Name: ______________________ Class: ________

# WHITEBRIDGE HIGH SCHOOL

# 2006

## Higher School Certificate

## Trial HSC Examination

# MATHEMATICS EXTENSION 2

Time Allowed: 3 hours

*(Reading time: 5 minutes)*

**Directions to Candidates**

- All questions of equal value.
- Commence each question on a new page.
- Marks may be deducted for careless or badly arranged work.

**Question 1** (15 marks) Commence each question on a SEPARATE page

a. Find $\int \frac{dx}{x \log_e x}$ **2**

b. Find $\int \frac{dx}{\sqrt{3 + 2x - x^2}}$ **2**

c. Find $\int \frac{dx}{(x+1)(x^2+4)}$ **4**

d. Using the substitution $t = \tan \frac{x}{2}$, calculate $\int \frac{15}{17 + 8\cos x}\,dx$, leaving your answer in terms of $t$. **3**

e. i. Differentiate $\frac{x}{\sqrt{x-3}}$. **1**

ii. Hence evaluate $\int_4^7 \frac{2x - 9}{2(x-3)\sqrt{x-3}}\,dx$ **3**

**Question 2** (15 marks) Commence each question on a SEPARATE page

a. i. Express $-1 + i\sqrt{3}$ in modulus argument form. **2**

ii. Hence evaluate $(-1 + i\sqrt{3})^{-6}$ **2**

b. If $z$ is a non-zero complex number such that $z + \frac{1}{z}$ is real, prove that $Im(z) = 0$ or $|z| = 1$. **3**

c. Sketch the region where the inequalities $-\frac{\pi}{2} \leq \arg(z - 1 - 2i) \leq \frac{\pi}{4}$, and $|z| \leq \sqrt{5}$ both hold. **3**

d. Let $z$ be a complex number for which $|z| = 1$ and $\arg z = \theta$, where $0 < \theta < \frac{\pi}{2}$.

i. Show that $|1 - z| = \sqrt{2 - 2\cos\theta}$ and $|1 + z| = \sqrt{2 + 2\cos\theta}$ **3**

ii. Hence find the value of $\left|\frac{2}{1 - z^2}\right|$ in terms of $\theta$. **2**

**Question 3** (15 marks) Commence each question on a SEPARATE page

a. Show that $\int_0^{\frac{\pi}{4}} x \sin x \, dx = \frac{\sqrt{2}}{8}(4 - \pi)$ **3**

b. The shape of a particular cake can be represented by rotating the region between the curve $y = \sin x$ and the $x$-axis, from $x = 0$ and $x = \frac{\pi}{4}$, about the line $x = \frac{\pi}{4}$. Using the method of cylindrical shells, find the volume of the cake. **4**

c. The hyperbola $H$ has equation $9x^2 - 4y^2 = 36$.

i. Find the co-ordinates of the foci, S and S′. **2**

ii. Find the equations of the directrices. **1**

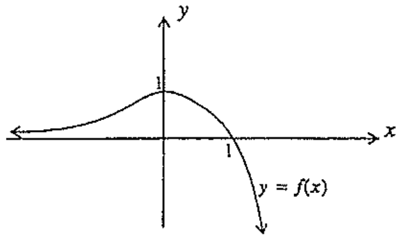
iii. Find the equations of the asymptotes. **1**

iv. Sketch the curve $H$ indicating the information obtained in i. to iii. **1**

v. The point $P(x_0, y_0)$ lies on $H$. Prove that the equation of the tangent at P is $9x_0x - 4y_0y = 36$. **3**

**Question 4** (15 marks) Commence each question on a SEPARATE page

a. The graph of $y = f(x)$ is sketched below.
There is a stationary point at (0, 1)




Use this graph to sketch the following without calculus, showing essential features.

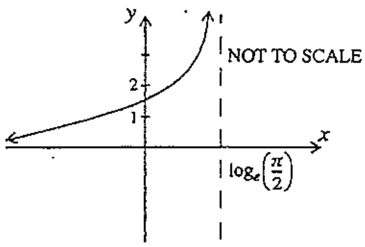
i. $y = f\left(\frac{x}{2}\right)$ **1**

ii. $y = x + f(x)$ **2**

iii. $y = \frac{1}{f(x)}$ **2**

iv. $y = f\left(\frac{1}{x}\right)$ **2**

b. The diagram shows part of the curve $y = \tan(e^x)$, where $x < \log_e\left(\frac{\pi}{2}\right)$. The part to the right of $\log_e\left(\frac{\pi}{2}\right)$ has not yet been drawn.




i. By considering values of x greater than $\log_e\left(\frac{\pi}{2}\right)$, find the smallest positive solution to the equation $\tan(e^x) = 0$. **1**

ii. Copy the diagram and hence sketch the curve $y = \tan(e^x)$ for $x < \log_e\left(\frac{3\pi}{2}\right)$. **1**

iii. How many solutions are there to the equation $\tan(e^x) = 0$ in the domain $1 < x < 3$? **2**

iv. Find the equation of the inverse function of the $y = \tan(e^x)$ for the case when

α. $x < \log_e\left(\frac{\pi}{2}\right)$. **2**

β. $\log_e\left(\frac{\pi}{2}\right) < x < \log_e\left(\frac{3\pi}{2}\right)$ **2**

**Question 5** (15 marks) Commence each question on a SEPARATE page

a. i. Prove that $\tan^{-1} n - \tan^{-1}(n - 1) = \tan^{-1} \dfrac{1}{n^2 - n + 1}$, where $n$ is a positive integer. **2**

ii. Hence evaluate $\tan^{-1} 1 + \tan^{-1} \dfrac{1}{3} + \ldots + \tan^{-1} \dfrac{1}{n^2 - n + 1}$. **2**

iii. Hence find the limit $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{n^2 - n + 1}$. **1**

b. A food package of mass $m$ kg has a parachute device attached. It is released from rest from the top of a cliff 100 metres high. During its fall, the only forces acting are gravity, and owing to the parachute, a resistive force of magnitude $\dfrac{1}{10}mv^2$, where $v$ metres per second is the speed of the package

After $\dfrac{1}{2}\ln 99$ seconds, the parachute disintegrates, and then the only force acting on the particle is due to gravity.

