G. HARRISON

WHITEBRIDGE HIGH SCHOOL

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VIAM QUAERE VITAE

HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

2005

1999

MATHEMATICS

4 UNIT (ADDITIONAL)

Time Allowed: Three hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt all questions
- ALL questions are of equal value
- All necessary working should be shown. Marks may be deducted for careless or badly arranged work.
- Standard integrals are provided
- Board-approved calculators may be used
- Each question is to be returned on a separate sheet of paper clearly labelled, showing your Name and Student Number.

Question 1

Mark

2 a) Find $\int \frac{e^x dx}{e^{2x}+1}$

2 b) Find $\int x^2 \cos\ x\ dx$

3 c) Evaluate $\int_0^{\pi/2} \frac{dx}{2+\cos x}$

4 d) Evaluate $\int_0^1 \frac{3x^2-2x+1}{(x^2+1)(x^2+2)}\,dx$

4 e) Find $\int \frac{x+3}{\sqrt{x^2-2x+5}}\,dx$

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Question 2

a) 2 (i) Express $-1+\sqrt{3}i$ in Modulus-Argument form.

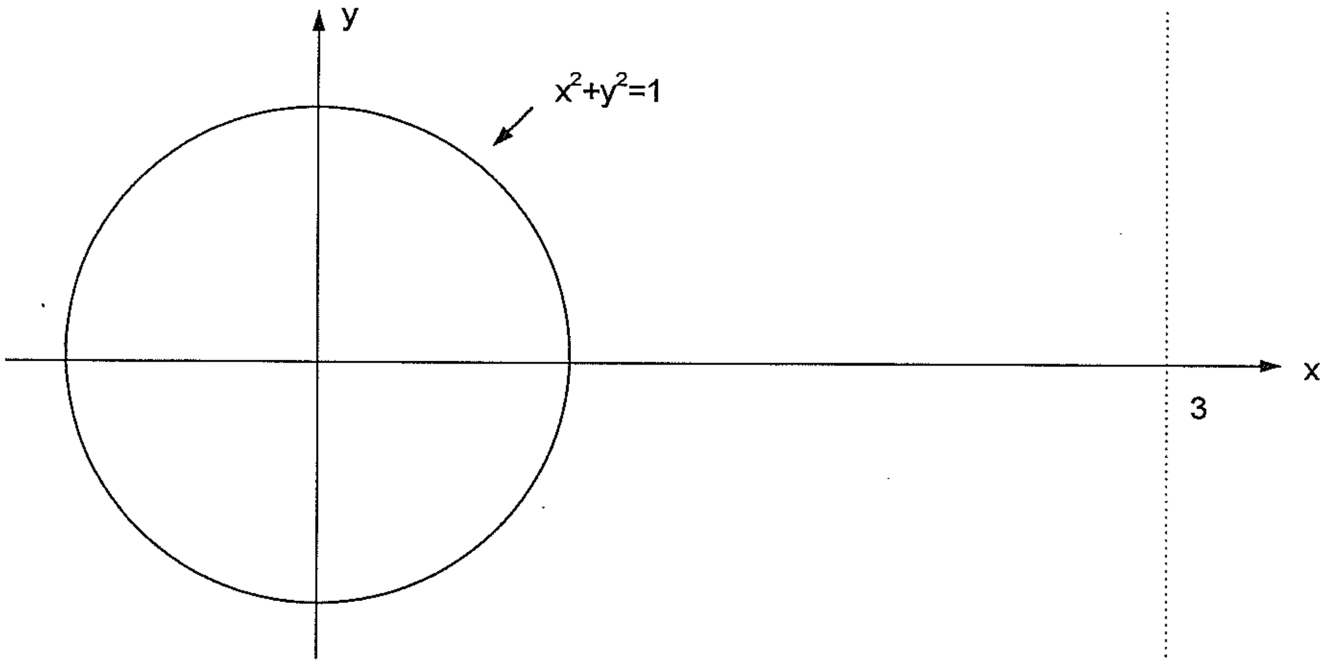
2 (ii) Hence evaluate $\left(-1+\sqrt{3}i\right)^6$

4 b) Find the complex 5th roots of -1.

3 c) Sketch the regions where the inequalities $|z-3+i|\le 5$ and $|z+1|\le|z-1|$ both hold.

4 d) If $\operatorname{Arg}(z-2) = \operatorname{Arg}(z+2) + \frac{\pi}{3}$, show that the locus of the point P representing z on an Argand diagram is an arc of a circle and find the centre and radius of this circle.

Question 3



The circle $x^2+y^2=1$ is rotated about the line $x = 3$ to form a ring (Torus).

Find the volume of the ring by

8 a) the slice technique,

7 b) the method of cylindrical shells.

Question 4

a) The polynomial $z^3 - 7z^2 + 25z - 39$ has one zero equal to $2 + 3i$. Write down its three

3 linear factors.

b) If $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ has a triple zero, find all the zeros and factor $P(x)$ over

4 the real numbers.

c) The equation $x^4 - px^3 + qx^2 - pqx + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Show that

4 $(\alpha+\beta+\gamma)\ (\alpha+\beta+\delta)\ (\alpha+\gamma+\delta)\ (\beta+\gamma+\delta) = 1$

d) α, β, γ are the roots of $x^3 + qx + r = 0$. Find in terms of q, r the equation with roots

4 $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

Question 5

a) Let $f(x) = -x^2 + 5x - 4$

On separate diagrams and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

1 (i) $y = f(x)$

2 (ii) $y = |f(x)|$

2 (iii) $y^2 = f(x)$

2 (iv) $y = \dfrac{1}{f(x)}$

2 (v) $y = e^{f(x)}$

b) Solve the following inequality with the aid of an appropriate sketch.

3 $$|x-1| + |x+1| > 3$$

c) Sketch $y = x\cos x$ where $-2\pi \le x \le 2\pi$

3

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Question 6

a) For the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$

Find 1 (I) the coordinates of the foci

1 (ii) the equations of the directrices

1 (iii) the equations of the asymptotes

3 (iv) the equation of the tangent at $P(2\sec\theta, 2\sqrt{3}\tan\theta)$

3 (v) given that $0 < \theta < \pi/2$ show that the point of intersection of the above tangent and the nearer directrix has $y-$ coordinate $\dfrac{6\cancel{3} - 12\cos\theta}{2\sqrt{3}\sin\theta}$

Question 6 continued

b) $P(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R.

3 (I) Find the equation of the normal at P

3 (ii) Show that P is the midpoint of QR

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Question 7

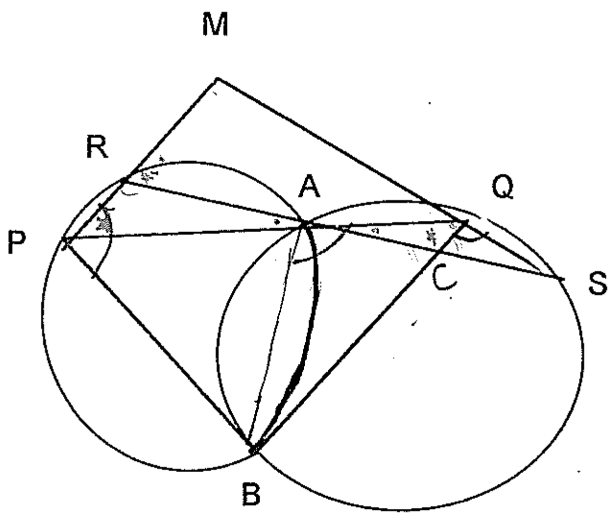
a) If x, y, z are positive real numbers, prove the following:

1 (I) $x^2 + y^2 \geq 2xy$

2 (ii) $\frac{x}{y} + \frac{y}{x} \geq 2$

4 (iii) $x^3 + y^3 + z^3 \geq 3xyz$

b)



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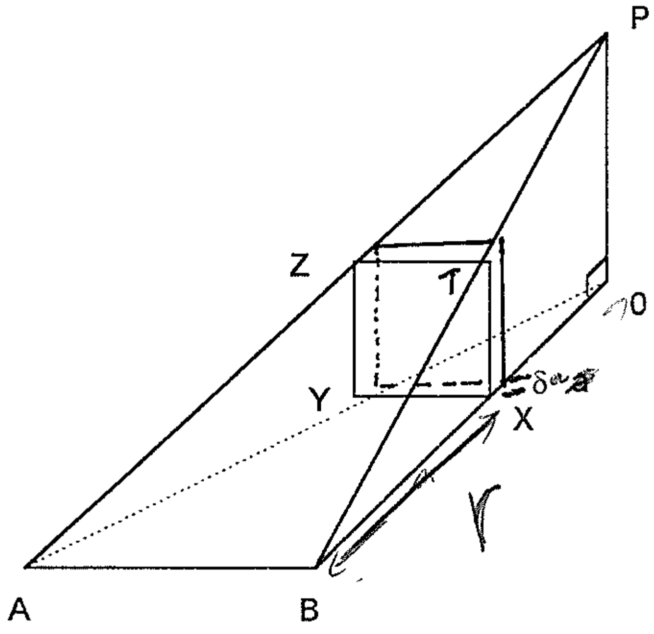
Two circles ARPB, AQSB intersect at A & B. PAQ and RAS are straight lines. PR and SQ are produced to meet at M.

