HSC NUMBER:______________________________

TAYLORS COLLEGE SYDNEY CAMPUS

YEAR 12 HSC ASSESSMENT TASK

MATHEMATICS EXTENSION 2

TRIAL HSC EXAMINATION

August 2003

WEIGHTING: 40%

Time Allowed: 3 hours (plus 5 mins reading time)

INSTRUCTIONS

- START EACH QUESTION IN A NEW ANSWER BOOKLET
- WRITE YOUR HSC NUMBER AT THE TOP OF ANSWER BOOKLET
- SHOW ALL NECESSARY WORKING
- APPROVED TEMPLATES AND CALCULATORS MAY BE USED

Question 1 | *Marks*

(a) Find:

(i) $\int \sec^2 x(\tan^2 x + 2)\, dx$ | **2**

(ii) $\int \frac{x}{1+x^4}\, dx$ | **2**

(iii) $\int \frac{dx}{2+\cos x}$ using the substitution $t = \tan\frac{x}{2}$ | **3**

(b) Find the exact value of $\int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx$ | **2**

(c) (i) Let $I_n = \int_0^1 x^n e^x dx$, where $n \geq 0$. Show that | **3**

$$I_n = e - n I_{n-1}, \text{ for all } n \geq 1$$

(ii) Hence evaluate $\int_0^{\frac{1}{5}} y^3 e^{5y}\, dy$ | **3**

Question 2 *(START A NEW ANSWER BOOK)*

(a) Let $z = 3 - 4i$ and $w = 2 + 5i$. Express the following in the form $x + iy$, where x and y are real numbers:

(i) z^2 | **1**

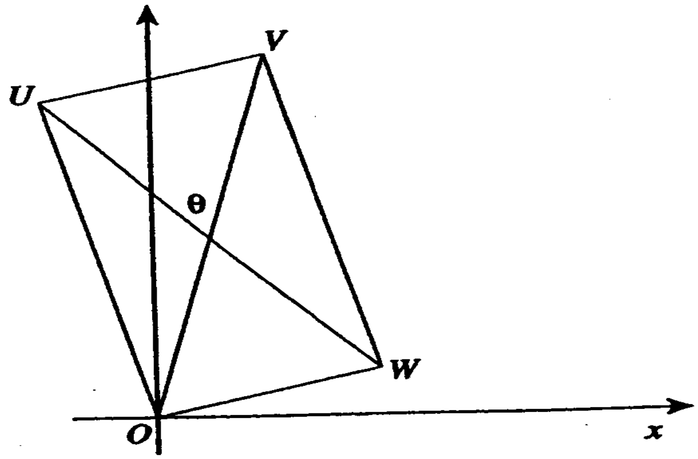
(ii) $\frac{z}{w}$ | **2**

(b) (i) On an Argand diagram shade the region containing all the points representing complex numbers z such that both | **2**

$$|z| \leq 1 \text{ and } |z-1| \leq \sqrt{2}$$

(ii) Find the exact value of the area of the shaded region. | **2**

(c) **Marks**



The diagram above shows a parallelogram $OUVW$ in the complex plane.
Let u, v and w be the complex numbers represented by the points U, V and W respectively.

Suppose that $u\overline{w}+\overline{u}w=0$

(i) Show that $\text{Re}(u\overline{w})=0$ and hence that $\text{Re}(\frac{u}{w})=0$ *3*

(ii) Show that $OUVW$ is a rectangle. *2*

(iii) Suppose now that $\frac{u}{w}=2i$

(α) Express $\frac{u-w}{u+w}$ in the form $a+ib$, where a and b are real numbers. *2*

(β) Hence find the value of $\tan\theta$, where θ is the acute angle between the diagonals of $OUVW$. *1*

Question 3 *(START A NEW ANSWER BOOKLET)*

(a) The polynomial $P(z)$ is defined by $P(z)=z^4-2z^3-z^2+2z+10$
Given that $z-2+i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors. *3*

Marks

(b) A sequence $u_1, u_2, u_3, u_4 \ldots\ldots$ satisfies the relationship $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$.