The acceleration due to gravity is taken as $10\ \text{m s}^{-1}$. At time $t$ seconds after being dropped, the package has fallen a distance of $x$ metres from the plane, and its speed is $v\,\text{m s}^{-1}$.

i. Show that while the parachute is operating, $\ddot{x} = 10 - \dfrac{v^2}{10}$. Hence show that $v = 10\left(\dfrac{e^{2t} - 1}{e^{2t} + 1}\right)$ and $x = 5\ln\left(\dfrac{100}{100 - v^2}\right)$ **5**

ii. Find the exact speed of the package and the exact vertical distance fallen just before the parachute disintegrates. **2**

iii. Find the speed of the package just before it reaches the ground. Give your answer correct to two significant figures. **3**

**Question 6** (15 marks) Commence each question on a SEPARATE page

a. The polynomial $P(z)$ has equation $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$ **3**

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of two quadratic factors with real coefficients.

b. A particle moves in a straight line. It is placed at the origin on the x-axis and is then released from rest. When at position $x$, its acceleration is given by

$$\ddot{x} = -9x + \frac{5}{(2-x)^2}.$$

i. Show that $v^2 = \dfrac{x(3x-5)(3x-1)}{2-x}$. **2**

ii. Prove that the particle moves between two points on the $x$-axis, and find these points. **4**

c. An athlete is throwing a javelin. The horizontal and vertical components of the speed of the javelin after t seconds are:

$$\dot{x} = V + 3V\cos\theta \text{ and } \dot{y} = 3V\sin\theta - gt$$

where $V$ is a positive constant, $\theta$ is an acute angle, and $x$ and $y$ are the horizontal and vertical displacements from the point of projection.

(Assume when $t = 0$, $x = 0$ and $y = 0$)

Show that:

i. $x = Vt + 3Vt\cos\theta$ and $y = 3Vt\sin\theta - \frac{1}{2}gt^2$. **2**

ii. the range of the javelin, $R$ metres, is given by $R = \dfrac{6V^2\sin\theta}{g}(3\cos\theta + 1)$. **2**

iii. the angle $\theta$ which will yield maximum range is $\theta = \cos^{-1}\left(\dfrac{\sqrt{73}-1}{12}\right)$. **3**

**Question 7** (15 marks) Commence each question on a SEPARATE page

a. Given that the quartic polynomial $p(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a zero of multiplicity three, factorise the polynomial completely and find all its zeroes. **3**

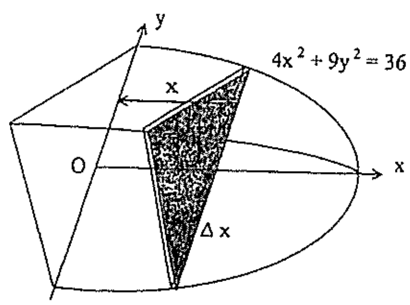
b. Let $Q(x) = x^3 + px + q$, where $p$ and $q$ are real and non-zero. Two of the zeroes of $Q(x)$ are $a + ib$ and $k$, where $a$, $b$ and $k$ are real and non-zero and $k < 0$.
It is known that the graph of $y = Q(x)$ has two turning points.

i. By a consideration of $Q'(x)$, show that $p < 0$. **1**

ii. Deduce that $a > 0$. **2**

iii. Show that $q = 8a^3 + 2ap$ **3**

c.




The base of the solid **K** shown in the diagram is the region in the $x$-$y$ plane enclosed between the semi-ellipse $4x^2 + 9y^2 = 36$ and the $y$-axis. Each cross section perpendicular to the $x$-axis is an equilateral triangle.

i. Consider a slice of the solid with thickness $\Delta x$ and distance $x$ from the $y$-axis. Find the area of this slice in terms of $x$. **2**

ii. Find the volume of the solid **K**. **2**

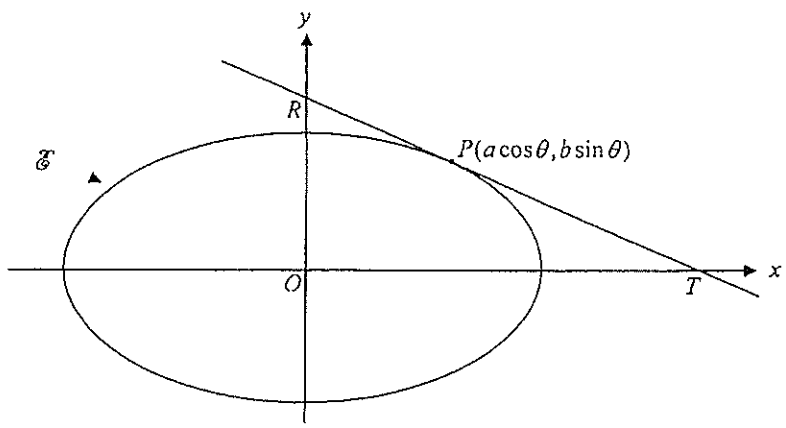
iii. Solid **J** has the same base as solid **K** but its perpendicular cross sectional slice is an isosceles right angled triangle with its hypotenuse in the $x$-$y$ plane. Find the ratio of volumes of solid **K** to solid **J**. **2**

**Question 8** (15 marks) Commence each question on a SEPARATE page

a. A particle $P$ of mass $m$ moves with constant angular velocity $\omega$ on a circle of radius $r$. Its position at time $t$ is given by: $x = r\cos\theta \quad y = r\sin\theta$, where $\theta = \omega t$. **3**

Show that there is an inward radial force of magnitude $mr\omega^2$ acting on $P$.

b.




The ellipse E with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ shown in the diagram above, has a tangent at the point P($a\cos\theta$, $b\sin\theta$). The tangent cuts the $x$-axis at $T$ and the $y$-axis at R.

i. Show that the equation of the tangent at the point $P$ is **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

ii. If $T$ is the point of intersection between the tangent at point $P$ and one of the directrices of the ellipse, show that $\cos\theta = e$. **2**

iii. Hence find the angle that the focal chord through $P$ makes with the $x$-axis. **1**

iv. Using similar triangles or otherwise, show that $RP = e^2RT$. **3**

c. Let $I_n = \displaystyle\int_0^1 x(x^2 - 1)^n \, dx$ for $n = 0, 1, 2, \ldots$

Use integration by parts to show that $I_n = \dfrac{-n}{n+1} I_{n-1}$ for $n \geq 1$. **4**

## Standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$

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Question 1

a. $\int \frac{dx}{x \ln x}$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\therefore dx = x\,du$