Prove that MPBQ is a cyclic quadrilateral.

4 c) Prove that $\cos^6\theta - \sin^6\theta = \cos 2\theta\left(1 - \frac{1}{4}\sin^2 2\theta\right)$.

Question 8

a)



δa

Let ABO be an isosceles triangle. AO = BO = r, AB = b

Let PABO be a triangular pyramid with height OP = h and OP perpendicular to the plane of ABO as in the diagram.

Consider a slice S of the pyramid of width δa as in the diagram.

The slice S is prependicular to the plane of ABO at XY with XY∥AB and XB = a.

Note XT∥OP.

5 (I) Show that the volume of S is $(\frac{r-a}{r})b(\frac{ah}{r})\delta$ a. When δa is small.

(You may assume that the slice is approximately a rectangular prism of base XYZT and height δa)

3 (ii) Hence show that the pyramid PABO has volume $\frac{1}{6}hbr$.

b) For n = 1, 2, 3, . . . , let Sn = $1+\sum_{r=1}^{n}\frac{1}{r!}$

5 (I) Prove by mathematical induction that $e - Sn = e\int_0^1 \frac{x^n}{n!}e^{-x}dx$

Question 8 continued

2 (ii) From (I) deduce that $0 < e - Sn < \dfrac{3}{(n+1)!}$ for $n = 1,2,3,\ldots$

(N.B. $e < 3$ and $e^{-x} \leq 1$ for $x \geq 0$)

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1999 4-UNIT TRIAL SOLUTIONS

WHITEBRIDGE HIGH

QUESTION 1:

(a) $\int \frac{e^x\,dx}{e^{2x}+1}$ Let $u = e^x$ $\therefore \frac{du}{dx} = e^x$ $du = e^x\,dx$

$= \int \frac{du}{u^2+1}$

$= \tan^{-1} u$

$= \tan^{-1} e^x + C$

(b) $\int x^2 \cos x = x^2 \sin x - \int 2x \sin x$

$= x^2 \sin x - 2\left\{-x\cos x - \int -\cos x \times 1\right.$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

(c) $\int_0^{\pi/2} \frac{dx}{2+\cos x}$ LET $t = \tan \frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2 \frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{2\,dt}{1+t^2} \div 2 + \frac{1-t^2}{1+t^2}$

$= \int_0^1 \frac{2}{1+t^2} \times \frac{1+t^2}{3+t^2}\,dt$

$\int_0^1 \frac{2}{3+t^2}\,dt$

$\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\left\{\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right\}$

$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6}$

$= \frac{\pi}{3\sqrt{3}}$

(d) $\int_0^1 \frac{3x^2-2x+1}{(x^2+1)(x^2+2)}$

$\frac{3x^2-2x+1}{(x^2+1)(x^2+2)} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+2} =$

$= \frac{ax^3+bx^2+2ax+2b+cx^3+dx^2+cx+d}{(x^2+1)(x^2+2)} =$

$= \frac{(a+c)x^3+(b+d)x^2+(2a+c)x+2b+d}{(x^2+1)(x^2+2)} =$

$$\left.\begin{aligned} a + c &= 0 \\ b + d &= 3 \\ 2a + c &= -2 \\ 2b + d &= 1 \end{aligned}\right\} \quad \begin{aligned} a &= -2 \\ c &= 2 \\ b &= -2 \\ d &= 5 \end{aligned}$$

$\therefore \int_0^1 \frac{-2x-2}{x^2+1} + \frac{2x+5}{x^2+2}$

$\int_0^1 \frac{2x}{x^2+2} + \int_0^1 \frac{5}{x^2+2}\,dx$

$= -\int_0^1 \frac{2x}{x^2+1} - \int_0^1 \frac{2}{1+x^2} + \int_0^1 \frac{2x}{x^2+2} + \ln(x^2+2) + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$

$= \left[-\ln(x^2+1) - 2\tan^{-1}x + \ln 3 + \frac{5}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right) - \left(-\ln 1 - 2\tan^{-1} 0 + \ln 2 + \frac{5}{\sqrt{2}}\tan^{-1} 0\right)$

$= \left(-\ln 2 - 2\tan^{-1} 1 + \ln 3 + \frac{5}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} - \ln 2\right.$

$= -\ln 2 - \pi/2 + \ln 3 + 5/\sqrt{2}\,\tan^{-1}\frac{1}{\sqrt{2}}$

$= \ln 3 - 2\ln 2 - \pi/2 + 5/\sqrt{2}\,\tan^{-1}\frac{1}{\sqrt{2}}$

$= \ln 3/4 - \pi/2 + 5/\sqrt{2}\,\tan^{-1}\frac{1}{\sqrt{2}}$

(e) $\int \frac{x+3}{\sqrt{x^2-2x+5}}\,dx = \int \frac{x-1 \;+\; 4}{\sqrt{x^2-2x+5}}\,dx$

$= \int \frac{x-1}{\sqrt{x^2-2x+5}} + \int \frac{4}{\sqrt{x^2-2x+5}}\,dx$

$\int \frac{x-1}{\sqrt{x^2-2x+5}}\,dx$ Let $u = x^2-2x$

$\frac{du}{dx} = 2x-2$

$= 2(x-1)$

$\therefore \frac{du}{2} = (x-1)\,dx$

$= \frac{1}{2}\int \frac{du}{\sqrt{u+5}}$

$= \frac{1}{2}\int (u+5)^{-1/2}\,du$

$= \frac{1}{2} \times 2(u+5)^{\frac{1}{2}}$

$= \sqrt{u+5}$

$= \sqrt{x^2-2x+5}$

$\int \frac{4}{\sqrt{x^2-2x+5}} = \int \frac{4}{\sqrt{x^2-2x+1+4}}$

$= \int \frac{4}{\sqrt{(x-1)^2+4}}\,dx$

$= 4\ln\left[(x-1) + \sqrt{(x-1)^2+4}\right]$

$\therefore \int \frac{x+3}{\sqrt{x^2-2x+5}} = \sqrt{x^2-2x+5} + 4\ln\left[(x-1)+\sqrt{x^2-2x+5}\right] + C$

QUESTION 2:

(a) (i) $|-1+\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = 2$

$ARG(-1+\sqrt{3}i) = \theta$ WHERE $\tan\theta = -\sqrt{3}$

$\therefore \theta = \frac{2\pi}{3}$

$\therefore -1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(ii) $(-1+\sqrt{3}i)^6 = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^6$

$= 2^6\left[\cos 4\pi + i\sin 4\pi\right]$

$= 2^6\left[1+0\right]$

$= 64$

(b) LET $Z^5 = -1$

Let $Z = r(\cos\theta + i\sin\theta)$

$\therefore r^5(\cos 5\theta + i\sin 5\theta) = 1(\cos\pi + i\sin\pi)$

$\therefore r^5 = 1$

$r = 1$

$\therefore 5\theta = \pi + 2k\pi$

$\therefore \theta = \frac{\pi}{5} + \frac{2k\pi}{5}$

If $k=1$: $\theta_1 = \frac{\pi}{5} + \frac{2\pi}{5} = \frac{3\pi}{5}$ $\therefore Z_1 = \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

