WHITEBRIDGE HIGH SCHOOL

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VIAM VITAE QUAERE

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

2003
MATHEMATICS

EXTENSION 2

Time Allowed: Three hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt all questions
- ALL questions are of equal value
- All necessary working should be shown. Marks may be deducted for careless or badly arranged work.
- Standard integrals are provided
- Board-approved calculators may be used
- Each question is to be returned on a separate sheet of paper clearly labelled, showing your Name and Student Number.

Question 1.

Marks

(a) Evaluate $\int_1^3 x^2 \ln x \, dx$ *2*

(b) Find the partial fraction decomposition of $\dfrac{16x}{x^4 - 16}$. Hence show that *3*

$$\int_4^6 \frac{16x}{x^4 - 16}\, dx = \log_e\left(\frac{4}{3}\right)$$

(c) Evaluate $\displaystyle\int_{-4}^{4} \frac{x + 6}{\sqrt{x + 5}}\, dx$ *3*

(d) Find $\displaystyle\int \frac{5x - 2\, dx}{\sqrt{5 + 2x - x^2}}$ *3*

(e) (i) Prove that $\displaystyle\int \sec^n x = \frac{\sec^{n-2} x \tan x}{n - 1} + \frac{n - 2}{n - 1}\int \sec^{n-2} x \, dx$ *2*

(ii) Hence find $\displaystyle\int \sqrt{a^2 + x^2}\, dx$ (let $x = a \tan \theta$) *2*

Question 2

A function *f(x)* is defined by $f(x) = \dfrac{\log_e x}{x}$ *for* $x > 0$

(a). Prove that the graph of *f(x)* has a relative maximum turning point at $x = e$ and *A* point of inflection at $x = e^{3/2}$. *4*

(b). Discuss the behaviour of *f(x)* in the neighbourhood of $x = 0$ and for large values of *x*. *1*

Marks

(c) Hence draw a clear sketch of $f(x)$ indicating on it all these features. *3*

(d) Draw separate sketches of the graphs of

(i) $y = \left|\dfrac{\log_e x}{x}\right|$ *3*

(ii) $y = \dfrac{x}{\log_e x}$ *3*

(Hint: There is no need to find any further derivatives to answer this part.)

(e) What is the range of the function $y = \dfrac{x}{\log_e x}$ *1*

Question 3

(a) Solve the following equation for Z giving your answer in modulus – argument form. *2*

$$Z^2 + Z + 1 = 0$$

(b) If $Z_1 = 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ and

$Z_2 = \left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

express your answer to the following in the form $a + ib$

(i) $Z_1 Z_2$ (ii) $\dfrac{Z_1}{Z_2}$ *2*

(c) The equation $Z^3 - 3Z^2 + 7Z - 5 = 0$ has one root equal to $(1 - 2i)$. Factorise the equation. *2*

(d) Sketch the locus of Z such that $|z - 2 - 2i| = 2$ *2*

(i) find the range of $|Z|$ *1*

(ii) find the range of $ARG\ Z$ *1*

Marks

(e) *(i)* express $(1 - i)^{-7}$ in the form $x + iy$ **2**

(ii) find the locus of Z if $W = \dfrac{Z - 2}{Z}$, given that W is purely imaginary. **3**

QUESTION 4.

(a) A solid has a base in the shape of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. If every cross section perpendicular to the base is a semi circle, with its diameter at right angles to the major axis of the ellipse, find the volume of the solid by slicing. **3**

(b) The circle $x^2 + y^2 = 4$ is rotated about the line $x = 3$ to form a torus. Show that the volume of the torus is $24\pi^2$. **3**

(c) *(continued on next page)*

(c) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 shows this position.

Figure 1

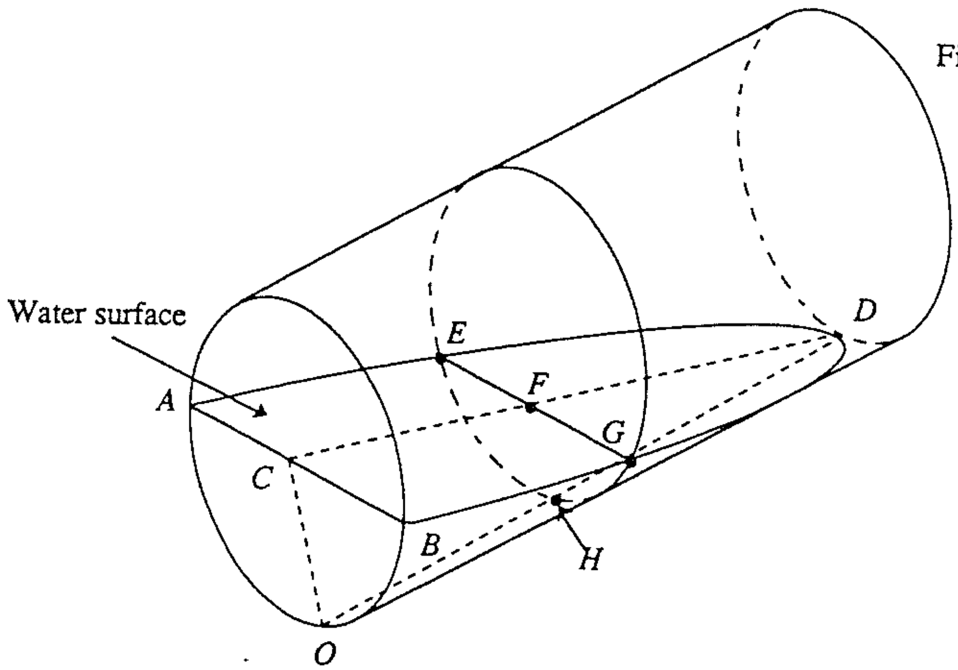


Figure 2

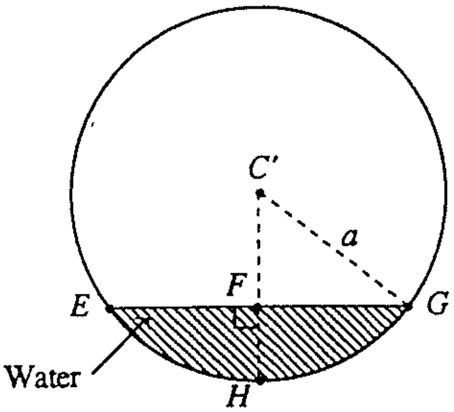
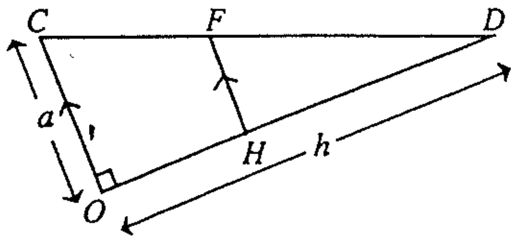


Figure 3



Note: $EG \perp C'H$ at F

Note: $FH \parallel CO$, $CO = a$, and $OD = h$

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.