$= \int \frac{x\,du}{x\,u}$

$= \int \frac{1}{u}\,du$

$= \ln(\ln x) + c$ 2

b. $\int \frac{dx}{\sqrt{3+2x-x^2}}$

$= \int \frac{dx}{\sqrt{4-1+2x-x^2}}$

$= \int \frac{dx}{\sqrt{4-(x^2-2x+1)}}$

$= \int \frac{dx}{\sqrt{4-(x-1)^2}}$

$= \sin^{-1}\frac{x-1}{2} + c$ 2

c. $\int \frac{dx}{(x+1)(x^2+4)}$

Now, $\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

$\therefore a(x^2+4) + (bx+c)(x+1) = 1$

Let $x=-1 \therefore 5a + 0 = 1$

$a = \frac{1}{5}$

$x = 0 \therefore 4a + c(1) = 1$

$\frac{4}{5} + c = 1$

$c = \frac{1}{5}$

$x = 1 \therefore 5(\frac{1}{5}) + (b+\frac{1}{5})(2) = 1$

$1 + 2b + \frac{2}{5} = 1$

$2b = -\frac{2}{5}$

$b = -\frac{1}{5}$

$\therefore \int \frac{\frac{1}{5}}{x+1} + \frac{-\frac{1}{5}x + \frac{1}{5}}{x^2+4}\,dx$

$= \frac{1}{5}\int \frac{1}{x+1} + \frac{1-x}{x^2+4}\,dx$

$= \frac{1}{5}\int \frac{1}{x+1} + \frac{1}{x^2+4} - \frac{x}{x^2+4}\,dx$ 4

$= \frac{1}{5}\left[\ln|x+1| + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\ln|x^2+4|\right] + c$

d. $\int \frac{15}{17+8\cos x}\,dx$

As $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dt = \frac{1}{2}(1+t^2)\,dx$

$\therefore dx = \frac{2\,dt}{1+t^2}$

Also, $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore \int \frac{15}{17+8\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$

$= \int \frac{30\,dt}{17+17t^2+8-8t^2}$

$= \int \frac{30\,dt}{25+9t^2}$

$= \frac{30}{9}\int \frac{dt}{\frac{25}{9}+t^2}$

$= \frac{30}{9} \cdot \frac{3}{5}\tan^{-1}\frac{3t}{5} + c$

$= 2\tan^{-1}\frac{3t}{5} + c$ 3

e. i. $\frac{d}{dx}\left[\frac{x}{\sqrt{x-3}}\right]$

$= \frac{\sqrt{x-3}\cdot 1 - \frac{1}{2}(x-3)^{-\frac{1}{2}}\cdot x}{x-3}$

$= \frac{\sqrt{x-3} - \frac{x}{2\sqrt{x-3}}}{x-3}$

$$= \frac{x-3-\frac{x}{2}}{(x-3)\sqrt{x-3}}$$

$$= \frac{2x-6-x}{2(x-3)\sqrt{x-3}}$$

$$= \frac{x-6}{2(x-3)\sqrt{x-3}}$$ 1

ii. $\displaystyle\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}\,dx$

$$= \int_4^7 \frac{x-6}{2(x-3)\sqrt{x-3}}\,dx + \int_4^7 \frac{\cancel{x-3}}{2\cancel{(x-3)}\sqrt{x-3}}\,dx$$

$$= \left.\frac{x}{\sqrt{x-3}}\right]_4^7 + \frac{1}{2}\int_4^7 (x-3)^{-\frac{1}{2}}\,dx$$

$$= \left(\frac{7}{2}-4\right) + \frac{1}{\cancel{2}}\left[\frac{(x-3)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}\right]_4^7$$

$$= -\frac{1}{2} + (2-1)$$

$$= -\frac{1}{2}+1$$ 3

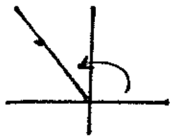
$$= \frac{1}{2}$$

Question 2

a. i. $r = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= 2$

$\tan\theta = \dfrac{\sqrt{3}}{-1}$



$\therefore \theta = \dfrac{2\pi}{3}$

$\therefore -1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ 2

ii. $(-1+\sqrt{3}i)^{-6}$

$$= 2^{-6}\left[\cos -6\left(\frac{2\pi}{3}\right) + i\sin -6\left(\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{64}\left[\cos(-4\pi) + i\sin(-4\pi)\right]$$

$$= \frac{1}{64}\left[1 + i.0\right]$$

$$= \frac{1}{64}$$ 2

b. Let $z = x+iy$

$$\therefore z+\frac{1}{z} = x+iy + \frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$

$$= x+iy + \frac{x-iy}{x^2+y^2}$$

$$= \left[x+\frac{x}{x^2+y^2}\right] + i\left[y-\frac{y}{x^2+y^2}\right]$$

Now if $z+\frac{1}{z}$ is real, then

$$y - \frac{y}{x^2+y^2} = 0$$

$\therefore y(x^2+y^2) - y = 0$

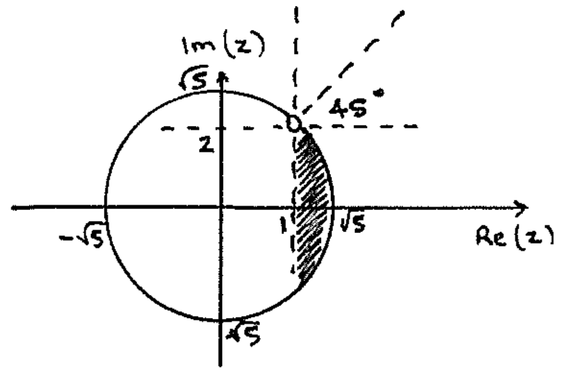
$\therefore y(x^2+y^2-1) = 0$

ie $y=0$ or $x^2+y^2=1$

but $\text{Im}(z) = y$ $\therefore \text{Im}(z) = 0$

but $|z| = \sqrt{x^2+y^2}$ $\therefore |z| = \sqrt{1}$ $\therefore |z| = 1$ 3

c.

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d. Let $z = \cos\theta + i\sin\theta$

$\therefore |1-z| = |1-\cos\theta - i\sin\theta|$

$$= \sqrt{(1-\cos\theta)^2 + (-\cancel{i}\sin\theta)^2}$$

$$= \sqrt{1-2\cos\theta+\cancel{\cos^2\theta+\sin^2\theta}}$$

$$= \sqrt{2-2\cos\theta}$$

$|1+z| = |1+\cos\theta + i\sin\theta|$

$$= \sqrt{(1+\cos\theta)^2+\sin^2\theta}$$

$$= \sqrt{1+2\cos\theta+\cancel{\cos^2\theta+\sin^2\theta}}$$ 3

$$= \sqrt{2+2\cos\theta}$$

$\therefore |1-z| = \sqrt{2-2\cos\theta} \quad |1+z| = \sqrt{2+2\cos\theta}$

ii. $\left|\frac{2}{1-z^2}\right| = \frac{2}{|1-z||1+z|}$

$= \frac{2}{\sqrt{2-2\cos\theta}\cdot\sqrt{2+2\cos\theta}}$

$= \frac{2}{\sqrt{(2-2\cos\theta)(2+2\cos\theta)}}$

$= \frac{\cancel{2}}{\cancel{2}\sqrt{1-\cos^2\theta}}$

$= \frac{1}{\sin\theta}$ 2

$= \operatorname{cosec}\theta$

Question 3

a. $\int_0^{\pi/4} x\sin x\, dx$ $\quad u = x$, $u' = 1$, $v' = \sin x$, $v = -\cos x$