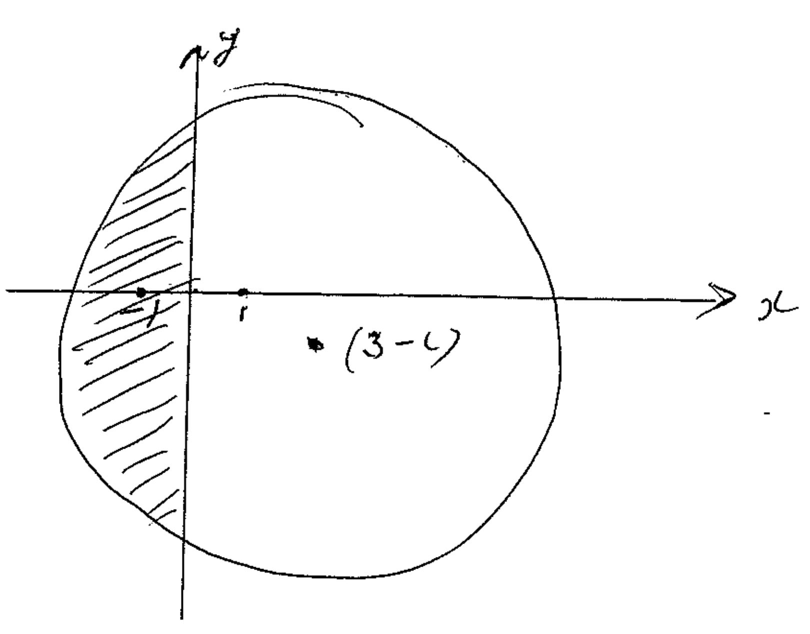
If $k=-1$: $\theta_2 = \frac{\pi}{5} - \frac{2\pi}{5} = -\pi/5$ $\therefore Z_2 = \left(\cos\frac{-\pi}{5} + i\sin\frac{-\pi}{5}\right)$

If $k=0$: $\theta_3 = \pi/5$ $\therefore Z_3 = \left(\cos\frac{\pi}{5} + i\sin\pi/5\right)$

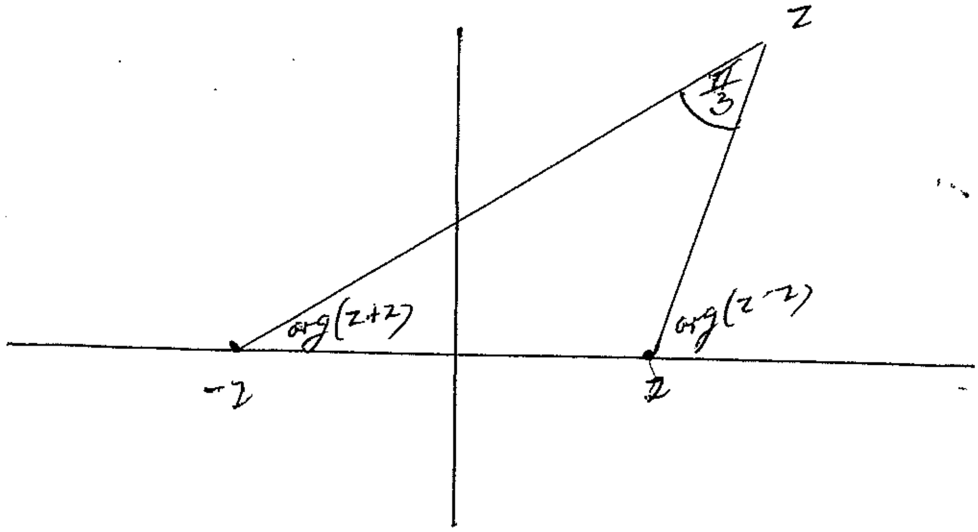
If $k=2$: $\theta_4 = \frac{\pi}{5} + \frac{4\pi}{5} = \pi$ $\therefore Z_4 = (\cos\pi + i\sin\pi)$

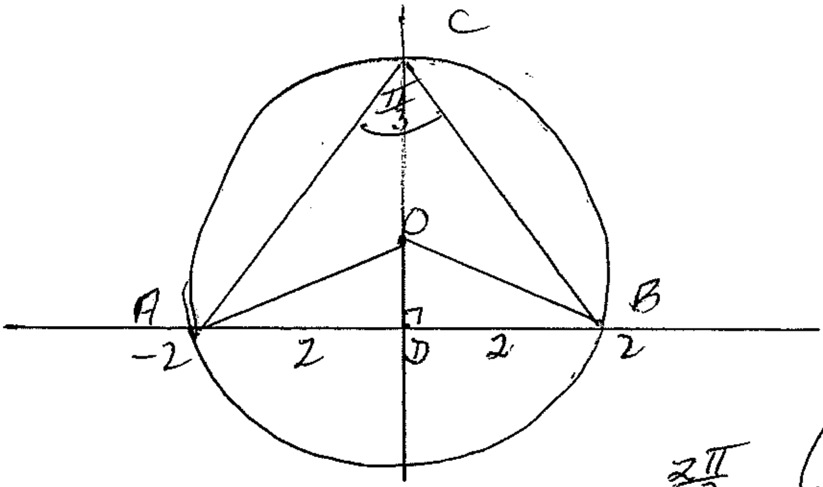
If $k=-2$: $\theta_5 = \frac{\pi}{5} - \frac{4\pi}{5} = -\frac{3\pi}{5}$ $\therefore Z_5 = \left(\cos\frac{-3\pi}{5} + i\sin\frac{-3\pi}{5}\right)$

(c)



(d)





$\text{ANGLE AT CENTRE} = \frac{2\pi}{3}$ (AB = CHORD SUBTENDING A CONSTANT ANGLE $\frac{\pi}{3}$ i.e. ON CIRCUMFERENCE OF CIRCLE)

$\therefore \hat{DOB} = \frac{\pi}{3}$

$\therefore \tan\frac{\pi}{3} = \frac{2}{OD}$

$\therefore OD = \frac{2}{\sqrt{3}}$

$\therefore \text{CENTRE } (0, 2/\sqrt{3})$

$\text{RADIUS} = \sqrt{(2-0)^2 + (0-2/\sqrt{3})^2}$

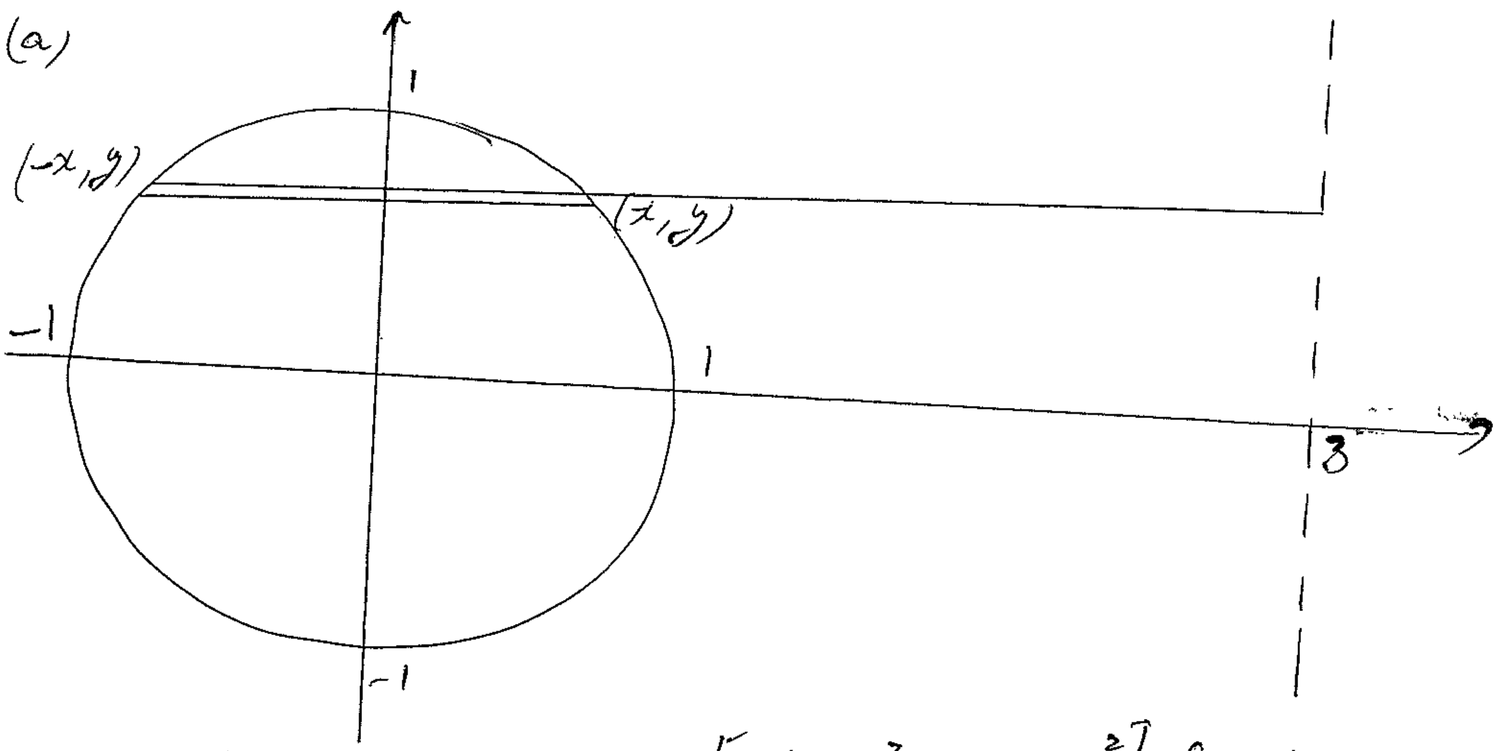
$= \sqrt{4 + 4/3}$

$= \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}$

$\therefore$ ~~EQUATION~~

QUESTION 3 : $\underline{\underline{5}}$

(a)