Figure 2 shows a cross-section of the tilted glass parallel to its base. The centre of this circular section is C' and EFG shows the water level. The section cuts the lines CD and OD of Figure 1 in F and H respectively.

Figure 3 shows the section COD of the tilted glass.

(i) Use Figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$. 1

(ii) Use Figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$. 2

(iii) Use (ii) to show that the area of the shaded segment EGH is 3

$$a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1 - \left(\frac{x}{h}\right)^2}\,\right].$$

(iv) Given that $\int \cos^{-1}\theta\, d\theta = \theta\cos^{-1}\theta - \sqrt{1 - \theta^2}$, find the volume of water in the tilted glass of Figure 1. 3

Marks

Question 5

(a) The ellipse E has equation $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$.

(i) sketch the curve E, showing on your diagram the co ordinates of the foci and the equation of each directrix. **2**

(ii) find the equation of the normal to the ellipse at the point $P\,(5, 7.5)$. **3**

(iii) find the equation of the circle that is tangential to the ellipse at P and Q (5, - 7.5) **4**

(b) The tangent to the hyperbola $xy = c^2$ at the point $P\,(ct, \frac{c}{t})$ intersects the axes in Q and R and the normal at P intersects the line $y = x$ *in S*.
Prove that $PQ = PR = PS$. **6**

Question 6

(a) Given that $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5 = 0$ has a triple root, find all its real roots. **3**

(b) If $\cancel{\partial, B, 8}$ α, β, γ are the roots of $2x^3 + 3x^2 + x - 5 = 0$, form the equation whose roots are $\cancel{\partial^2, B^2, 8^2}$. $\alpha^2, \beta^2, \gamma^2$ **4**

(c) When a polynomial P(x) is divided by (x – 3) the remainder is 5 and when it is divided by x – 4 the remainder is 9. find the remainder when P(x) is divided by (x – 4) (x – 3). **4**

(d) If $Z = \cos\theta + i\sin\theta$, prove that $Z^n + Z^{-n} = 2\cos n\theta$, hence solve the equation $3Z^4 - Z^3 + 4Z^2 - Z + 3 = 0$ **4**

Question 7

(a)

i. Prove that $\dfrac{a + b}{2} \geq \sqrt{ab}$ if a and b are positive real numbers. **2**

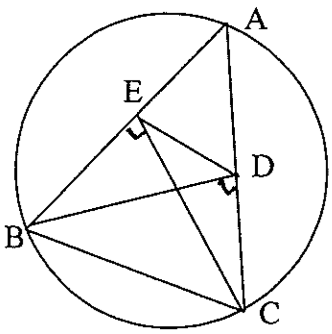
ii. Given that for $x + y = c$ prove that $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{c}$ for $x > 0, y > 0$ **3**

Marks

(b) Show that for n > 0, $2n + 3 > 2\sqrt{(n+1)(n+2)}$

Hence, by induction prove that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$ *4*

(c) In the diagram, BC is a fixed chord of a circle, A is a variable point on the major arc on the chord BC. BD ⊥ AC and CE ⊥ AB.
Prove that:



(i) BCDE is a cyclic quadrilateral *2*

(ii) As A varies, the segment ED has constant length. *2*

(iii) The locus of the midpoint of ED is a circle whose centre is the midpoint of BC. *2*

Question 8

(a) Find the general solution to the equation $\sin 2x + \sin 4x = \sin 6x$. *3*

(b) If $Z_1 = 3 + 4i$ *and* $|Z_2| = 13$, find the greatest value of $|Z_1 + Z_2|$. *If* $|Z_1 + Z_2|$ takes its greatest value, express Z_2 in the form a + *ib*. *2*

(c) If α *and* β are the roots of $x^2 - 2x + 4 = 0$ prove that
$\partial^n + B^n = 2^{n+1} \cos\frac{n\pi}{3}$. 5

$\alpha^n + \beta^n$

(d) Consider the function $f(x) = e^x\left(1 - \frac{x}{10}\right)^{10}$

(i) Find the turning points of the graph of y = f(x). *2*

(ii) Sketch the curve y = f(x) and label the turning points and any asymptotes. *1*

(iii) From your graph deduce that $e^x < \left(1 - \frac{x}{10}\right)^{-10}$ for $x < 10$ *2*

(iv) Using (iii) show that $\left(\frac{11}{10}\right)^{10} \le e \le \left(\frac{10}{9}\right)^{10}$ *2*

Standard integrals

Marks

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

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WHITEBRIDGE HIGH SCHOOL

2003 H.S.C. TRIAL

MATHEMATICS EXTENSION 2:

QUESTION 1:

(a) $\int_1^3 x^2 \ln x = \left[\ln x \cdot \frac{x^3}{3}\right]_1^3 - \int_1^3 \frac{x^3}{3} \times \frac{1}{x}\, dx$

$= 9\ln 3 - \frac{1}{3}\ln 1 - \frac{1}{3}\int_1^3 x^2\, dx$

$= 9 \ln 3 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^3$

$= 9 \ln 3 - \frac{1}{9}(27 - 1)$

$= 9 \ln 3 - 26/9$ (6.999 TO 3 DEC PL)

(b) $\frac{16x}{x^4 - 16} = \frac{16x}{(x^2-4)(x^2+4)} = \frac{16x}{(x-2)(x+2)(x^2+4)}$

LET $\frac{16x}{(x-2)(x+2)(x^2+4)} = \frac{a}{x-2} + \frac{b}{x+2} + \frac{cx+d}{x^2+4}$

$\therefore 16x = a(x+2)(x^2+4) + b(x-2)(x^2+4) + (cx+d)(x-2)(x+2)$

WHEN $x=2$: $32 = 32a$

$\therefore a = 1$

WHEN $x=-2$: $-32 = -32b$

$b = 1$

EQUATING COEFFICIENTS OF x^3:

$0 = a + b + c$

$0 = 1 + 1 + c$

$\therefore c = -2$

EQUATING CONSTANTS

$0 = 8a - 8b - 4d$

$0 = 8 - 8 - 4d$

$d = 0$

$\therefore \frac{16x}{x^4-16} = \frac{1}{x-2} + \frac{1}{x+2} - \frac{2x}{x^2+4}$

$\therefore \int_4^6 \frac{16x}{x^4-16} = \int_4^6 \frac{1}{x-2} + \frac{1}{x+2} - \frac{2x}{x^2+4}$

$= \left[\ln(x-2) + \ln(x+2) - \ln(x^2+4)\right]_4^6$

$= \left[\ln \frac{(x-2)(x+2)}{x^2+4}\right]_4^6 = \ln \frac{32}{40} - \ln\frac{12}{20} = \ln \frac{32}{40}\cdot\frac{20}{12}$