$\therefore \int uv'\,dx = uv - \int u'v\,dx$

$= -x\cos x\Big]_0^{\pi/4} + \int_0^{\pi/4}\cos x\,dx$

$= -\pi/4\cdot\cos\pi/4 - 0 + \sin x\Big]_0^{\pi/4}$

$= -\pi/4\cdot\frac{1}{\sqrt2} + [\sin\pi/4 - \cancel{\sin 0}]$

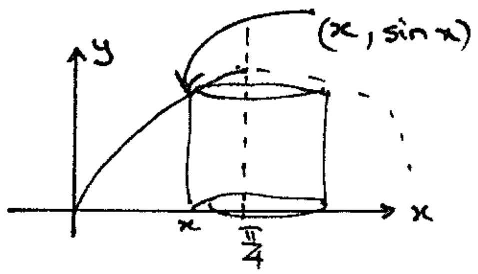
$= \frac{-\pi}{4\sqrt2} + \frac{1}{\sqrt2}$

$= \left(\frac{-\pi}{4}+1\right)\frac{1}{\sqrt2}\times\frac{\sqrt2}{\sqrt2}$

$= \frac{\sqrt2}{2}\left(1-\frac{\pi}{4}\right)$

$= \frac{\sqrt2}{8}(4-\pi)$ 3

b.




$\therefore$ Volume of shell is $\delta V$, with thickness $\delta x$, height $y = \sin x$

$\therefore$ circumference $= 2\pi\left(\frac{\pi}{4}-x\right)\sin x\,\delta x$

$\therefore \delta V = 2\pi(\pi/4 - x)\sin x\,\delta x$

$\therefore V = 2\pi\int_0^{\pi/4}(\pi/4 - x)\sin x\,dx$

$= 2\pi\left[\frac{\pi}{4}\int_0^{\pi/4}\sin x\,dx - \int_0^{\pi/4}x\sin x\,dx\right]$

$= \frac{\pi^2}{2}\int_0^{\pi/4}\sin x\,dx - 2\pi\int_0^{\pi/4}x\sin x\,dx$

$= \frac{\pi^2}{2}\Big[-\cos x\Big]_0^{\pi/4} - 2\pi\cdot\frac{\sqrt2}{8}(4-\pi)$

from a.

$= \frac{\pi^2}{2}\left[-\frac{1}{\sqrt2}+1\right] - \frac{\sqrt2\pi}{4}(4-\pi)$

$= \frac{-\pi^2}{2\sqrt2} + \frac{\pi^2}{2} - \sqrt2\pi + \frac{\sqrt2\pi^2}{4}$

$= \cancel{-\frac{\sqrt2\pi^2}{4}} + \frac{\pi^2}{2} - \sqrt2\pi + \cancel{\frac{\sqrt2\pi^2}{4}}$

$= \frac{\pi}{2}(\pi - 2\sqrt2)$ units$^3$ 4

c. i. $9x^2 - 4y^2 = 36$

$\therefore \frac{x^2}{4} - \frac{y^2}{9} = 1$

$\therefore a = 2$ and $b = 3$

$\therefore b^2 = a^2(e^2-1)$

$\therefore 9 = 4(e^2-1)$

$e^2 - 1 = \frac{9}{4}$

$e^2 = \frac{13}{4}$

$e = \frac{\sqrt{13}}{2}$

$\therefore$ foci $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$ 2

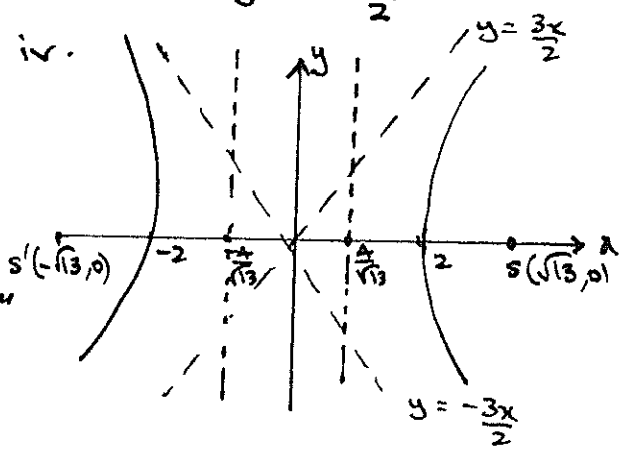
ii. Directrices:

$x = \frac{4}{\sqrt{13}}$, $x = -\frac{4}{\sqrt{13}}$ 1

iii. Asymptotes:

$y = \pm\frac{3x}{2}$ 1

iv.

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v. $9x^2 - 4y^2 = 36$

Differentiate with respect to $x$:

$$18x - 8y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{18x}{8y}$$

At $P(x_0, y_0)$: $m = \frac{18x_0}{8y_0} = \frac{9x_0}{4y_0}$

$$\therefore y - y_0 = \frac{9x_0}{4y_0}(x - x_0)$$

$$4y_0y - 4y_0^2 = 9x_0x - 9x_0^2$$

$$\therefore 9x_0x - 4y_0y = 9x_0^2 - 4y_0^2 \quad (1)$$

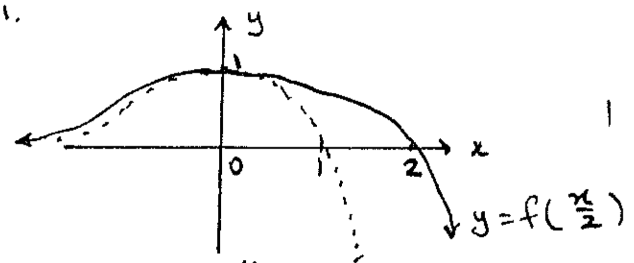
But as $(x_0, y_0)$ lies on $9x^2 - 4y^2 = 36$

$$\therefore 9x_0^2 - 4y_0^2 = 36$$

$\therefore$ (1) becomes $\therefore 9x_0x - 4y_0y = 36$ 3

## Question 4:

a. i.

