A, B, C, D, E respectively. Given that $z_0 = 2$:
(i) Express z_2 in modulus-argument form.
(ii) Find the value of z_2^5.
(iii) Show that the perimeter of the pentagon is $20\sin\frac{\pi}{5}$.

3. **(a)** Let α, β and γ be the roots of $x^3 - 7x^2 = 18x - 7 = 0$.
(i) Find a cubic equation that has roots $1+\alpha^2, 1+\beta^2$ and $1+\gamma^2$.
(ii) Hence, or otherwise, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$.
(b) a lifebelt mould is made by rotating the circle $x^2+y^2=64$ through one complete revolution about the line $x=28$.
(i) Use the method of slicing to show that the volume, V, of the lifebelt is given by $V = 112\pi\int_{-8}^{8}\sqrt{64-y^2}\,dy$.

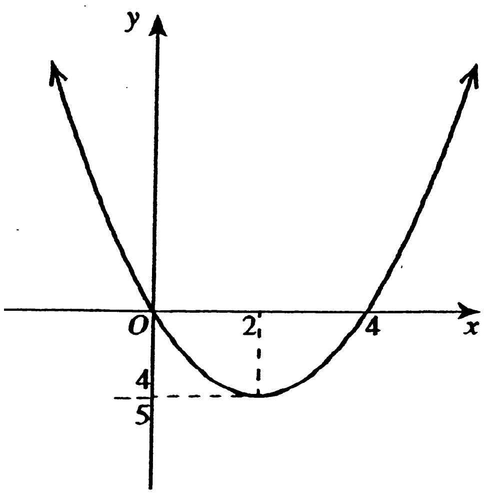
(ii) Find the exact volume of the lifebelt.

(c) Let $P(x) = x^4 + ax^3 + 36x^2 - 35x + b$, where a and b are real numbers. It is known that $x = 5$ and $x = \frac{1-i\sqrt{5}}{2}$ are zeroes of $P(x)$.

(i) Explain why $x^2 - x + 1$ must be a factor of $P(x)$.

(ii) Find a and b.

4. (a)

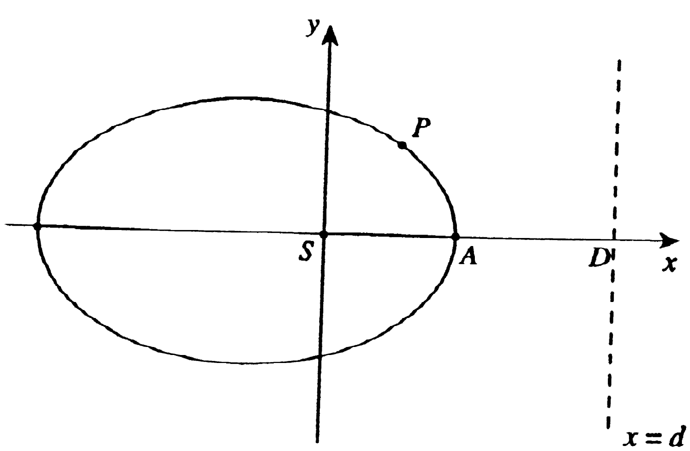


The sketch above shows the parabolic curve $y = f(x)$ where $f(x) = \frac{x^2-4x}{5}$. Without the use of calculus, draw sketches of the following, showing interceps, asymptotes and turning points:

(i) $y = |f(x)|$, **(ii)** $y = \frac{1}{f(x)}$, **(iii)** $y = \frac{x}{5}|x-4|$, **(iv)** $y = \tan^{-1}(f(x))$.

(b) Find the set of values of x for which the limiting sum exists for the series $1 + (\frac{2x-3}{x+1}) + (\frac{2x-3}{x+1})^2 + (\frac{2x-3}{x+1})^3 + \cdots$

5. (a)

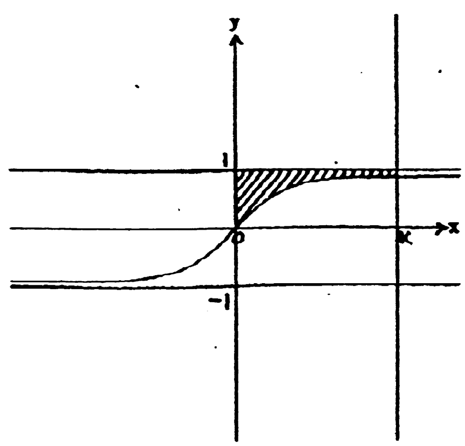


Consider the ellipse sketch obove of eccentricity e with one focus S at the origin and its corresponding directrix at $x = d$.