(i) Show that $u_1 u_2 + u_2 u_3 = u_3^2 - u_1^2$ **2**

(ii) Use induction to show that, for $n \geq 1$,

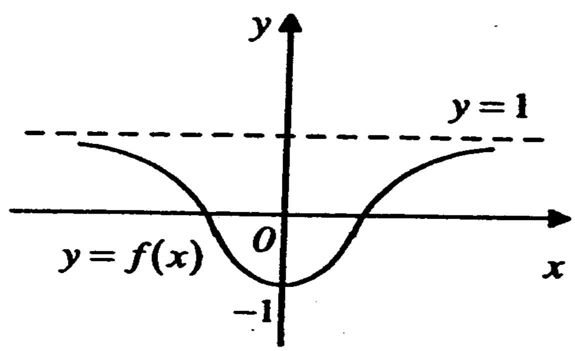
$$u_1 u_2 + u_2 u_3 + u_3 u_4 + \ldots + u_{2n-1} u_{2n} + u_{2n} u_{2n+1} = u_{2n+1}^2 - u_1^2$$ **5**

(c) (i) Suppose that $x = \alpha$ is a double root of the polynomial equation $P(x) = 0$. Show that $P'(\alpha) = 0$ **2**

(ii) The polynomial $Q(x) = mx^7 + nx^6 + 1$ is divisible by $(x+1)^2$. Find the values of m and n, where m and n are real numbers. **3**

Question 4 *(START A NEW ANSWER BOOKLET)*

(a) The diagram below shows the graph of $y = f(x)$ where $f(x) = 1 - 2e^{-x^2}$.



(i) Find the x intercepts **1**

(ii) Draw, on separate diagrams, neat sketches of:

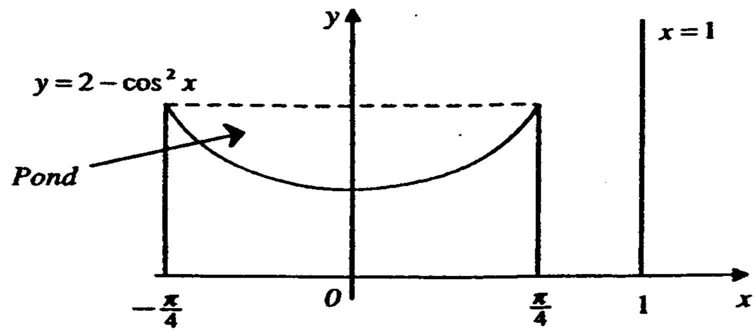
(α) $y = |f(x)|$ **2**

(β) $y = \dfrac{1}{f(x)}$ **3**

(χ) $y = \cos^{-1} f(x)$ **2**

Marks

(b)



A mould for a circular fish pond is made by rotating the region bounded by the curve $y = 2 - \cos^2 x$ and the x axis between $x = -\dfrac{\pi}{4}$ and $x = \dfrac{\pi}{4}$ through one complete revolution about the line $x = 1$. All measurements are in metres.

(i) Use the method of cylindrical shells to show that the volume of the fish pond is given by

$$V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)\cos 2x\,dx$$ **4**

(ii) Hence find the capacity of the fish pond to the nearest litre. **3**

Question 5 *(START A NEW ANSWER BOOKLET)*

(a) Consider the rectangular hyperbola $x^2 - y^2 = 4$.

(i) Find the coordinates of the foci S and S' and the equations of the asymptotes. **2**

(ii) Sketch the curve, showing vertices, foci and asymptotes. **1**

(b) (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. **1**

(ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. **2**

(iii) Hence, or otherwise, prove that

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$$

where a, b and c are distinct positive real numbers. **2**

Marks

(c) The normal at a point $P(cp, \frac{c}{p})$ on the hyperbola $xy = c^2$ meets the x axis at Q. Let M be the midpoint of PQ.

(i) Show that the normal at P has equation $p^3x - py = c(p^4 - 1)$. **2**

(ii) Show that M has coordinates $(\frac{c(2p^4 - 1)}{2p^3}, \frac{c}{2p})$. **2**

(iii) Hence, or otherwise, find the equation of the locus of M. 2

Question 6 *(START A NEW ANSWER BOOKLET)*

(a) Consider the function $f(x) = \frac{x}{1 - x^2}$

(i) Show that the function is increasing for all values of x in its domain. **2**

(ii) Sketch the graph of $y = f(x)$ showing the intercepts on the axes and the equations of any asymptotes. **2**

(iii) Find the values of k such that the equation $\frac{x}{1 - x^2} = kx$ has three distinct real roots. **2**

(b) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has a magnitude $\frac{1}{10}mv^2$ Newtons when the speed of the particle is v ms^{-1}.
After t seconds, the particle has fallen x metres, and has a velocity v ms^{-1} and acceleration a ms^{-2}.
The particle hits the ground $\ln(1 + \sqrt{2})$ seconds after it is dropped.
Take $g = 10$ ms^{-2}.