~~AREA~~ VOLUME OF SLICE $= \left[\pi(3+x)^2 - \pi(3-x)^2\right] \times \delta y$

$$\therefore V = \pi \int_{-1}^{1} 9 + x^2 + 6x - 9 - x^2 + 6x$$

$$= \pi \int_{-1}^{1} 12x \, dy$$

$$= 12\pi \int_{-1}^{1} \sqrt{1-y^2} \, dy$$

Let $y = \sin\theta$

$$\frac{dy}{d\theta} = \cos\theta$$

$$dy = \cos\theta \, d\theta$$

$$= 12\pi \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta} \cos\theta \, d\theta$$

$$= 12\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

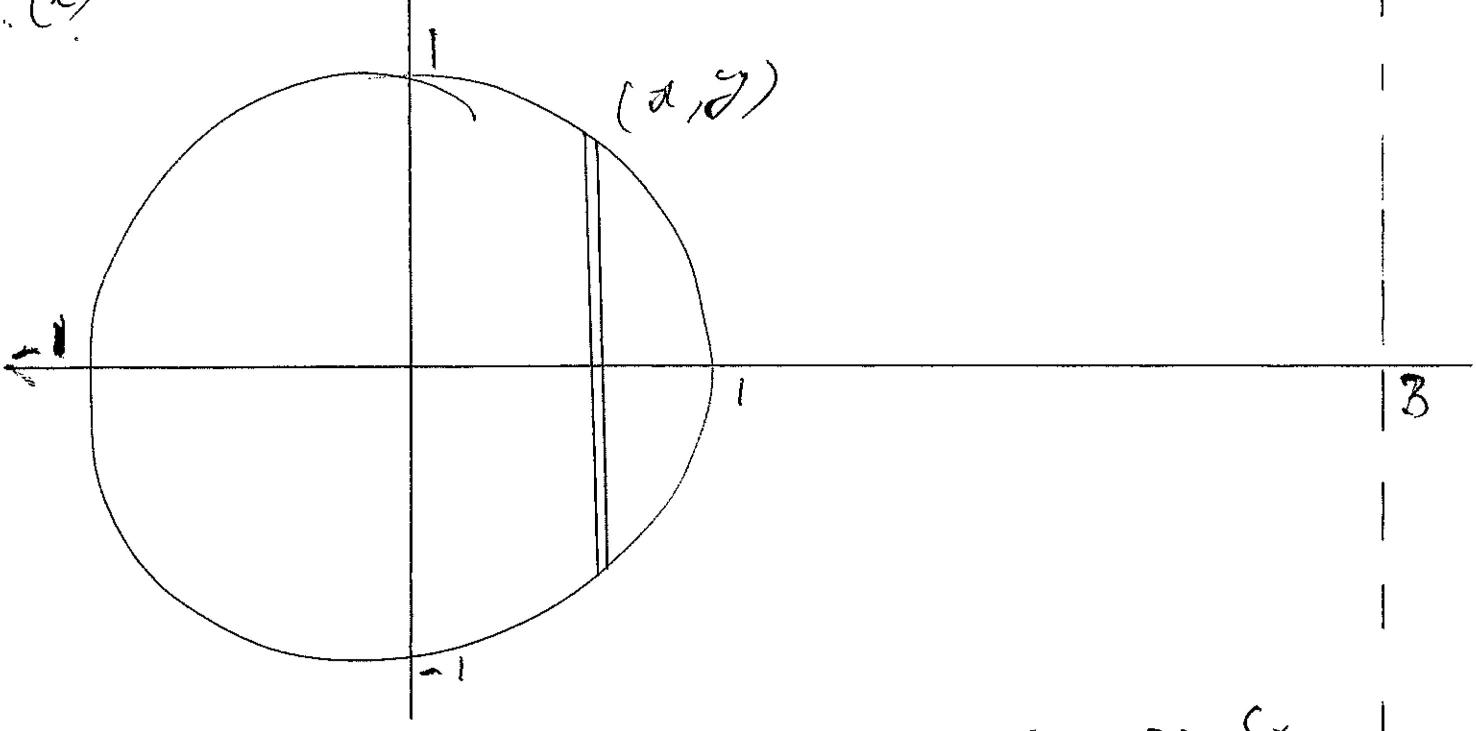
$$= 12\pi \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) \, d\theta$$

$$= 6\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 6\pi \left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right)\right]$$

$$= 6\pi \times \pi = 6\pi^2 \text{ units}^3$$

(b) 6



~~AREA~~ VOLUME OF SLICE $= 2\pi(3-x) \times 2y\,\delta x$

$V = \int_{-1}^{1} 2\pi(3-x)\,2y\,dx$

$= 4\pi\int_{-1}^{1}(3-x)\sqrt{1-x^2}\,dx$

$= 12\pi\int_{-1}^{1}\sqrt{1-x^2}\,dx - 4\pi\int_{-1}^{1} x\sqrt{1-x^2}\,dx$

$= 6\pi^2$ (FROM PART (a)) $-$ Let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$\therefore -\frac{du}{2} = x\,dx$

$\therefore -4\pi\int_{-1}^{1} x\sqrt{1-x^2}\,dx = +\frac{4\pi}{2}\int_{0}^{0}\sqrt{u}\,du$

$=$ no solution

$\therefore V = 6\pi^2$ units^3

QUESTION 4: 7

(a) If $2+3i$ is a root then $2-3i$ is also a root as polynomial has real coefficients.

$\therefore$ Roots are $2+3i$, $2-3i$ & α

Sum of roots $= 2+3i+2-3i+\alpha = 7$

$\therefore \alpha = 3$

$\therefore$ Factors are $[z-(2+3i)][z-(2-3i)][z-3]$

$= (z-2-3i)(z-2+3i)(z-3)$

(b) If $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ has a triple root

$P'(x) = 4x^3 - 9x^2 - 12x + 28$ has a double root

& $P''(x) = 12x^2 - 18x - 12$ has a single root

$\therefore$ If $12x^2 - 18x - 12 = 0$

$2x^2 - 3x - 2 = 0$

$(2x+1)(x-2) = 0$

$x = 2, -\frac{1}{2}$

$P'(2) = 0 \quad \therefore x = 2$ is the triple root

$\therefore$ Roots are $2, 2, 2, \alpha$

Sum of roots $= 6 + \alpha = 3$

$\alpha = -3$

$\therefore$ Zeros are $2, 2, 2, -3$

Factors are $(x-2)^3(x+3)$

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(4) (d) $x^3 + qx + r = 0$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = q$

$\alpha\beta\gamma = -r$

Sum of roots $= \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$= \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{(\alpha\beta\gamma)^2}$

$= \frac{q + 2r \times 0}{r^2}$

$= \frac{q}{r^2}$

Double $= \frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\beta^2\gamma^2}$

$= \frac{\gamma^2 + \alpha^2 + \beta^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{(\alpha + \beta + \gamma)^2 - 2(\alpha\gamma + \beta\alpha + \beta\gamma)}{(\alpha\beta\gamma)^2}$

$= \frac{0 - 2q}{r^2}$

$= \frac{-2q}{r^2}$

Triple $= \frac{1}{\alpha^2\beta^2\gamma^2}$

$= \frac{1}{(\alpha\beta\gamma)^2}$

$= \frac{1}{r^2}$

$\therefore x^3 - \frac{q}{r^2}x^2 - \frac{2q}{r^2}x - \frac{1}{r^2} = 0$

$r^2x^3 - qx^2 - 2qx - 1 = 0$

(c) $x^4 - px^3 + qx^2 - pqx + 1 = 0$

$\alpha + \beta + \gamma + \delta = p$

$\therefore \alpha + \beta + \gamma = p - \delta$

$\therefore (\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta) = (p-\delta)(p-\gamma)(p-\beta)(p-\alpha)$

$= (p^2 - p(\delta+\gamma) + \delta\gamma)(p^2 - p(\alpha+\beta) + \alpha\beta)$

$= p^4 - p^3(\delta+\gamma) + p^2\delta\gamma - p^3(\alpha+\beta) + p^2(\delta+\gamma)(\alpha+\beta) - p(\alpha\delta\gamma + \delta\gamma\beta)$

$+ p^2\alpha\beta - p(\alpha\beta\delta + \alpha\beta\gamma) + \alpha\beta\delta\gamma$

$= p^4 - p^3(\alpha+\beta+\delta+\gamma) + p^2(\delta\gamma + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \alpha\beta)$

$- p(\alpha\delta\gamma + \delta\gamma\beta + \alpha\beta\delta + \alpha\beta\gamma) + \alpha\beta\gamma\delta.$

$= p^4 - p^3 \times p + p^2 \times q - p \times pq + 1$

$= p^4 - p^4 + p^2 q - p^2 q + 1$

$= 1$

$=$ R.H.S.