$= \ln 4/3$

(c) $\displaystyle\int_{-4}^{4} \frac{x+6}{\sqrt{x+5}}\,dx$ 2/1 LET $u = x+5$

$\therefore du = dx$

~~LET~~ $\therefore = \displaystyle\int_{1}^{9} \frac{u+1}{\sqrt{u}}\,du$

$= \displaystyle\int_{1}^{9} u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \left[\frac{2}{3}u^{3/2} + 2u^{\frac{1}{2}}\right]_{1}^{9}$

$= \left(\frac{2}{3}\cdot 9^{3/2} + 2\cdot 9^{1/2}\right) - \left(\frac{2}{3} + 2\right)$

$= (18 + 6) - 2\frac{2}{3}$

$= 21\frac{1}{3}$

$=$

(d) $\displaystyle\int \frac{5x-2}{\sqrt{5+2x-x^2}}\,dx = \int \frac{-5/2(2-2x) - 2 + 5/2\cdot 2}{\sqrt{5+2x-x^2}}\,dx$

$= -\frac{5}{2}\displaystyle\int \frac{2-2x}{\sqrt{5+2x-x^2}} + 3\int \frac{dx}{\sqrt{5+2x-x^2}}$

$-\frac{5}{2}\displaystyle\int \frac{2-2x}{\sqrt{5+2x-x^2}}\,dx$ LET $u = 5+2x-x^2$

$\frac{du}{dx} = 2-2x$

$= -\frac{5}{2}\displaystyle\int \frac{du}{\sqrt{u}} = -\frac{5}{2}\int u^{-1/2}\,du$

$= -5/2 \times 2u^{1/2}$

$= -5\sqrt{5+2x-x^2}$

$3\displaystyle\int \frac{dx}{\sqrt{5+2x-x^2}} = 3\int \frac{dx}{\sqrt{6-(x-1)^2}}$ Let $v = x-1$

$\therefore \frac{dv}{dx} = 1$

$= 3\displaystyle\int \frac{dv}{\sqrt{6-v^2}}$

$= 3\sin^{-1}\frac{v}{\sqrt{6}}$

$= 3\sin^{-1}\frac{x-1}{\sqrt{6}}$

$\therefore \displaystyle\int \frac{5x-2}{\sqrt{5+2x-x^2}} = -5\sqrt{5+2x-x^2} + 3\sin^{-1}\frac{x-1}{\sqrt{6}} + C$

(9) i $\int \sec^n x \, dx = \int \sec^{n-2}x \sec^2 x$

$= \left[\sec^{n-2}x \tan x\right] - \left[\tan x \times n-2 \sec^{n-3} x \times \sec x \tan x\right]$

$= \left[\sec^{n-2}x \tan x\right] - (n-2)\int \tan^2 x \sec^{n-2} x \, dx$

$= \left[\sec^{n-2}x \tan x\right] - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x \, dx$

$= (\sec^{n-2}x \tan x) - (n-2)\int \sec^n x - \sec^{n-2} x \, dx$

$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2} x \, dx$

$= \sec^{n-2}x \tan x + n-2 \int \sec^{n-2} x \, dx$

$(n-1)\int \sec^n x = \sec^{n-2}x \tan x$

$\therefore \int \sec^n x = \dfrac{\sec^{n-2}x \tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x \, dx$

ii $\int \sqrt{a^2 + x^2} \, dx$ Let $x = a \tan\theta$

$\dfrac{dx}{d\theta} = a \sec^2\theta$

$= \int \sqrt{a^2 + a^2 \tan^2\theta} \times a \sec^2\theta \, d\theta.$

$= \int a^2 \sqrt{1 + \tan^2\theta} \sec^2\theta \, d\theta$

$= a^2 \int \sec^3\theta \, d\theta$

$= a^2 \left[\dfrac{\sec x \tan x}{2} + \dfrac{1}{2}\int \sec x \, dx\right]$

$= a^2 \left[\dfrac{\sec x \tan x}{2} + \dfrac{1}{2}\int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\right]$

$= a^2 \left[\dfrac{\sec x \tan x}{2} + \dfrac{1}{2}\ln(\sec x + \tan x)\right] + C.$

$= a^2 \left[\dfrac{\sec x \tan x}{2}\right.$

QUESTION 2: 4

(a) $f(x) = \dfrac{\log x}{x}$ for $x > 0$

FOR STAT PTS $f'(x) = 0$

$f'(x) = \dfrac{x \times \frac{1}{x} - \log x \times 1}{x^2}$

$= \dfrac{1 - \log x}{x^2} = 0$

$\therefore 1 - \log x = 0$

$\log x = 1$

$\left.\begin{aligned} x &= e \\ y &= \tfrac{1}{e} \end{aligned}\right\}$ 1

$f''(x) = \dfrac{x^2 \times -\frac{1}{x} - (1 - \log x) \times 2x}{x^4}$

$= \dfrac{-x - 2x + 2x \log x}{x^4}$

$= \dfrac{-3x + 2x \log x}{x^4}$

$= \dfrac{-3 + 2 \log x}{x^3}$

$f''(e) = \dfrac{-3 + 2 \log e}{e^3} = \dfrac{-1}{e^3} < 0$ 1

$\therefore (e, \tfrac{1}{e})$ IS A MAXIMUM T.P.

P.I. OCCUR WHEN $f''(x) = 0$

$\therefore -3 + 2 \log x = 0$

$\log x = 3/2$

$x = e^{3/2}$ 1

$y = \dfrac{\log e^{3/2}}{e^{3/2}} = \dfrac{3}{2e^{3/2}}$

TEST PI 1

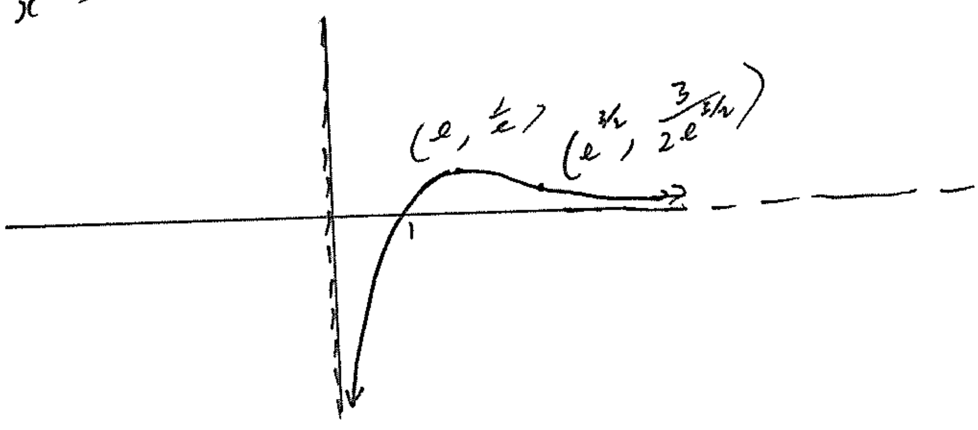
x	4	$e^{3/2}$	5
$f''(x)$	$-$	0	$+$

AS THERE IS A CHANGE IN CONCAVITY $\left(e^{3/2}, \dfrac{3}{2e^{3/2}}\right)$ IS P.I.