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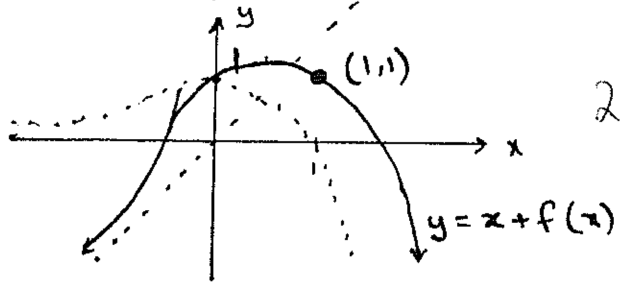
ii.

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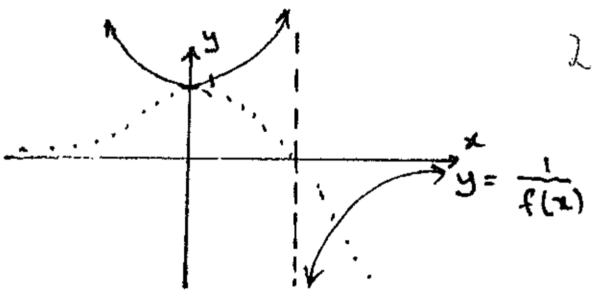
iii.

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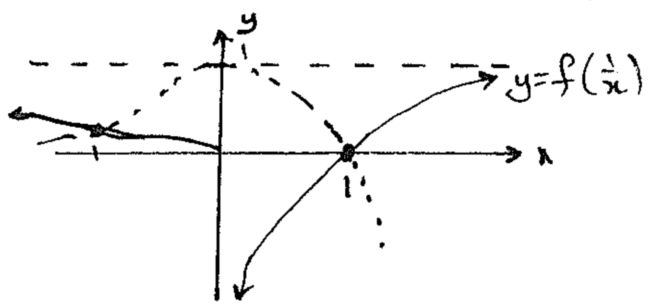
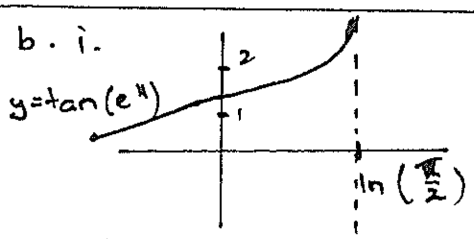
iv.




2

b. i.



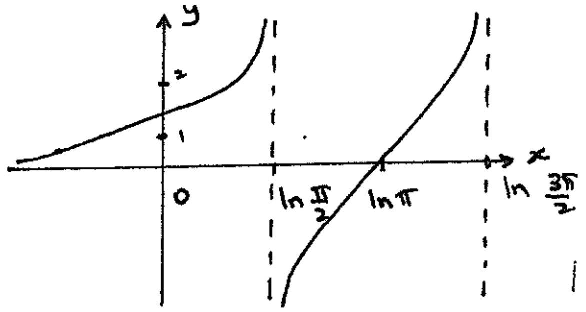

$\tan e^x = 0$

$\therefore e^x = \cancel{0}, \pi, \ldots$

$\therefore e^x = \pi$

$\therefore x = \ln\pi$ 1

ii.




1

iii. $\tan e^x = 0$

$\therefore e^x = \cancel{0}, \pi, 2\pi, \ldots$

$x = \ln\pi, \ln 2\pi, \ldots$

Now $\ln 6\pi = 2.93$ and $\ln 7\pi > 3$

$\therefore$ there are 6 solutions in the domain $1 < x < 3$. 2

iv. α. $y = \tan e^x$

$\therefore x = \tan e^y$

$\therefore e^y = \tan^{-1}x$

$\therefore y = \ln(\tan^{-1}x)$ 2

β. When $\ln\frac{\pi}{2} < x < \ln\frac{3\pi}{2}$ then the equation of the original function is

$y = \tan(e^x - \pi)$

$\therefore x = \tan(e^y - \pi)$

$e^y - \pi = \tan^{-1}x$

$e^y = \pi + \tan^{-1}x$

$y = \ln(\pi + \tan^{-1}x)$ 2

**Question 5** 

a. i. Let $\alpha = \tan^{-1} n$

$\tan\alpha = n$

and $\beta = \tan^{-1}(n-1)$

$\tan\beta = n-1$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{n-(n-1)}{1+n(n-1)}$$

$$= \frac{1}{1+n^2-n}$$

$$= \frac{1}{n^2-n+1}$$

$$\tan^{-1}\cancel{\tan}(\alpha-\beta) = \tan^{-1}\frac{1}{n^2-n+1}$$

$$\therefore \alpha-\beta = \tan^{-1}\frac{1}{n^2-n+1} \qquad 2$$

$$\therefore \tan^{-1} - \tan^{-1}(n-1) = \tan^{-1}\frac{1}{n^2-n+1}$$

ii. Let $n=1$

$$\therefore \tan^{-1}1 - \tan^{-1}0 = \tan^{-1}\frac{1}{1^2-1+1}$$

$$\therefore \tan^{-1}1 = \tan^{-1}1$$

Let $n=2$

$$\therefore \tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{2^2-2+1}$$

$$= \tan^{-1}\frac{1}{3}$$

Let $n=3$

$$\therefore \tan^{-1}3 - \tan^{-1}2 = \tan^{-1}\frac{1}{3^2-3+1}$$

$$= \tan^{-1}\frac{1}{7}$$

$$\therefore \tan^{-1}1 + \tan^{-1}\frac{1}{3} + \cdots + \tan^{-1}\frac{1}{n^2-n+1} \qquad 2$$

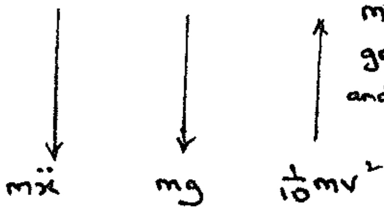
$$= \cancel{\tan^{-1}1} + \cancel{\tan^{-1}2} - \cancel{\tan^{-1}1} + \cancel{\tan^{-1}3} - \cancel{\tan^{-1}2} + \cdots + \tan^{-1}n + \tan^{-1}\cancel{(n-1)}$$

$$= \tan^{-1}n$$

iii. $$\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{n^2-n+1} = \lim_{n\to\infty} \tan^{-1}n$$

$$= \frac{\pi}{2} \qquad 1$$

b. i.




motion is down, gravity is down and resistance is upward.