(i) If P corresponds to the complex number z, where $z = r(\cos\theta + i\sin\theta)$, use the

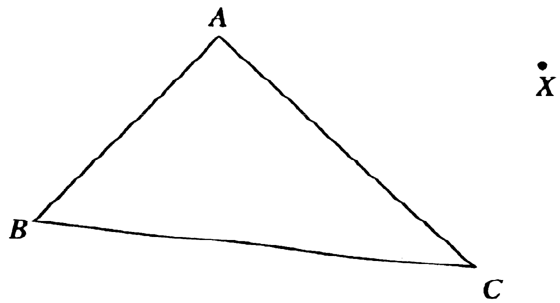
focus-directrix definition of an ellipse to show that $r = \frac{ed}{1+e\cos\theta}$.
(ii) Hence draw the ellipse represented by $r = \frac{33}{5+3\cos\theta}$ showing the coordinates of the points A and D. [There is no need to find the coordinates of any other point, or to write the Cartesian equation of the ellipse.]
(b) Consider the curve defined by the equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$.
(i) Show that $\frac{dy}{dx} = \frac{3x-y-4}{x-y}$.
(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$.
(c) Consider the complex number $z = \cos\theta + i\sin\theta$.
(i) Using de Moivre's theorem, show that $z^n + \frac{1}{z^2} = 2\cos n\theta$, for any integer n.
(ii) Hence or otherwise express $(z+\frac{1}{z})^6$ in the form $A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$, where A, B, C and D are real constants.
(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \cos^6\theta \, d\theta$.

6. (a)



The sketch above shows the curve with equation $y = \frac{e^{2x}-1}{e^{2x}+1}$ which has asymptotes at $y = \pm 1$. The line $x = K$ (where $K > 0$) is also shown.
(i) Using the substitution $u = e^{2x}$ or otherwise, show that the shaded area is given by $A = \ln 2 + 2K - \ln(e^{2K} + 1)$
(ii) Explain why this area is always less than $\ln 2$ no matter how large K is.
(b) **(i)** Use the substitution $u = 1 + x$ to evaluate $\int_0^1 x(1+x)^n \, dx$
(ii) Use the binomial theorem to write an expansion of $x(1+x)^n$
(iii) Prove that $\sum_{r=0}^{n} \frac{1}{r+2} {}^nC_r = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$.
(iv) Find the largest integer value of n such that $\sum_{r=0}^{n} \frac{1}{r+2} {}^nC_r < 50$

7. (a)



The diagram above shows a point X outside a triangle ABC. Show that $AX + BX + CX > \frac{AB+BC+CA}{2}$.

(b) (i) Show that the normal at the point $P(cp, \frac{c}{p})$ to the rectangular hyperbola $xy = c^2$ is given by $p^3x - py = c(p^4 - 1)$.

(ii) If this normal meets the hyperbola again at $Q(cq, \frac{c}{q})$, show that $p^3q = -1$.

(iii) Hence find the area of the triangle PQR, where R is the point of intersection of the tangent at P with the y-axis. You may assume that the equation of the tangent is given by $x + p^2y = 2cp$.

(iv) What is the value of p that produces a triangle of minimum area?

(c) (i) Using t results, or otherwise, find $\int \frac{1}{1+\sin x}\ dx$.

(ii) Hence find $\int_0^{\frac{\pi}{3}} \frac{2}{2+\sin x+\sqrt{3}\cos x}\ dx$.

8. (a) P, Q represent complex numbers α, β respectively in an Argand diagram, where O is the origin and O, P, Q are not collinear. In $\triangle OPQ$, the line from O to the midpoint M of PQ meets the line from Q to the midpoint N of OP in the point R, where R represents the complex number z.

(i) Show this information on a sketch.

(ii) Explain why there are positive real numbers k, l such that $kz = \frac{1}{2}(\alpha + \beta)$ and $l(z - \beta) = \frac{1}{2}\alpha - \beta$

(iii) Show that $z = \frac{1}{3}(\alpha + \beta)$

(iv) If S is the midpoint of OQ show that R lies on PS.

(b) Let $J_n = \int_0^1 x^n e^{-x}\ dx$, where $n \geq 0$.

(i) Show that $J_0 = 1 - \frac{1}{e}$.

(ii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$, for $n \geq 1$.

(iii) Show that $J_n \to 0$ as $n \to \infty$.

(iv) Deduce by the principle of mathematical induction that for all $n \geq 0$, $J_n = n! - \frac{n!}{e}\sum_{r=0}^{n} \frac{1}{r}$.

(v) Conclude the $e = \lim\limits_{n\to\infty} (\sum_{r=0}^{n} \frac{1}{r!})$.