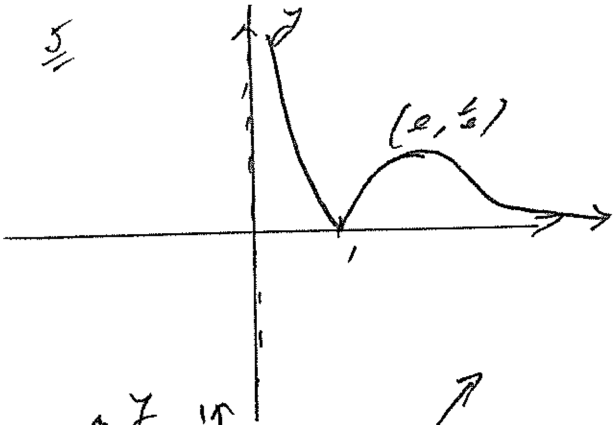
(b) as $x \to 0$ $f(x) \to -\infty$

as $x \to \infty$ $f(x) \to 0$

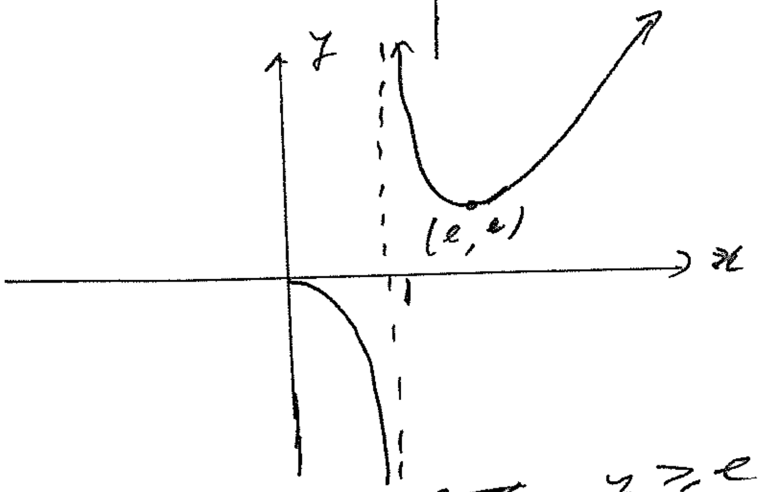
(c)



(d) i $y = \left|\frac{\log x}{x}\right|$



ii $y = \frac{x}{\log x}$



(e) Range is $y < 0$ & ~~or~~ $y \geq e$

QUESTION 3:

(a) $z^2 + z + 1 = 0$

$z = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{3i^2}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$z_1 = \frac{-1+\sqrt{3}i}{2} = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z_2 = \frac{-1-\sqrt{3}i}{2} = 1\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

(b) $z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z_2 = 1\left(\cos\frac{\pi}{3} + i\sin\pi/3\right)$

i $z_1 z_2 = 2(\cos\pi + i\sin\pi)$

$= 2(-1+0)$

$= -2$

ii $\frac{z_1}{z_2} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2\left(\frac{1}{2} + i \times \sqrt{3}/2\right)$

$= 1 + \sqrt{3}i$

$\underline{\underline{6}}$

(c) $z^3 - 3z^2 + 7z - 5 = 0$

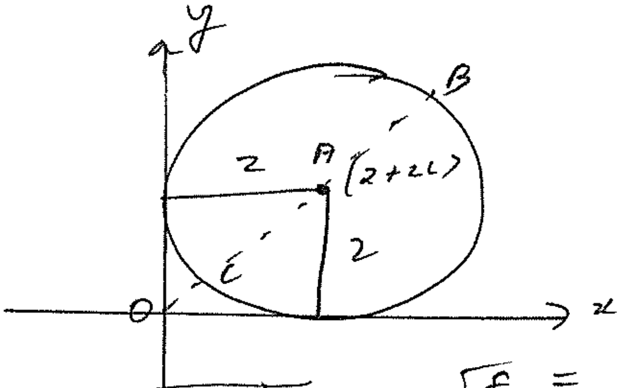
IF $(1-2i)$ IS A ROOT SO IS $1+2i$

$\therefore (1-2i) + (1+2i) + x = 3$

$\therefore x = 1$

$\therefore \left[z-(1-2i)\right]\left[z-(1+2i)\right]\left[z-1\right] = 0.$

(d)



$\underline{\underline{i}}$ $OA = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\therefore OC = 2\sqrt{2} - 2$

$OB = 2\sqrt{2} + 2$

$\therefore$ Range of $|z|$ is $2\sqrt{2} - 2 \le |z| \le 2\sqrt{2} + 2$

$\underline{\underline{ii}}$ $0 \le \text{ARG } z \le \pi/2$

(e) $\underline{\underline{i}}$ $(1-i)^{-7} = \left[\sqrt{2}\left(\cos -\frac{\pi}{4} + i \sin -\pi/4\right)\right]^{-7}$

$= \frac{1}{(\sqrt{2})^7}\left[\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right]$

$= \frac{1}{8\sqrt{2}}\left[\cos -\frac{\pi}{4} + i \sin -\pi/4\right]$

$= \frac{\sqrt{2}}{16}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{16} - i\frac{1}{16}$

$\underline{\underline{ii}}$ $w = \frac{z-2}{z} = \frac{x+iy-2}{x+iy}$

$= \frac{(x-2)+iy}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x^2 - 2x + y^2 + \cancel{\ldots} 2iy}{x^2+y^2}$

IF PURELY IMAGINARY THEN REAL PART OF $w = 0$

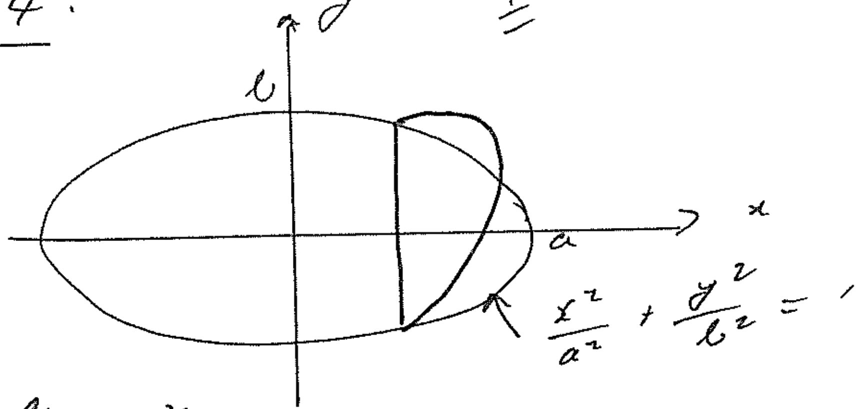
$\therefore \frac{x^2-2x+y^2}{x^2+y^2} = 0$

i.e. $x^2 - 2x + y^2 = 0.$

QUESTION 4:

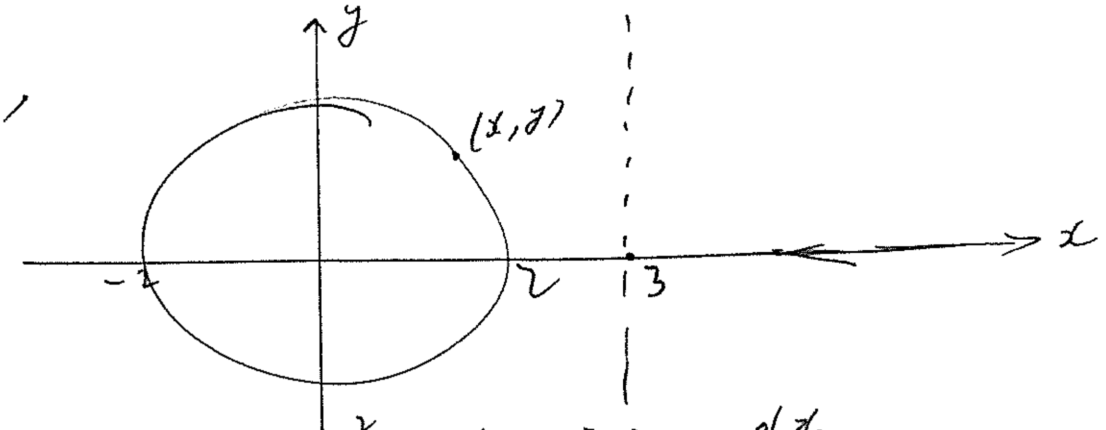
$\underline{\underline{1}}$

(a)



$$V = 2\int_0^a \pi \frac{y^2}{2}\, dx$$
$$= \pi \int_0^a y^2\, dx$$
$$= \pi \int_0^a b^2 - \frac{b^2x^2}{a^2}\, dx$$
$$= \pi \left[b^2x - \frac{b^2x^3}{3a^2} \right]_0^a$$
$$= \pi \left\{ b^2a - \frac{b^2a^3}{3a^2} \right\}$$
$$= \pi \left\{ b^2a - \frac{b^2a}{3} \right\}$$
$$= \frac{2\pi b^2 a}{3} \text{ units}^3$$

(b)



$$V = \int_{-2}^{2} 2\pi(3-x)\, 2y\, dx$$
$$= 4\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\, dx$$
$$= 4\pi \int_{-2}^{2} 3\sqrt{4-x^2} \quad - 4\pi \int_{-2}^{2} x\sqrt{4-x^2}\, dx$$

LET $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$12\pi \int_{-\pi/2}^{\pi/2} \sqrt{4-4\sin^2\theta} \times 2\cos\theta\, d\theta$

$= 48\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta = 48\pi \int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}$

$= 24\pi^2$

LET $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$-4\pi \int_0^0 \sqrt{u}\, du$

$= 0$

$\therefore$ SOL^N IS $24\pi^2$ units^3

(c) i Δ's DFH & DCO ARE SIMILAR

$\therefore \dfrac{FH}{a} = \dfrac{h-x}{h}$

$\therefore FH = \dfrac{a}{h}(h-x)$

ii IN FIG 2: $C'F = C'H - FH$

$= a - \dfrac{a}{h}(h-x)$

$= \dfrac{ah - ah + ax}{h}$

$= \dfrac{ax}{h}$

$\cos H\hat{C'}G = \dfrac{ax}{h} \div a$

$= \dfrac{x}{h}$

$\therefore HC'G = \cos^{-1}\left(\dfrac{x}{h}\right)$

iii AREA OF EGH $= 2 \times$ AREA OF FGH

$= 2 \times \{$AREA OF SECTOR $HC'G - \Delta FC'G\}$

$= 2 \times \left\{\text{AREA OF SECTOR } \cos^{-1}\left(\dfrac{x}{h}\right) - \dfrac{1}{2} \cdot \dfrac{ax}{h} \cdot \sqrt{a^2 - (ax/h)^2}\right\}$

$= 2 \times \left\{\dfrac{1}{2} \times a^2 \times \cos^{-1}\left(\dfrac{x}{h}\right) - \dfrac{ax}{h} \times \dfrac{\sqrt{a^2h^2 - a^2x^2}}{h}\right.$

$= a^2 \cos^{-1}\left(\dfrac{h}{x}\right) - \dfrac{ax}{h} \times \dfrac{a\sqrt{h^2-x^2}}{h}$

$= a^2 \cos^{-1}\left(\dfrac{h}{x}\right) - \dfrac{a^2x}{h^2}\sqrt{h^2-x^2}$

$= a^2 \cos^{-1} h/x - a^2 \dfrac{x}{h}\sqrt{1-(x/h)^2}$

$= a^2 \cos^{-1} \dfrac{h}{x} - a^2 \dfrac{x}{h}\sqrt{1-(x/h)^2}$

$= a^2 \left\{\cos^{-1}\dfrac{h}{x} - \dfrac{x}{h}\sqrt{1-(x/h)^2}\right\}$

iv $V = \displaystyle\int_0^h a^2 \left\{\cos^{-1}\cancel{\dfrac{x}{h}} - \dfrac{x}{h}\sqrt{1-(x/h)^2}\right\} dx$

LET $\theta = x/h \qquad \therefore dx = h\,d\theta$

$d\theta = \dfrac{1}{h}dx$

$\therefore V = \displaystyle\int_0^1 a^2 \left\{\cos^{-1}\theta - \theta\sqrt{1-\theta^2}\right\} h\,d\theta \}$

$= a^2 h \left\{\displaystyle\int_0^1 \cos^{-1}\theta - \int_0^1 \theta\sqrt{1-\theta^2}\,d\theta\right\}$

$= a^2 h \left\{\left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2}\right]_0^1 + \dfrac{1}{3}\left[(1-\theta^2)^{3/2}\right]_0^1\right\}$ ← SUB.