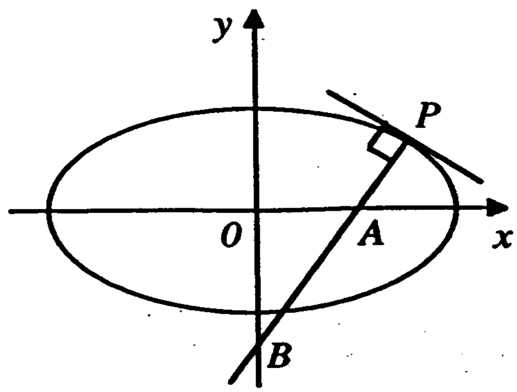
(i) Draw a diagram showing the forces acting on the particle and deduce that $a = \frac{1}{10}(100 - v^2)$. **2**

(ii) Express v as a function of t. Hence find the speed with which the particle hits the ground. Give your answer in simplest exact form. **4**

(iii) Find, in simplest exact form, the distance fallen by the particle before it hits the ground. **3**

Question 7 ***(START A NEW ANSWER BOOKLET)***

(a)



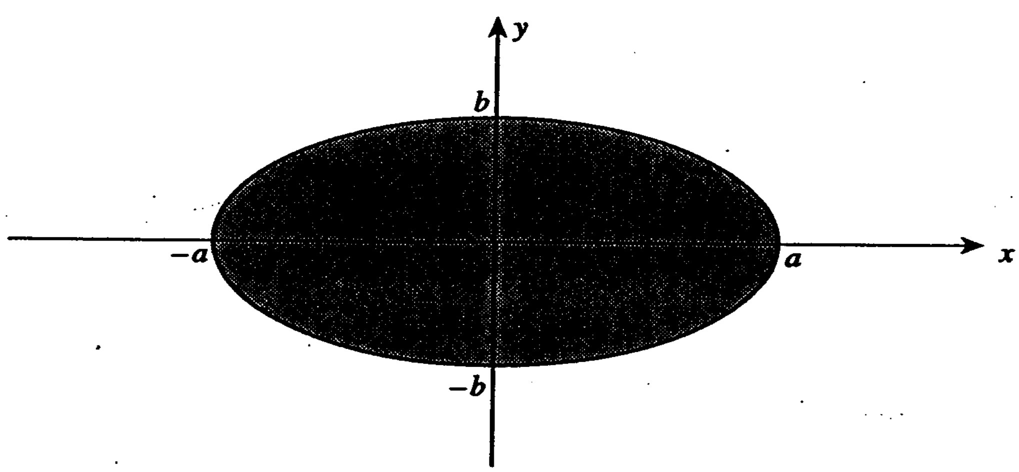
$P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. The normal at P cuts the x axis at A and the y axis at B.

(i) Show that the normal at P has equation $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$ **1**

(ii) Show that triangle OAB has area $\frac{(a^2 - b^2)^2}{2ab}\sin\theta\cos\theta$ **2**

(iii) Find the maximum area of triangle OAB and the coordinates of P when this maximum occurs. **3**

(b)



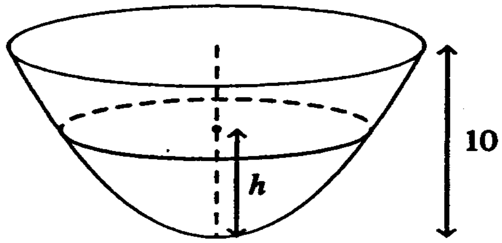
The diagram above shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with major diameter $2a$ and minor diameter $2b$, where a and b are positive real numbers.

(i) Show that the shaded area of the ellipse is given by $\frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$. **2**

(ii) Hence show that the shaded area is πab square units. **2**

Marks

(iii)



The diagram above shows a solid of height 10 cm. At height h cm above the vertex, the cross-section of the solid is an ellipse with major diameter $10\sqrt{h}$ cm and minor diameter $8\sqrt{h}$ cm.

α. Show that the cross-section at height h cm above the vertex has area $20\pi h$ cm^2. **2**

β. Find the volume of the solid. **3**

Question 8 *(START A NEW ANSWER BOOKLET)*

(a) (i) Express the roots of the equation $z^5 + 32 = 0$ in modulus-argument form. *3*

(ii) Hence show that *2*

$$z^4 - 2z^3 + 4z^2 - 8z + 16 = \left\{z^2 - (4\cos\frac{\pi}{5})z + 4\right\}\left\{z^2 - (4\cos\frac{3\pi}{5})z + 4\right\}$$

(iii) Hence find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in simplest surd form. *4*

(b) (i) Show that $\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$

Hence show that

$$\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\{\sin(2n+1)\theta - \sin\theta\}$$ *3*

(ii) Hence evaluate $\sum_{r=1}^{100}\cos^2(\frac{r\pi}{100})$ *3*