$$\therefore m\ddot{x} = mg - \frac{1}{10}mv^2$$

$$\therefore \ddot{x} = g - \frac{v^2}{10}$$

$$\therefore \ddot{x} = 10 - \frac{v^2}{10} \qquad 1$$

$$\frac{dv}{dt} = 10 - \frac{v^2}{10}$$

$$= \frac{100-v^2}{10}$$

$$\frac{dt}{dv} = \frac{10}{100-v^2}$$

$$t = \int \frac{10}{100-v^2}\,dv$$

$$= 10\int \frac{1}{(10-v)(10+v)}\,dv$$

Now, by partial fractions:

$$\frac{a}{10-v} + \frac{b}{10+v}$$

$$= a(10+v) + b(10-v) = 1$$

$v = -10 \therefore 20b = 1 \therefore b = \frac{1}{20}$

$v = 10 \therefore 20a = 1 \therefore a = \frac{1}{20}$

$$\therefore t = 10\int \frac{1}{20(10-v)} + \frac{1}{20(10+v)}\,dv$$

$$= \frac{1}{2}\left[-\log(10-v) + \log(10+v)\right] + c$$

$$= \frac{1}{2}\left[\log\frac{10+v}{10-v}\right] + c$$

$t=0,\ v=0 \therefore 0 = \frac{1}{2}\log^{0} 1 + c$

$c = 0$

$$\therefore t = \frac{1}{2}\left[\log\frac{10+v}{10-v}\right]$$

$$\frac{10+v}{10-v} = e^{2t}$$

$$10+v = e^{2t}(10-v)$$

$$v + ve^{2t} = 10e^{2t} - 10 \qquad 2$$

$$v = \frac{10(e^{2t}-1)}{e^{2t}+1}$$

$$v\frac{dv}{dx} = 10 - \frac{v^2}{10}$$

$$\therefore \frac{dv}{dx} = \frac{10}{v} - \frac{v}{10}$$

$$= \frac{100 - v^2}{10v}$$

$$\frac{dx}{dv} = \frac{10v}{100 - v^2}$$

$$x = -5\ln(100 - v^2) + c$$

$x = 0, v = 0$

$$\therefore 0 = -5\ln 100 + c$$

$$c = 5\ln 100$$

$$\therefore x = 5\left[\ln 100 - \ln(100 - v^2)\right]$$

$$x = 5\ln\frac{100}{100 - v^2}$$ 2

ii. $t = \frac{1}{2}\ln 99$

$$\therefore v = 10\left[\frac{e^{2\cdot\frac{1}{2}\ln 99} - 1}{e^{2\cdot\frac{1}{2}\ln 99} + 1}\right]$$

$$= 10\left[\frac{99 - 1}{99 + 1}\right]$$

$$= \frac{98}{10}$$

$= 9.8$ $\therefore 9.8\ ms^{-1}$

$$x = 5\ln\left(\frac{100}{100 - 9.8^2}\right)$$ 2

$= 5\ln\frac{2500}{99}$ metres

iii. After parachute disintegrates, only gravity is acting.

$$\therefore \ddot{x} = 10$$

$$\therefore v\frac{dv}{dx} = 10$$

$$\frac{dv}{dx} = \frac{10}{v}$$

$$\frac{dx}{dv} = \frac{v}{10}$$

$$\therefore x = \frac{v^2}{20} + c$$

From ii, $v = 9.8$, $x = 5\ln\frac{2500}{99}$

$$\therefore 5\ln\frac{2500}{99} = \frac{9.8^2}{20} + c$$

$$\therefore c = 5\ln\frac{2500}{99} - 4.802$$

$$\therefore x = \frac{v^2}{20} + 5\ln\frac{2500}{99} - 4.802$$

When $x = 100$

$$\therefore 100 = \frac{v^2}{20} + 5\ln\frac{2500}{99} - 4.802$$

$\therefore v = 42$ (2 sig figs)

$\therefore 42\ ms^{-1}$ 3

Question 6

a. As $P(z)$ has real coefficients and $z - 2 + i$ is factor, then $z - 2 - i$ is also factor (conjugate pairs)

$\therefore (z - 2 + i)(z - 2 - i)$ is also factor

Now, as $2 - i$ and $2 + i$ are roots $\therefore$ sum of roots: 4

product of roots: 5

$\therefore z^2 - 4z + 5$ is a factor

$$\begin{array}{r} z^2 + 2z + 2 \\ z^2 - 4z + 5 \enclose{longdiv}{z^4 - 2z^3 - z^2 + 2z + 10} \\ \underline{z^4 - 4z^3 + 5z^2} \\ 2z^3 - 6z^2 + 2z \\ \underline{2z^3 - 8z^2 + 10z} \\ 2z^2 - 8z + 10 \\ \underline{2z^2 - 8z + 10} \\ 0 \end{array}$$

$$\therefore P(z) = (z^2 - 4z + 5)(z^2 + 2z + 2)$$ 3

b. i. $\ddot{x} = -9x + \frac{5}{(2 - x)^2}$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + \frac{5}{(2 - x)^2}$$

$$\frac{1}{2}v^2 = \frac{-9x^2}{2} + 5\frac{(2 - x)^{-1}}{-1\cdot -1} + c$$

$$\frac{1}{2}v^2 = \frac{-9x^2}{2} + \frac{5}{2 - x} + c$$

$v = 0$, $x = 0$

$\therefore 0 = 0 + \frac{5}{2} + c$ $\therefore c = -\frac{5}{2}$

$$\frac{1}{2}v^2 = \frac{-9x^2}{2} + \frac{5}{2 - x} - \frac{5}{2}$$

$$v^2 = -9x^2 + \frac{10}{2 - x} - 5$$

$$\therefore v^2 = \frac{-9x^2(2-x)+10-5(2-x)}{2-x}$$

$$= \frac{-18x^2+9x^3+10-10+5x}{2-x}$$

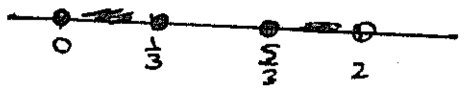
$$= \frac{9x^3-18x^2+5x}{2-x}$$

$$= \frac{x(9x^2-18x+5)}{2-x}$$

$$= \frac{x(3x-1)(3x-5)}{2-x} \qquad 2$$

ii. Now, $v^2=0 \quad \therefore x = 0, \frac{1}{3}, \frac{5}{3}$ and $v^2$ is undefined when $x=2$.