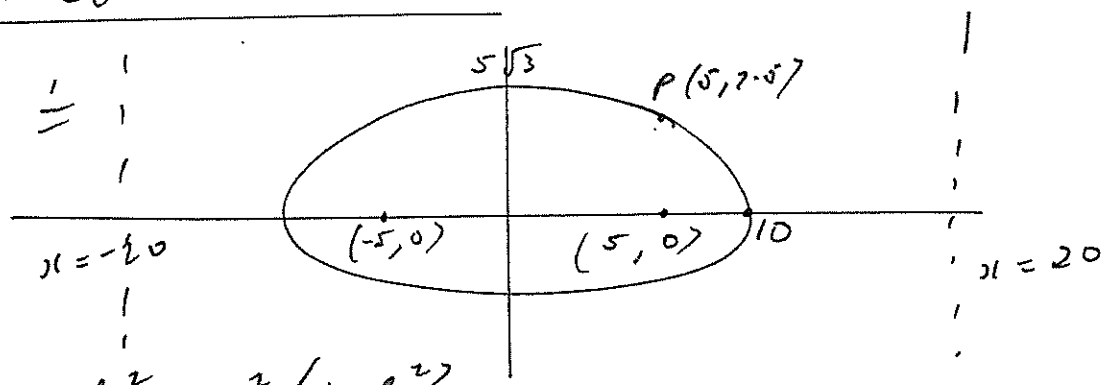
$= a^2 h \left\{\left[\cos^{-1} 1 + 1\right] + \dfrac{1}{3}\left[0 - 1\right]\right\}$

$= a^2 h \left\{1 - \dfrac{1}{3}\right\}$

$= \dfrac{2a^2h}{3}$ UNITS^3.

QUESTION 5: 9

(a) i



$b^2 = a^2(1-e^2)$
$75 = 100(1-e^2)$
$100e^2 = 25$
$e^2 = 1/4$
$e = \frac{1}{2}$

ii $\frac{x^2}{100} + \frac{y^2}{75} = 1$

$\frac{2x}{100} + \frac{2y}{75}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{100} \times \frac{75}{2y}$

$= \frac{-3x}{4y}$

at $P(5, 7.5)$ grad $= \frac{-3 \times 5}{4 \times 7.5} = \frac{-15}{30} = -\frac{1}{2}$

$\therefore$ grad of $\perp$ is 2.

$\therefore$ Equ of $\perp$ is $y - 7.5 = 2(x-5)$

$y = 2x - 2.5$

iii SIMILARLY EQU^N OF NORMAL AT $Q(5, -7.5)$ IS $y = -2x + 2.5$

SOLVING THE TWO NORMALS SIMULTANEOUSLY YOU OBTAIN $(1\frac{1}{4}, 0)$ & THIS WOULD BE THE CENTRE OF THE CIRCLE.

RADIUS $= \sqrt{(5 - 1\frac{1}{4})^2 + (7\frac{1}{2} - 0)^2}$

$= \sqrt{(3\frac{3}{4})^2 + (7\frac{1}{2})^2}$

$= \frac{15\sqrt{5}}{4}$

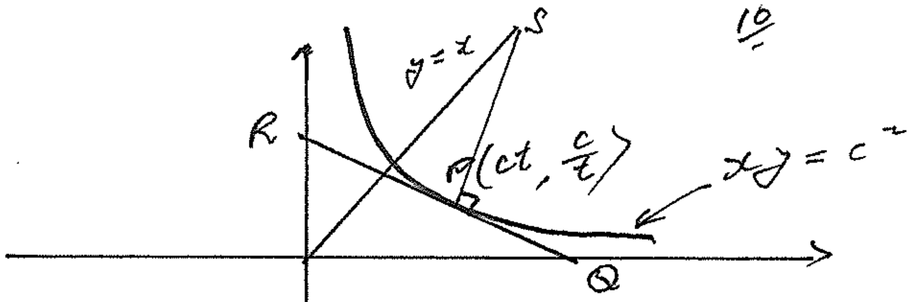
$\therefore$ EQU^N OF CIRCLE IS

$(x - 1\frac{1}{4})^2 + y^2 = \frac{1125}{16}$

$\left(\frac{4x-5}{4}\right)^2 + y^2 = \frac{1125}{16}$

$(4x-5)^2 + y^2 = 1125.$

(ii) 10



GRAD OF TAN IS $\frac{dy}{dx} = -\frac{c^2}{x^2}$

at $P(ct, \frac{c}{t})$ grad $= \frac{-c^2}{c^2t^2} = -\frac{1}{t^2}$

EQU^N OF TAN IS $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ 1

$t^2y - tc = -x + tc$

$x + t^2y = 2ct$

FOR Q SUB $y=0$; $x = 2ct$ $\therefore Q(2ct, 0)$ 1

FOR R SUB $x=0$; $t^2y = 2ct$ $\therefore R(0, \frac{2c}{t})$

AS $P(ct, \frac{c}{t})$ IS THE MIDPOINT OF QR 1

THEN $RP = PQ$

EQU^N OF $\perp$ AT P IS

$y - \frac{c}{t} = t^2(x - ct)$ 1

$ty - c = t^3x - ct^4$ ——— (1)

$t^3x - ty = ct^4 - c$ ——— (2)

$y = x$

SUB (2) in (1): $t^3x - tx = ct^4 - c$

$x = \frac{ct^4 - c}{t^3 - t} = \frac{c(t^2-1)(t^2+1)}{t(t^2-1)}$

$= \frac{c(t^2+1)}{t}$

$\therefore y = \frac{c(t^2+1)}{t}$ 1

$\therefore S\left[\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right]$

$SP = \sqrt{\left[\frac{c(t^2+1)}{t} - ct\right]^2 + \left[\frac{c(t^2+1)}{t} - \frac{c}{t}\right]^2}$

$= \sqrt{\frac{c^2}{t^2} + c^2t^2}$

& $PQ = \sqrt{(ct - 2ct)^2 + (\frac{c}{t} - 0)^2}$ 1

$= \sqrt{c^2t^2 + \frac{c^2}{t^2}}$

$= SP$

$\therefore PQ = PR = SP.$

QUESTION 6: 18

(a) $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5 = 0$ HAS A ROOT OF MULTIPLICITY 3

$\therefore P'(x) = 4x^3 + 6x^2 - 24x + 14$ HAS A ROOT OF MULTIPLICITY 2

$\therefore P''(x) = 12x^2 + 12x - 24$ HAS A ROOT OF MULT. 1

FOR $12x^2 + 12x - 24 = 0$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

MULTIPLE ROOT MUST BE 1 (FACTOR OF 5)

$\therefore P(x) = (x-1)^3(x-a)$

PRODUCT OF ROOTS IS $+1 \times +1 \times +1 \times a = -5$

$\therefore a = -5$

$\therefore$ ALL REAL ROOTS ARE $-5, 1$

(b) IF α, β, γ ARE ROOTS OF $2x^3 + 3x^2 + x - 5 = 0$

$\alpha + \beta + \gamma = -3/2$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{2}$