(d) Let $x = \alpha, \beta, \gamma$ ARE ROOTS OF $x^3 + qx + r = 0$

Let $y = \frac{1}{x^2}$ TO REPRESENT ROOTS $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

$\therefore x^2 = \frac{1}{y}$

$x = \frac{1}{\sqrt{y}}$

$\therefore \left(\frac{1}{\sqrt{y}}\right)^3 + q\left(\frac{1}{\sqrt{y}}\right) + r = 0$

$\frac{1}{y\sqrt{y}} + \frac{q}{\sqrt{y}} + r = 0$

$\therefore \left(\frac{1}{y\sqrt{y}} + \frac{q}{\sqrt{y}}\right)^2 = -r$

$\therefore \frac{1}{y^3} + \frac{q}{y} + \frac{2q}{y^2} = r^2$

$\therefore 1 + qy^2 + 2qy = r^2 y^3$

Let $y = x$:

$\therefore r^2x^3 - qx^2 - 2qx - 1 = 0$

QUESTION 5:

(a) i $f(x) = -x^2 + 5x - 4$

CURVE CUTS X AXIS WHEN $y = 0$

$\therefore -x^2 + 5x - 4 = 0$

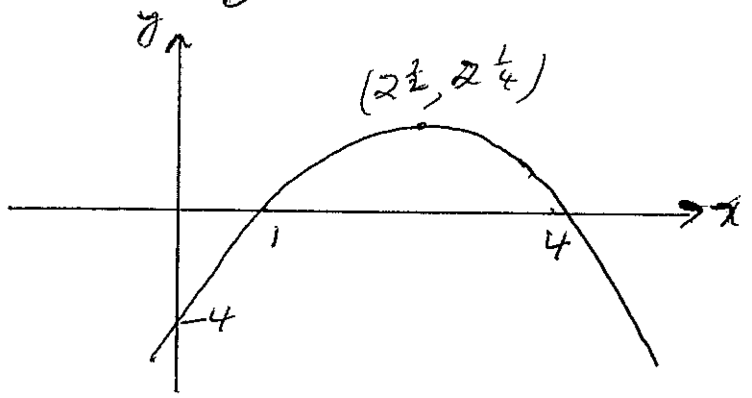
$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 1, 4$

AXIS OF SYM IS $x = \frac{-5}{-2} = 2\frac{1}{2}$

$y = 2\frac{1}{4}$



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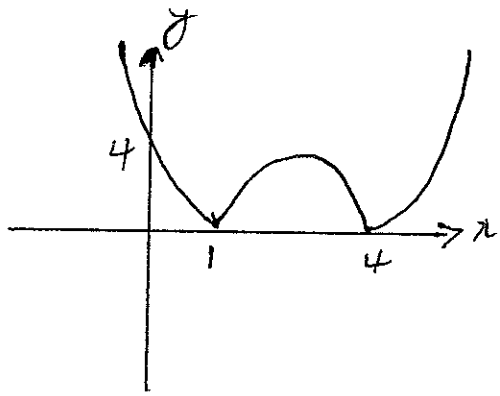
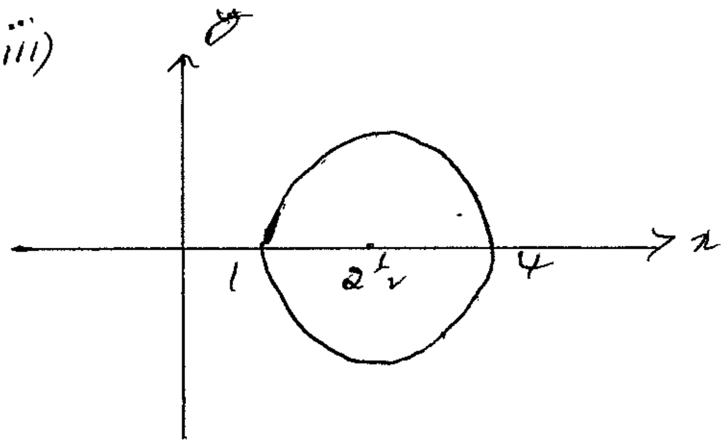
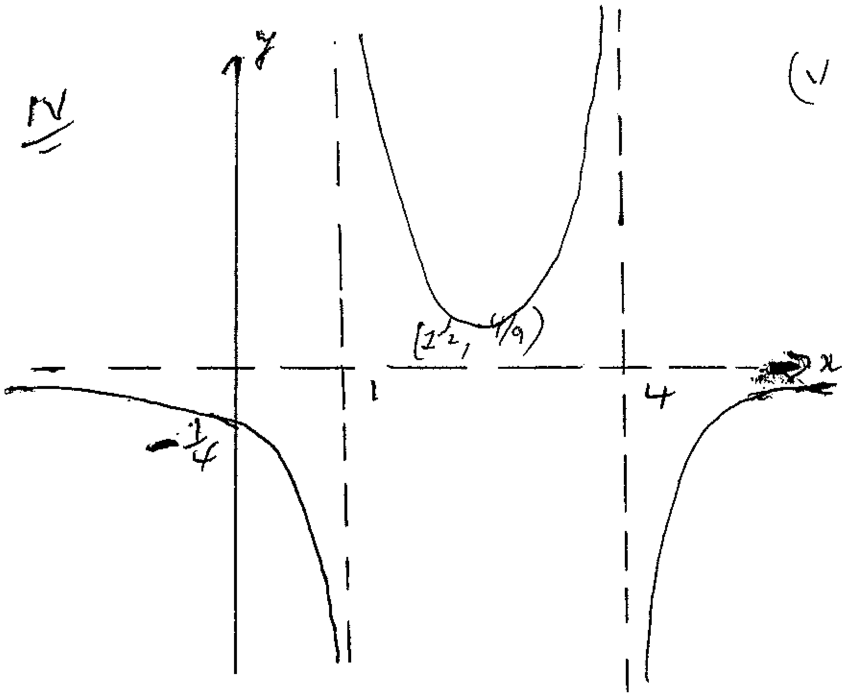


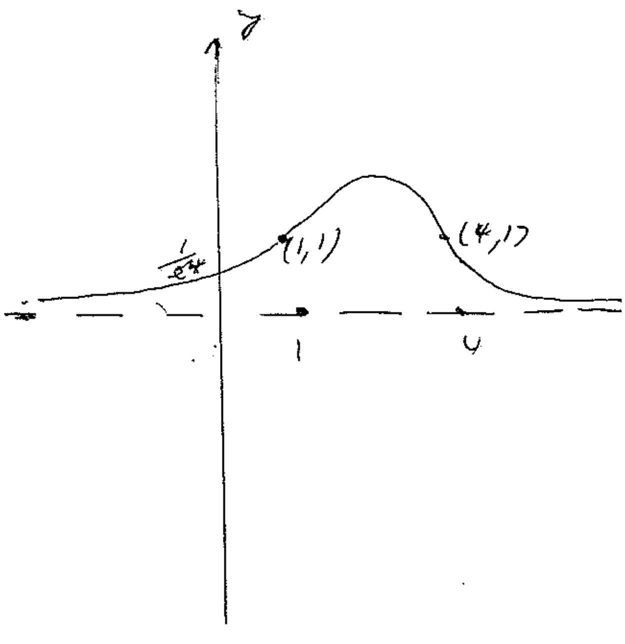
(iii)