Now, motion is possible only when $v^2 \geq 0 \quad \therefore x(3x-1)(3x-5) \geq 0$



$\therefore \ 0 \leq x \leq \frac{1}{3}$ and $\frac{5}{3} \leq x < 2$

But, initially $x=0$ $\qquad 4$

$\therefore$ travel between $x=0$ & $x=\frac{1}{3}$

c. i. $\dot{x} = V + 3V\cos\theta$

$\therefore x = Vt + 3Vt\cos\theta + c_1$

$x=0,\ t=0 \ \therefore 0 = 0+0+c_1 \ \therefore c_1=0$

$\therefore x = Vt + 3Vt\cos\theta$

Also, $\dot{y} = 3V\sin\theta - gt$

$y = 3Vt\sin\theta - \frac{1}{2}gt^2 + c_2$

$y=0,\ t=0 \ \therefore 0 = 0 - 0 + c_2 \ \therefore c_2 = 0$

$\therefore y = 3Vt\sin\theta - \frac{1}{2}gt^2 \qquad 2$

ii. Range $\therefore y=0$

$\therefore 3Vt\sin\theta - \frac{1}{2}gt^2 = 0$

$t(3V\sin\theta - \frac{1}{2}gt) = 0$

$$t=0, \quad t = \frac{3V\sin\theta}{\frac{1}{2}g}$$

$$= \frac{6V\sin\theta}{g}$$

$\therefore$ Subs in $x$

ie $x = Vt + 3Vt\cos\theta$

$= t(V + 3V\cos\theta)$

$$= \frac{6V\sin\theta}{g}\cdot V(1+3\cos\theta)$$

$$= \frac{6V^2\sin\theta}{g}(1+3\cos\theta) \qquad 2$$

iii. $$\frac{dR}{d\theta} = \frac{6V^2\sin\theta}{g}(-3\sin\theta) + (1+3\cos\theta)\cdot\frac{6V^2\cos\theta}{g}$$

$$= \frac{6V^2}{g}\left[-3\sin^2\theta + \cos\theta + 3\cos^2\theta\right]$$

$$= \frac{6V^2}{g}\left[-3(1-\cos^2\theta) + \cos\theta + 3\cos^2\theta\right]$$

$$= \frac{6V^2}{g}\left[-3 + 3\cos^2\theta + \cos\theta + 3\cos^2\theta\right]$$

$$= \frac{6V^2}{g}\left[6\cos^2\theta + \cos\theta - 3\right]$$

Now $\frac{dR}{d\theta} = 0 \ \therefore 6\cos^2\theta + \cos\theta - 3 = 0$

$$\therefore \cos\theta = \frac{-1 \pm \sqrt{1-4(6)(-3)}}{12}$$

$$= \frac{-1 \pm \sqrt{73}}{12} \qquad \text{But } \cos\theta > 0$$

$$\therefore \theta = \cos^{-1}\left[\frac{\sqrt{73}-1}{12}\right]$$

Now check max/min

| $\theta$ | $\cos^{-1}\frac{\sqrt{73}-1}{12}^-$ | $\cos^{-1}\frac{\sqrt{73}-1}{12}$ | $\cos^{-1}\frac{\sqrt{73}-1}{12}^+$ |
|---|---|---|---|
| $\frac{dR}{d\theta}$ | + | 0 | − |

3

$\therefore$ Max when $\theta = \cos^{-1}\left[\frac{\sqrt{73}-1}{12}\right]$

**Question 7:**

a. $p(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$

Let triple root be $\alpha$, other root is $\beta$

$\therefore p'(x) = 4x^3 - 15x^2 - 18x + 81$

$p''(x) = 12x^2 - 30x - 18 = 0$

$2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$

$x = -\frac{1}{2}, 3$

$\therefore \alpha = -\frac{1}{2}$ or $3$

Now, $P(-\frac{1}{2}) \neq 0 \therefore \alpha = 3$

Now, sum of roots:

$$3(\alpha) + \beta = 5$$
$$9 + \beta = 5$$
$$\beta = -4$$

$\therefore$ roots are $3, 3, 3, -4$ 3

b. i. $Q(x) = x^3 + px + q$, $Q'(x) = 3x^2 + p$

Now, as $Q(x)$ has 2 turning points,

$\therefore$ $Q'(x)$ has 2 distinct real solns.

$$\therefore 3x^2 + p = 0$$
$$3x^2 = -p$$
$$x^2 = -\frac{p}{3}$$

$\therefore p$ must be negative 1

ie $p < 0$

ii. As $p, q$ real $\therefore$ $a - ib$ is also root

$\therefore$ Sum of roots: $a + i\cancel{b} + a - i\cancel{b} + k = 0$

$$2a + k = 0$$
$$a = -\frac{k}{2} \quad \text{—— ①}$$

But $k < 0$ $\therefore a > 0$ 2

iii. Product of roots $= -q$

$$\therefore k(a+ib)(a-ib) = -q$$
$$\therefore k(a^2 + b^2) = -q \quad \text{—— ②}$$

Also, roots 2 at a time:

$$\therefore k(a+ib) + k(a-ib) + (a+ib)(a-ib) = p$$
$$ka + ik\cancel{b} + ka - ik\cancel{b} + a^2 + b^2 = p$$
$$2ka + a^2 + b^2 = p$$
$$\therefore b^2 = p - 2ka - a^2 \quad \text{—— ③}$$

From ① $k = -2a$ —— ④

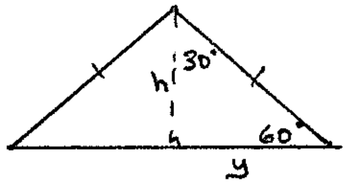
Subs ④ and ③ in ②

$$\therefore -2a(a^2 + p - 2a(-2a) - a^2) = -q$$
$$-2a(a^2 + p + 4a^2 - a^2) = -q$$
$$\therefore q = 2a(4a^2 + p)$$
$$\therefore q = 8a^3 + 2ap$$

3

c.

i.




Consider the triangle above, taken as a slice of the solid with thickness $\Delta x$

$\therefore \tan 30 = \frac{y}{h}$ $\quad \therefore h = \frac{y}{\tan 30}$

$$\therefore h = \sqrt{3}y$$

$\therefore$ Area of slice $= \frac{1}{2} \times 2y \times \sqrt{3}y$

$$= y^2\sqrt{3}$$

Now, $4x^2 + 9y^2 = 36$

$$\therefore y^2 = \frac{36 - 4x^2}{9}$$

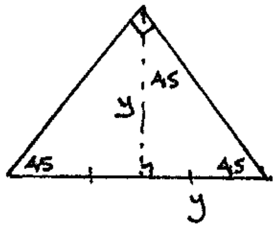
$\therefore$ Area $= \sqrt{3}\left[\frac{36 - 4x^2}{9}\right]$ units$^2$ 2

ii. $\text{Vol} = \frac{\sqrt{3}}{9}\int_0^3 36 - 4x^2 \, dx$

$$= \frac{\sqrt{3}}{9}\left[36x - \frac{4x^3}{3}\right]_0^3$$
$$= \frac{\sqrt{3}}{9}\left[108 - 36 - 0\right]$$
$$= 8\sqrt{3} \text{ units}^3$$

2

iii.