$\alpha\beta\gamma = 5/2$

$$\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= (-3/2)^2 - 2 \times 1/2 \\
&= 1\tfrac{1}{4} = 5/4 \\
&=
\end{aligned}$$

$$\begin{aligned}
\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 &= (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) \\
&= \left(\tfrac{1}{2}\right)^2 - 2 \times 5/2 \times -3/2 \\
&= \tfrac{1}{4} + \tfrac{15}{2} \\
&= 7\,3/4 = 31/4 \\
&=
\end{aligned}$$

$$\begin{aligned}
\alpha^2\beta^2\gamma^2 &= (\alpha\beta\gamma)^2 \\
&= (5/2)^2 \\
&= 25/4 \\
&=
\end{aligned}$$

$\therefore$ EQUATION IS

$x^3 - 5/4\, x^2 + \frac{31}{4}x - \frac{25}{4} = 0$

$4x^3 - 5x^2 + 31x - 25 = 0$

(c) LET $P(x) = (x-4)(x-3) + ax + b$

$P(4) = 4a + b = 9$

$P(3) = 3a + b = 5$

SOLVING SIMULTANEOUSLY

$a = 4$

$b = -7$

$\therefore$ REMAINDER IS $4x - 7$.

(d) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ —— (A)

$\therefore z^n + z^{-n} = 2\cos n\theta$

$3z^4 - z^3 + 4z^2 - z + 3 = 0$ on rearranging becomes

$3(z^4+1) - (z^3+z) + 4z^2 = 0$

DIVIDING BY z^2:

$3\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 4 = 0$

USING (A) WITH $n = 2$ & $n = 1$

$3 \times 2\cos 2\theta - 2\cos\theta + 4 = 0$

$6\cos 2\theta - 2\cos\theta + 4 = 0$

$6(2\cos^2\theta - 1) - 2\cos\theta + 4 = 0$

$12\cos^2\theta - 2\cos\theta - 2 = 0$

$6\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta - 1)(3\cos\theta + 1) = 0$

$\therefore \cos\theta = \frac{1}{2}, -\frac{1}{3}$

WHEN $\cos\theta = \frac{1}{2}$ $\sin\theta = \pm\sqrt{3}/2$

WHEN $\cos\theta = -\frac{1}{3}$ $\sin\theta = \pm\sqrt{8}/3$

$\therefore$ THE ROOTS ARE $\frac{1}{2}(1 \pm \sqrt{3}i)$ & $\frac{1}{3}(-1 \pm \sqrt{8}i)$

QUESTION 7: 13/

(a) i $\left(\frac{a+b}{2}\right)^2 - ab = \frac{a^2+2ab+b^2}{4} - ab$

$= \frac{a^2-2ab+b^2}{4}$

$= \frac{(a-b)^2}{4}$

≥ 0

$\therefore \left(\frac{a+b}{2}\right)^2 \geq ab$

$\therefore \frac{a+b}{2} \geq \sqrt{ab}$

ii FROM (i) PUT $a = \frac{1}{x}$ & $b = \frac{1}{y}$

$\therefore \frac{1}{x} + \frac{1}{y} \geq 2\sqrt{\frac{1}{x} \times \frac{1}{y}}$

$\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$ ——— (1)

$\frac{x+y}{xy} \geq \frac{2}{\sqrt{xy}}$

$\frac{x+y}{2} \geq \sqrt{xy}$

$\therefore \frac{c}{2} \geq \sqrt{xy}$

$\therefore \frac{1}{\sqrt{xy}} \geq \frac{2}{c}$

$\therefore$ FROM (1) $\frac{1}{x} + \frac{1}{y} \geq 2 \times \frac{2}{c}$

$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{c}$

(b) $(2n+3)^2 - 4(n+1)(n+2) = 4n^2+12n+9-4n^2-12n-8$

$= 1$

> 0

$\therefore (2n+3)^2 > 4(n+1)(n+2)$

$\therefore 2n+3 > 2\sqrt{(n+1)(n+2)}$

PROVE THAT $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2(\sqrt{n+1} - 1)$ 14

STEP 1:
PROVE TRUE FOR $n = 1$

$LHS = \frac{1}{\sqrt{1}} = 1$

$RHS = 2(\sqrt{1+1} - 1)$

$= 2\sqrt{2} - 2$

$\doteq .8$

$\therefore LHS > RHS$

$\therefore$ true for $n = 1$

STEP 2: ASSUME TRUE FOR $n = k$

i.e. $S_k = \sum_{r=1}^{k} \frac{1}{\sqrt{r}} > 2(\sqrt{k+1} - 1)$

STEP 3: PROVE TRUE FOR $n = k+1$

(i.e. PROVE $S_{k+1} > 2(\sqrt{k+2} - 1)$

$S_{k+1} = S_k + T_{k+1}$

$> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}$

$> \frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}}$

$> \frac{2k+3 - 2\sqrt{k+1}}{\sqrt{k+1}}$

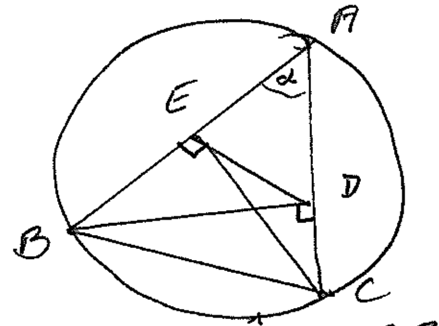
$> \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{k+1}}{\sqrt{k+1}}$ FROM PART (1)

$> 2\sqrt{k+2} - 2$

$> 2(\sqrt{k+2} - 1)$

STEP 4: THEREFORE TRUE FOR $n = k+1$ IF TRUE FOR $n = k$ BUT IT IS TRUE FOR $n = 1$ & THEREFORE TRUE FOR $n = 2$ etc & BY THE PROCESS OF MATHEMATICAL INDUCTION TRUE FOR ALL INTEGRAL VALUES OF n.

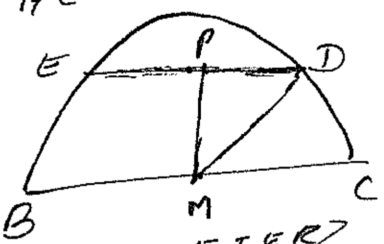
(c)



(i) $B\hat{E}C$ & $B\hat{D}C$ ARE BOTH 90° (GIVEN)
$\therefore$ E & D LIE ON A CIRCLE WHOSE DIAMETER IS BC ($\angle$'s IN A SEMI-CIRCLE = 90°)
$\therefore$ BCDE IS A CYCLIC QUAD

(ii) SINCE BC IS A CONSTANT LENGTH IT SUBTENDS A CONSTANT ANGLE α AT THE CIRCUMFERENCE.
NOW $A\hat{B}D = 90 - \alpha$ (ΔABD IS RT $\angle$'d)
THIS ANGLE MUST ALSO BE CONSTANT AS α IS CONSTANT & THIS IS SUBTENDED BY ED AT THE CIRCUMFERENCE OF CIRCLE EDCB.
$\therefore$ ED MUST BE A CONSTANT LENGTH.