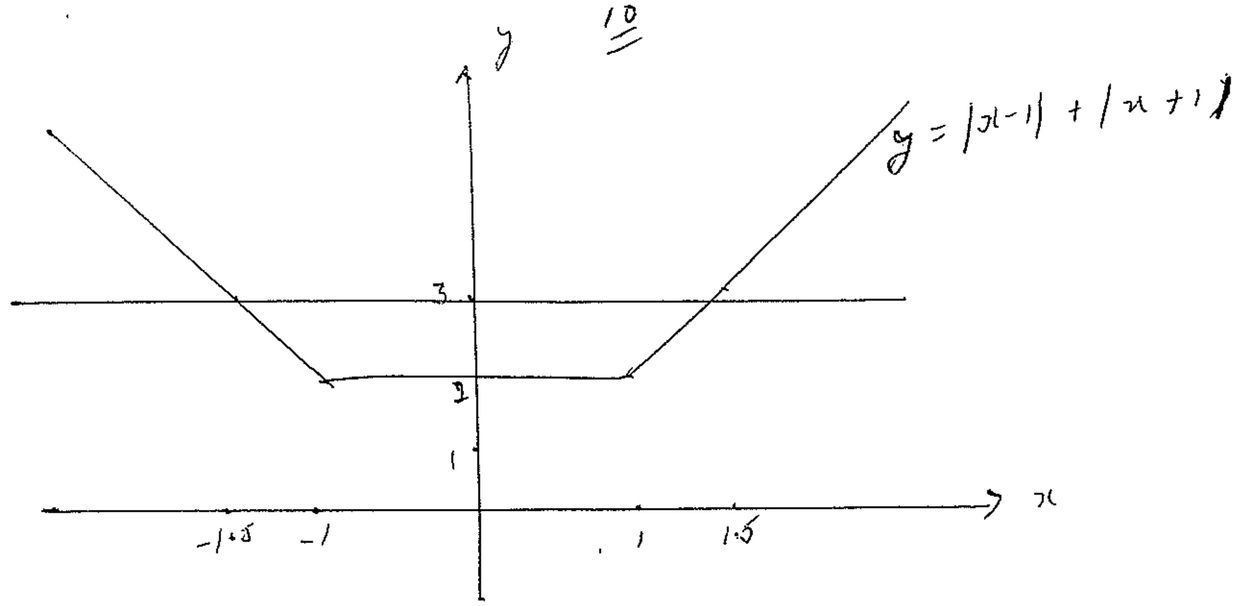
IV



(v)



(b)



C1: $x - 1 + x + 1 = 3$

$2x = 3$

$x = 1\frac{1}{2}$

∴ INEQUALITY HOLDS WHEN $x > 1\frac{1}{2}$

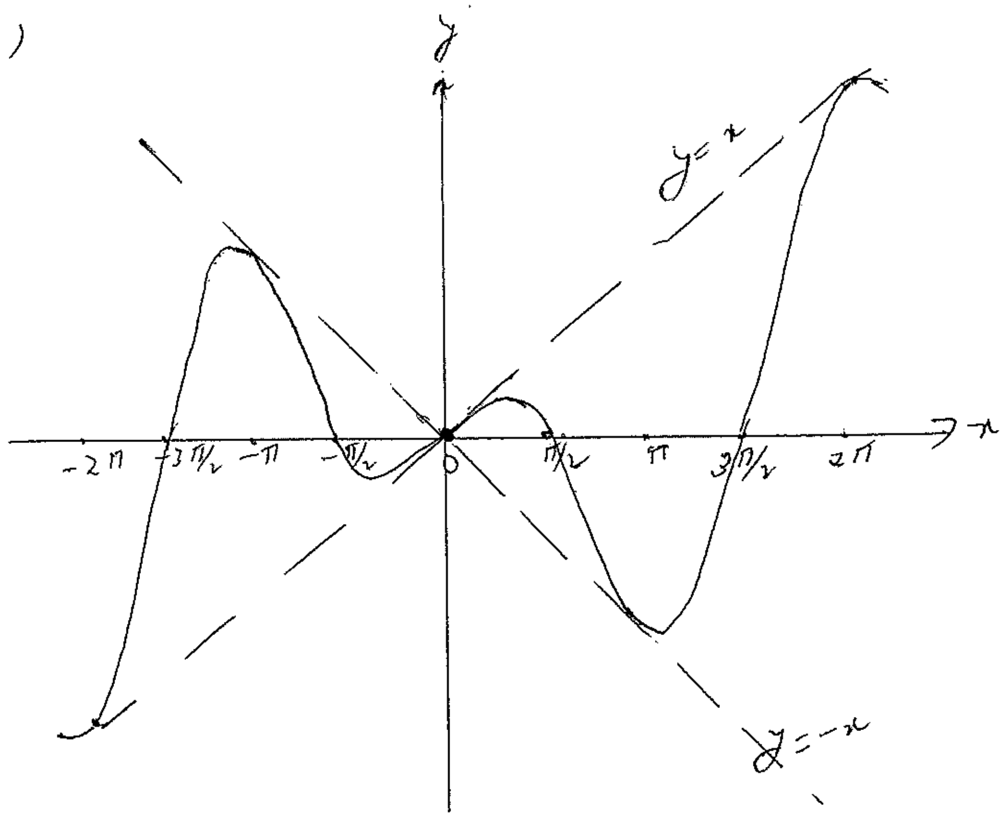
C2: $-x + 1 - x - 1 = 3$

$-2x = 3$

$x = -1\frac{1}{2}$

INEQUALITY HOLDS WHEN $x < -1\frac{1}{2}$

(c)



$f(x) = x \cos x$

$f(-x) = -x \cos(-x)$

$= -x \cos x$

$= -f(x)$

∴ ODD FUNCTION

∴ POINT SYMMETRY ABOUT O.

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QUESTION 6:

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

(i) $a = 2$, $b = \sqrt{12} = 2\sqrt{3}$

$b^2 = a^2(e^2 - 1)$

$12 = 4(e^2 - 1)$

$4e^2 = 16$

$e^2 = 4$, $e = 2$

$\therefore$ FOCI $(\pm ae, 0) = (\pm \cancel{4}, 0)$

(ii) DIRECTRICES: $x = \pm \frac{a}{e}$

$x = \pm 1$

(iii) ASSYMPTOTES: $y = \pm b/a \, x$

$= \pm \frac{2\sqrt{3}}{2} x$

$= \pm \sqrt{3} x$

(iv) GRAD OF TAN $= \frac{dy}{dx}$

$\frac{x}{2} - \frac{y}{6} \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{3x}{y}$

at $(2\sec\theta, 2\sqrt{3}\tan\theta)$ $\frac{dy}{dx} = \frac{6\sec\theta}{2\sqrt{3}\tan\theta}$

$= \frac{3}{\cos\theta} \div \frac{\sqrt{3}\sin\theta}{\cos\theta}$

$= \frac{\sqrt{3}}{\sin\theta}$

$\therefore$ EQU^n OF TANGENT IS

$y - 2\sqrt{3}\tan\theta = \frac{\sqrt{3}}{\sin\theta}(x - 2\sec\theta)$

$\sin\theta y - \frac{2\sqrt{3}\sin^2\theta}{\cos\theta} = \sqrt{3}x - \frac{2\sqrt{3}}{\cos\theta}$

$\therefore \cos\theta\sin\theta y - 2\sqrt{3}\sin^2\theta = \sqrt{3}x\cos\theta - 2\sqrt{3}$

$\cos\theta\sin\theta y - 2\sqrt{3}(1 - \cos^2\theta) = \sqrt{3}x\cos\theta - 2\sqrt{3}$

$\therefore \sqrt{3}x\cos\theta - \cos\theta\sin\theta y = 2\sqrt{3}\cos^2\theta$

$\frac{\sqrt{3}x}{\cos\theta} - \frac{\sin\theta y}{\cos\theta} = 2\sqrt{3}$

$\therefore \frac{x\sec\theta}{2} - \frac{y\tan\theta}{2\sqrt{3}} = 1$

(v) $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{2\sqrt{3}} = 1$ ——— (1)

$x = 1$ ——— (2)