$\therefore$ Area of slice

$$= \frac{1}{2} \times 2y \times y$$
$$= y^2$$

$$\therefore \text{Vol} = \int_0^3 y^2 \, dx$$
$$= \int_0^3 \frac{36 - 4x^2}{9} \, dx$$
$$= \frac{1}{9}\int_0^3 36 - 4x^2 \, dx$$
$$= \frac{1}{9}\left[36x - \frac{4x^3}{3}\right]_0^3$$
$$= \frac{1}{9}\left[108 - 36\right]$$
$$= 8$$

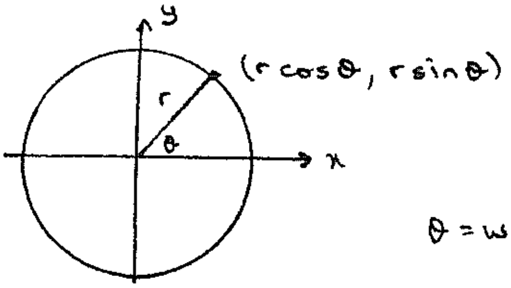
$\therefore$ Ratio of volume of K to volume of J

$$= 8\sqrt{3} : 8$$
$$= \sqrt{3} : 1$$

2

## Question 8:

a.




$\theta = wt$

$x = r\cos\theta$
$= r\cos wt$

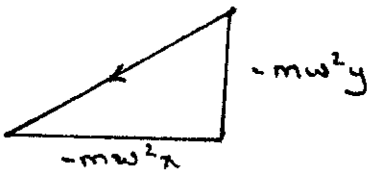
$y = r\sin\theta$
$= r\sin wt$

$\therefore \dot{x} = -rw\sin wt$ $\qquad$ $\therefore \dot{y} = rw\cos wt$

$\ddot{x} = -rw^2\cos wt$ $\qquad$ $\ddot{y} = -rw^2\sin wt$
$= -rw^2\cos\theta$ $\qquad$ $= -rw^2\sin\theta$
$= -w^2x$ $\qquad$ $= -w^2y$

$\therefore$ Horizontal: $F_x = m \cdot -w^2x$
$= -mw^2x$

Vertical: $F_y = -mw^2y$

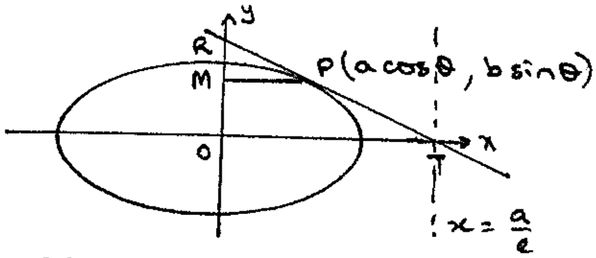



$\therefore \quad F = \sqrt{(mw^2x)^2 + (mw^2y)^2}$
$= \sqrt{m^2w^4x^2 + m^2w^4y^2}$
$= mw^2\sqrt{x^2+y^2}$

$\therefore F = mw^2r$ $\qquad$ 3

Force vector directed inwards

b.




i. Diff wrt x:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{a^2} \times \frac{b^2}{\cancel{2}y}$$

$$= \frac{-b^2x}{a^2y}$$

At $(a\cos\theta, b\sin\theta)$,

$$m = \frac{-b^{\cancel{2}}\cancel{a}\cos\theta}{a^{\cancel{2}}\cancel{b}\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$$\therefore y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cancel{\cos^2\theta})$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{—— (1)} \qquad 2$$

ii. As T lies on an asymptote

$\therefore T\left(\frac{a}{e}, 0\right)$

$\therefore$ subs in (1) $\qquad$ --

$$\therefore \frac{\frac{a}{e}\cos\theta}{a} + 0 = 1$$

$\frac{a}{e}\cos\theta = a$

$\frac{1}{e}\cos\theta = 1$

$\cos\theta = e$ $\qquad$ 2

iii. Focus $(ae, 0)$

and as $\cos\theta = e$

$\therefore P(ae, b\sin\theta)$

$\therefore$ focal chord is vertical

$\therefore$ makes $90°$ with x axis $\qquad$ 1

iv. For R, let $x = 0$ in tangent

$$\therefore \frac{y\sin\theta}{b} = 1$$

$\therefore y = \frac{b}{\sin\theta}$ $\quad \therefore R\left(0, \frac{b}{\sin\theta}\right)$

Let M lie on y axis, where $PM \perp RM$

$\therefore M(0, b\sin\theta)$

$\therefore \triangle ROT \parallel\!\mid \triangle RMP$

$$\therefore \frac{RP}{RT} = \frac{RM}{RO}$$

$$= \frac{\frac{b}{\sin\theta} - b\sin\theta}{\frac{b}{\sin\theta}}$$

$$= \frac{b - b\sin^2\theta}{\sin\cancel{\theta}} \times \frac{\sin\cancel{\theta}}{b} \qquad 3$$

$= 1 - \sin^2\theta$

$= \cos^2\theta = e^2$ $\quad \therefore RP = e^2RT$

b. $\int_0^1 x(x^2-1)^n \, dx$

i.

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = (x^2-1)^n$$

$$u' = n(x^2-1)^{n-1} \cdot 2x$$

$$v' = x$$

$$v = \frac{x^2}{2}$$

$$\therefore \left.\frac{x^2}{2}(x^2-1)^n\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot n(x^2-1)^{n-1} \cdot 2x \, dx$$

$$= \tfrac{1}{2}(0) - 0 - n\int_0^1 x^3(x^2-1)^{n-1} \, dx$$

$$= -n\int_0^1 \frac{x^3(x^2-1)^n}{x^2-1} \, dx$$

$$= -n\int_0^1 \frac{x^3}{x^2-1}\left[x^2-1\right]^n \, dx$$

$$\therefore x^2-1 \overline{\smash{)}\, x^3 + 0x^2 + 0x + 0}$$ quotient $x$; $x^3 \quad -x$; remainder $x$

$$= -n\int_0^1 \left(x + \frac{x}{x^2-1}\right)(x^2-1)^n \, dx$$

$$= -n\int_0^1 x(x^2-1)^n + \frac{x}{x^2-1}(x^2-1)^n \, dx$$

$$= -n\int_0^1 x(x^2-1)^n \, dx - n\int_0^1 x(x^2-1)^{n-1} \, dx$$

$$\therefore I_n = -nI_n - nI_{n-1}$$ if

$$\therefore (1+n)I_n = -nI_{n-1}$$

$$\therefore I_n = \frac{-n}{n+1} I_{n-1} \quad \text{for } n \geq 1$$