(iii) LET P & M BE THE MIDPOINTS OF ED & BC resp.
JOIN MP & MD



M IS THE CENTRE OF CIRCLE BCDE (BC IS THE DIAMETER)
$\therefore$ MP $\perp$ ED (LINE FROM CENTRE TO MIDPT OF CHORD MEETS IT AT 90°)
$\therefore MP^2 = MD^2 - PD^2$ (PYTH. THM)
BUT MD & PD ARE CONSTANT
$\therefore MP^2$ IS CONSTANT
HENCE LOCUS OF P IS A CIRCLE WITH THE CENTRE AT THE MIDPOINT OF BC.

16/

QUESTION 8:

(a) $\sin 2x + \sin 4x = \sin 6x$

$2\sin 3x \cos x = 2\sin 3x \cos 3x$

$2\sin 3x(\cos x - \cos 3x) = 0$

$\therefore 2\sin 3x = 0$ OR $\cos 3x = \cos x$

$\therefore 3x = n\pi$ $\qquad$ $3x = 2n\pi \pm x$

$x = \frac{n\pi}{3}$ $\qquad$ $2x = 2n\pi$ OR $4x = 2n\pi$

$x = n\pi$ OR $x = \frac{n\pi}{2}$

SINCE $n\pi$ IS INCLUDED IN $\frac{n\pi}{2}$

THE SOLUTIONS ARE $x = \frac{n\pi}{3}$ or $\frac{n\pi}{2}$

(b) $|z_1 + z_2| \le |z_1| + |z_2|$

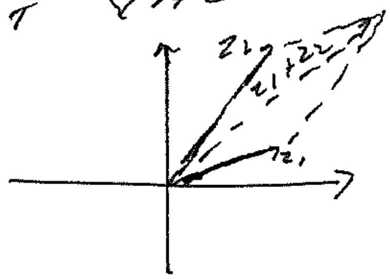
$= 5 + 13$

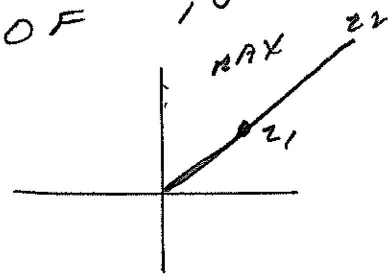
$= 18$

& THIS GREATEST VALUE OF 18

IS OBTAINED

WHEN $z_2 = kz_1$





$\therefore |z_2| = |kz_1|$

$13 = 5k$

$\therefore k = 13/5$

$\therefore z_2 = 13/5\,(3+4i)$

$= \frac{39}{5} + \frac{52i}{5}$

(c) $x^2 - 2x + 4 = 0$

$x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$

$\therefore \alpha = 1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

& $\beta = 1 - \sqrt{3}i = 2\left(\cos -\frac{\pi}{3} \cancel{+} i\sin -\frac{\pi}{3}\right)$

$= 2\left(\cos\frac{\pi}{3} - i\sin\pi/3\right)$

$\alpha^n + \beta^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$

$= 2 \cdot 2^n \cos\frac{n\pi}{3}$

$= 2^{n+1}\cos\frac{n\pi}{3}$

(d) $f(x) = e^{x}\left(1-\frac{x}{10}\right)^{10}$ 17

$\therefore$ FOR T.P. $f'(x) = 0$

$f'(x) = e^{x} \times 10\left(1-x/10\right)^{9} \times -\frac{1}{10} + e^{x}\left(1-x/10\right)^{10}$

$= -e^{x}\left(1-\frac{x}{10}\right)^{9} + e^{x}\left(1-\frac{x}{10}\right)^{10}$

$= e^{x}\left(1-\frac{x}{10}\right)^{9}\left[-1+1-\frac{x}{10}\right]$

$= -\frac{xe^{x}}{10}\left(1-\frac{x}{10}\right)^{9} = 0$

$\therefore \left.\begin{array}{l} x = 0 \\ y = 1 \end{array}\right\} \quad \left.\begin{array}{l} 10 \\ 0 \end{array}\right\}$ ARE T.P.

TEST $x=0$ IN $f'(x)$

x	-1	0	1
$f'(x)$	$+$	0	$-$

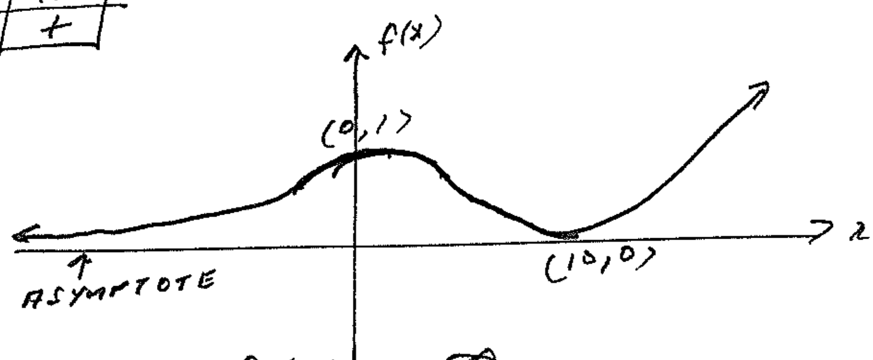
$\therefore$ $(0,1)$ IS A MAXIMUM T.P.

TEST $x = 10$

x	9	10	11
$f'(x)$	$-$	0	$+$

$\therefore$ $(10,0)$ IS A MINIMUM T.P

ii



AS $x \to \infty$ $f(x) \to \infty$

AS $x \to -\infty$ $f(x) \to 0$

(iii) FOR $x < 10$, $f(x) \le 1$

$\therefore e^{x}\left(1-\frac{x}{10}\right)^{10} \le 1$

$\therefore e^{x} \le \frac{1}{\left(1-x/10\right)^{10}}$ (NB $\left(1-\frac{x}{10}\right)^{10}$ IS POS)

$e^{x} \le \left(1-\frac{x}{10}\right)^{-10}$

(iv) LET $x=1$: $e \le (9/10)^{-10} = \left(\frac{10}{9}\right)^{10}$

LET $x=-1$: $e^{-1} \le (11/10)^{-10}$

$\frac{1}{e} \le \left(\frac{10}{11}\right)^{10}$

$\therefore e \ge (11/10)^{10}$

$\therefore \left(\frac{11}{10}\right)^{10} \le e \le \left(\frac{10}{9}\right)^{10}$.