$\therefore \frac{\sec\theta}{2} - \frac{y\tan\theta}{2\sqrt{3}} = 1$

$\therefore \sqrt{3}\sec\theta - y\tan\theta = 2\sqrt{3}$

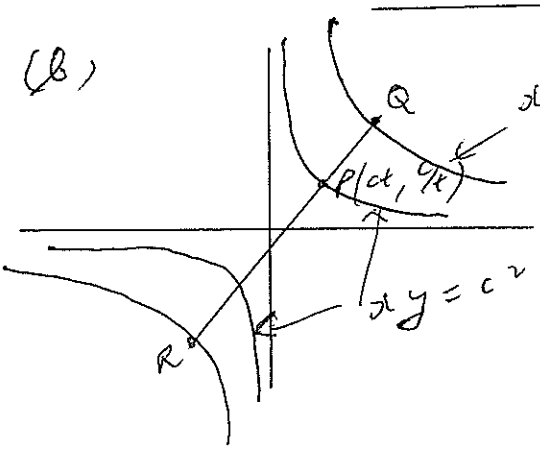
$\frac{\sqrt{3}}{\cos\theta} - \frac{y\sin\theta}{\cos\theta} = 2\sqrt{3}$

$\therefore y\sin\theta = \sqrt{3} - 2\sqrt{3}\cos\theta$

$y = \frac{\sqrt{3} - 2\sqrt{3}\cos\theta}{\sin\theta} \times \frac{2\sqrt{3}}{2\sqrt{3}}$

$= \frac{6 - 12\cos\theta}{2\sqrt{3}\sin\theta}$

(b)



$xy = c^2$

GRAD OF TAN $= \frac{dy}{dx}$

$x\frac{dy}{dx} + y = 0$

$\therefore \frac{dy}{dx} = -y/x$

at P $= -c/t / ct$

$= -\frac{1}{t^2}$

$\therefore$ GRAD OF $\perp = t^2$

$\therefore$ EQU OF $\perp$ IS

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3x - ct^4$

$ty = t^3x - ct^4 + c$

(ii) $ty = t^3x - ct^4 + c$ —— (1)

$x^2 - y^2 = a^2$ ——— (2)

$\therefore x^2 - \left(\frac{t^3x - ct^4 + c}{t}\right)^2 = a^2$

$\therefore t^2x^2 - \left[t^6x^2 - ct^7x + ct^3x - ct^7x + c^2t^8 - c^2t^4 + ct^3x - c^2t^4 + c^2\right] = a^2t^2$

$\therefore x^2(t^2 - t^6) + x(2ct^7 - 2ct^3) - c^2t^8 + 2c^2t^4 + c^2 - a^2t^2 = 0$

THE SOLUTIONS TO THIS EQUATION PRODUCES THE x COORDINATES OF Q & R. IF P IS THE MIDPOINT THEN ITS x COORDINATE $= \frac{Qx + Rx}{2}$ i.e. $\frac{\text{SUM OF ROOTS}}{2}$

SUM OF ROOTS $= \frac{-b}{a} = \frac{-(2ct^7 - 2ct^3)}{t^2 - t^6}$

$= \frac{2ct^3(-t^4 + 1)}{t^2(1 - t^4)}$

$= 2ct$

$= x$ COORDINATE OF P

$\therefore$ P IS THE MIDPOINT OF QR.

QUESTION 7:

(a) i

$$(x-y)^2 \geq 0$$
$$x^2 - 2xy + y^2 \geq 0$$
$$x^2 + y^2 \geq 2xy$$

ii

$$\frac{x}{y} + \frac{y}{x} = \frac{x^2+y^2}{xy}$$
$$\geq \frac{x^2+y^2}{\frac{x^2+y^2}{2}}$$
$$\geq 2$$

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$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

BUT $x^2 + y^2 \geq 2xy$

$\therefore x^2 - xy + y^2 \geq xy$

$$\therefore x^3 + y^3 \geq (x+y)xy$$
$$\geq \left(\frac{x}{z} + \frac{y}{z}\right) xyz$$
$$\geq 2xyz$$

SIMILARLY: $x^3 + z^3 \geq 2xyz$

& $y^3 + z^3 \geq 2xyz$

$\therefore 2x^3 + 2y^3 + 2z^3 \geq 6xyz$

$\therefore x^3 + y^3 + z^3 \geq 3xyz.$

(b) Construct AB

$R\hat{P}B = Q\hat{A}B$ (EXT $\angle$ OF CYCLIC QUAD = INT OPP $\angle$)

$Q\hat{A}B = B\hat{Q}S$ ($\angle$'s IN SAME SEGMENT OF CIRCLE ARE EQUAL)

$\therefore R\hat{P}B = B\hat{Q}S$

$\therefore$ MPBQ IS A CYCLIC QUAD AS EXTERIOR ANGLE $(B\hat{Q}S)$ EQUALS INTERIOR OPPOSITE ANGLE $(R\hat{P}B)$.

(c). L.H.S. $= \cos^6\theta - \sin^6\theta$ 14

$= (\cos^2\theta - \sin^2\theta)(\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta)$

$= \cos 2\theta \left[(\cos^2\theta + \sin^2\theta)^2 - \cos^2\theta\sin^2\theta \right]$

$= \cos 2\theta \left[1 - \left(\frac{2\cos\theta\sin\theta}{2} \right)^2 \right]$

$= \cos 2\theta \left[1 - \frac{1}{4}\sin^2 2\theta \right]$

$=$ R.H.S.

~~Question 8~~ :

QUESTION 8:

(i) $B\hat{T}X = B\hat{P}O$ (CORR ∠'s TX // OP)

$T\hat{X}B = P\hat{O}B$ (90°)

$\therefore \triangle TXB \;|||\; \triangle POB$

$$\therefore \frac{TX}{a} = \frac{h}{r}$$

$$\therefore TX = \frac{ah}{r} \qquad (1) \quad \text{(ONE SIDE OF RECTANGLE ZTXY)}$$

$\triangle PZT \;|||\; \triangle PAB$

$$\therefore \frac{ZT}{PT} = \frac{AB}{PB}$$

$$\therefore \frac{ZT}{\sqrt{h^2+r^2} - \sqrt{\frac{a^2h^2}{r^2}+a^2}} = \frac{b}{\sqrt{h^2+r^2}}$$

$$\therefore ZT = \frac{b}{\sqrt{h^2+r^2}} \times \left\{\sqrt{h^2+r^2} - \sqrt{\frac{a^2h^2+a^2r^2}{r^2}}\right\}$$

$$= \frac{b}{\sqrt{h^2+r^2}} \times \left\{\sqrt{h^2+r^2} - \frac{a\sqrt{h^2+r^2}}{r}\right\}$$

$$= \frac{b}{\sqrt{h^2+r^2}} \times \frac{\sqrt{h^2+a^2}\,(r-a)}{r}$$

$$= \frac{b(r-a)}{r} \qquad (2)$$

$$\therefore \text{Volume of } S \text{ is } \cancel{\left(\frac{r-a}{r}\right)b}\; TX \cdot ZT \cdot \delta a$$

$$= \left(\frac{r-a}{r}\right) b \left(\frac{ah}{r}\right) \delta a.$$

(ii) $$\therefore V = \int_0^r \left(\frac{r-a}{r}\right) b \left(\frac{ah}{r}\right) da$$

$$= \frac{b}{r^2}\int_0^r rha - ha^2 \, da$$

$$= \frac{b}{r^2}\left[\frac{rha^2}{2} - \frac{ha^3}{3}\right]_0^r$$

$$= \frac{b}{r^2}\left\{\frac{r^3h}{2} - \frac{r^3h}{3}\right\}$$

$$= \frac{b}{r^2} \times \frac{r^3h}{6}$$

$$= \frac{1}{6}hbr.$$

(b)(i) PROVE $e - S_n = e\int_0^1 \frac{x^n}{n!} e^{-x}\,dx$ 16

STEP 1: PROVE TRUE FOR $n=1$

$$\begin{aligned} LHS &= e - S_1 \\ &= e - \left(1 + \sum_{r=1}^{1} \frac{1}{r!}\right) \\ &= e - (1+1) \\ &= e - 2 \end{aligned}$$

$$\begin{aligned} RHS &= e\int_0^1 \frac{x^1}{1!} e^{-x}\,dx \\ &= e\int_0^1 x e^{-x}\,dx \\ &= e\left\{\left[x \times -e^{-x}\right]_0^1 - \int_0^1 -e^{-x} \times 1\,dx\right\} \\ &= e\left\{-\frac{1}{e} + \int_0^1 e^{-x}\,dx\right\} \\ &= e\left\{-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right\} \\ &= e\left\{-\frac{1}{e} + \left(-\frac{1}{e} - -e^0\right)\right\} \\ &= e\left\{-\frac{1}{e} - \frac{1}{e} + 1\right\} \\ &= e - 2 = LHS \end{aligned}$$

$\therefore$ TRUE FOR $n=1$

STEP 2: ASSUME TRUE FOR $n=k$

i.e. $e - S_k = e\int_0^1 \frac{x^k}{k!} e^{-x}\,dx$

i.e. $S_k = e - e\int_0^1 \frac{x^k}{k!} e^{-x}\,dx$

STEP 3: PROVE TRUE FOR $n=k+1$

i.e. AIM TO PROVE: $e - S_{k+1} = e\int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x}\,dx$

i.e. $S_{k+1} = e - e\int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x}\,dx$

NOW
$$\begin{aligned} S_{k+1} &= S_k + T_{k+1} \\ &= e - e\int_0^1 \frac{x^k}{k!} e^{-x} + \frac{1}{(k+1)!} \\ &= e - \frac{e}{k!}\left\{\left[\frac{x^{k+1}}{k+1} \times e^{-x}\right]_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \times -e^{-x}\right\} + \frac{1}{(k+1)!} \\ &= e - \frac{e}{k!}\left\{\frac{1}{k+1}\cdot\frac{1}{e} + \int_0^1 \frac{x^{k+1}}{k+1} e^{-x}\right\} + \frac{1}{(k+1)!} \\ &= e - \frac{1}{k!(k+1)} - \frac{e}{k!}\int_0^1 \frac{x^{k+1}}{k+1} e^{-x} + \frac{1}{(k+1)!} \\ &= e - \frac{1}{(k+1)!} - e\int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x} + \frac{1}{(k+1)!} \\ &= e - e\int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x}\,dx \end{aligned}$$

STEP 4: THEREFORE TRUE FOR $n=k+1$ IF TRUE FOR $n=k$; BUT IT IS TRUE FOR $n=1$ & THEREFORE TRUE FOR $n=2$ & THEREFORE FOR $n=3$ & SO ON. BY THE PROCESS OF MATHEMATICAL INDUCTION IT IS TRUE FOR ALL INTEGRAL VALUES OF n.

Q8 (b) (i) THIS IS THE SUM OF A GEOMET
SEQUENCE

$$\therefore S_n = \frac{1\left[1-(-x^2)^{2n+1}\right]}{1-(-x^2)}$$

$$= \frac{1+(x^2)^{2n+1}}{1+x^2} \quad \text{AS } 2n+1 \text{ IS ODD}$$

$$= \frac{1+x^{4n+2}}{1+x^2}$$

(ii) $1+x^2 \geq 1$ AS $x^2 \geq 0$

& AS $1-x^2+x^4-\dots+x^{4n} = \frac{1+x^{4n+2}}{1+x^2} = \frac{1}{1+x^2} + \frac{x^{4n+2}}{1+x^2}$

THEN $\frac{1}{1+x^2} \leq 1-x^2+x^4-\dots x^{4n}$ $\left(\frac{x^{4n+2}}{1+x^2} \text{ IS POS}\right)$

ALSO $\frac{1}{1+x^2} + \frac{x^{4n+2}}{1+x^2} \leq \frac{1}{1+x^2} + x^{4n+2}$

$\therefore \frac{1}{1+x^2} \leq 1-x^2+x^4-\dots+x^{4n} \leq \frac{1}{1+x^2} + x^{4n+2}$

(iii) $\therefore \int_0^y \frac{1}{1+x^2}\,dx \leq \int_0^y \frac{1+x^{4n+2}}{1+x^2}\,dx \leq \int_0^y \frac{1}{1+x^2} + x^{4n+2}$

$\therefore \tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \dots + \frac{y^{4n+1}}{4n+1} \leq \tan^{-1} y + \frac{y^{4n+3}}{4n+3}$

BUT IF $0 \leq y \leq 1$ THEN $y^{4n+3} \leq 1$

$\therefore \tan^{-1} y + \frac{y^{4n+3}}{4n+3} \leq \tan^{-1} y + \frac{1}{4n+3}$

$\therefore \tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \dots + \frac{y^{4n+1}}{4n+1} \leq \tan^{-1} y + \frac{1}{4n+3}$

(iv) SUBSTITUTE $y=1$, $n=250$

$\therefore \tan^{-1} 1 \leq 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{1001} \leq \tan^{-1} 1 + \frac{1}{1003}$

$\therefore \frac{\pi}{4} < 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{1001} < \frac{\pi}{4} + \frac{1}{1003}$

$\frac{\pi}{4}$ IRRATIONAL; RATIONAL; IRRATIONAL

$\therefore$ NO EQUALITY SIGN.

BUT $1003 > 1000$ $\therefore \frac{1}{1003} < \frac{1}{1000}$

$\therefore 0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{1001}\right) - \frac{\pi}{4} < \frac{1}{1000}$

$\therefore 0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{1001}\right) - \pi/4 < 10^{-3}$.

(v) $\therefore \dfrac{x \sec\theta}{2} - \dfrac{y \tan\theta}{2\sqrt{3}} = 1$ —— (1)

$x = 1$ —— (2)

$\therefore \dfrac{\sec\theta}{2} - \dfrac{y\tan\theta}{2\sqrt{3}} = 1$

$\sqrt{3}\sec\theta - y\tan\theta = 2\sqrt{3}$

$\dfrac{\sqrt{3}}{\cos\theta} - \dfrac{y\sin\theta}{\cos\theta} = 2\sqrt{3}$

$y\sin\theta = \sqrt{3} - 2\sqrt{3}\cos\theta$

$\therefore y = \dfrac{\sqrt{3} - 2\sqrt{3}\cos\theta}{\sin\theta} \times \dfrac{2\sqrt{3}}{2\sqrt{3}}$

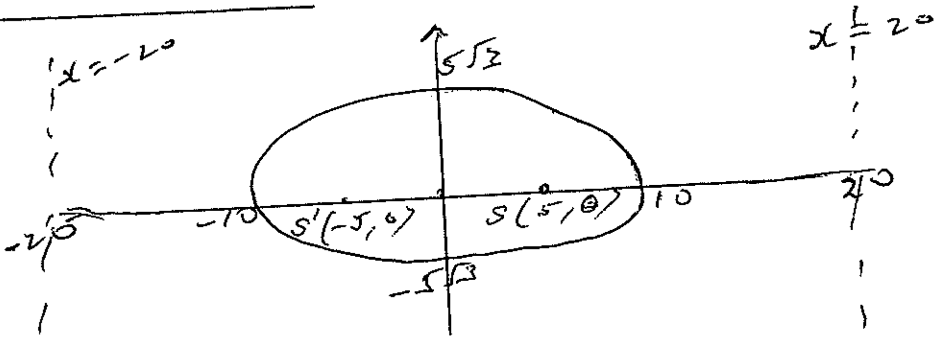
$= \dfrac{6 - 12\cos\theta}{2\sqrt{3}\sin\theta}$

(b) $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$

(i) $\therefore a = 10,\ b = 5\sqrt{3}$

$b^2 = a^2(1 - e^2)$

$\therefore e = 1/2$



(ii) $\dfrac{2x}{100} + \dfrac{2y}{75}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-3x}{4y}$

AT $P(5, 7.5)$ grad of tan $= -\frac{1}{2}$

$\therefore$ grad of $\perp = 2$

$\therefore$ EQU^{N} OF $\perp$ is $y - 7.5 = 2(x-5)$

$4x - 2y - 5 = 0$

(iii) BY SYMMETRY THE INTERSECTION OF THE TWO NORMALS WILL BE ON THE x-AXIS & THIS WILL BE THE CENTRE OF THE CIRCLE

$\therefore$ WHEN $y = 0$, $x = 5/4$ $\therefore$ CENTRE $(5/4, 0)$

RADIUS$^2 = (5 - 5/4)^2 + (7.5 - 0)^2$

$= \dfrac{1125}{16}$

$\therefore$ EQU^{N} OF CIRCLE IS $\left(x - \frac{5}{4}\right)^2 + y^2 = \dfrac{1125}{16}$.

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(ii) Since $x^n \geq 0$ for $x \geq 0$ & $e^{-x} > 0$

$$e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx = e\int_0^1 \text{FUNCTION IS ALWAYS POSITIVE}$$

$$> 0$$

$$\therefore \quad 0 < e - S_n$$

Now $e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx < 3\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$ as $e < 3$

$$< 3\int_0^1 \frac{x^n}{n!}\,dx \quad \text{as } e^{-x} \leq 1 \text{ for } x \geq 0$$

$$< \frac{3}{n!}\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$< \frac{3}{n!}\left\{\frac{1}{n+1}\right\}$$

$$< \frac{3}{(n+1)!}$$

$$\therefore \quad 0 < e - S_n < \frac{3}{(n+1)!}$$
