Name: ______________________ Class: ________

WHITEBRIDGE HIGH SCHOOL

2007

Trial HSC Examination

MATHEMATICS EXTENSION 2

Time Allowed: Two and a half hours

(Reading time: 5 minutes)

Directions to Candidates

- All questions to be completed on writing paper provided
- Commence each question on a new page.
- Marks may be deducted for careless or badly arranged work.
- Calculators may be used.

Question 1 (15 marks) Commence each question on a SEPARATE page **Marks**

a. Find $\int \frac{\sin x}{\cos^3 x}\, dx$. **2**

b. By completing the square, evaluate $\int \frac{1}{x^2 + 2x + 5}\, dx$. **2**

c. (i) Given that $\frac{4x - 6}{(x+1)(2x^2+3)}$ can be written as **3**

$$\frac{4x-6}{(x+1)(2x^2+3)} = \frac{a}{x+1} + \frac{bx+c}{2x^2+3},$$ where a, b and c are real numbers,

find a, b and c.

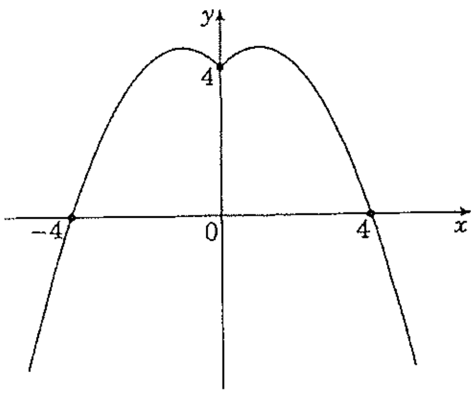
(ii) Hence find $\int \frac{4x-6}{(x+1)(2x^2+3)}\, dx$. **2**

d. Evaluate $\int_0^1 x \tan^{-1} x \; dx$. **3**

e. Use the substitution $t = \tan \frac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{3}} \frac{d\theta}{13 + 5\sin\theta + 12\cos\theta} = \frac{2}{5 + 25\sqrt{3}}$. **3**

Question 2 (15 marks) Commence each question on a SEPARATE page **Marks**

a. The even function $y = f(x)$ is shown below.



On a separate number plane sketch each of the following, showing all important features.

(i) $y = f(x - 4)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y^2 = f(x)$ **2**

b. The function $y = f(x)$ is defined by $f(x) = \dfrac{\ln x}{x}$, for $x > 0$.

(i) Determine any stationary points, points of inflexion and equations of possible asymptotes for $y = f(x)$. **4**

(ii) Draw a sketch of $y = f(x) = \dfrac{\ln x}{x}$, showing all relevant details determined from (i). **1**

(iii) Draw separate sketches of the graphs of

(α) $y = \left|\dfrac{\ln x}{x}\right|$ **2**

(β) $y = \dfrac{x}{\ln x}$ **2**

Question 3 (15 marks) Commence each question on a SEPARATE page | Marks

a. Let $z = 5 + 3i$ and $w = -3 + 2i$.

Express the following in the form $a + bi$, where a and b are real numbers.

(i) $z\bar{w}$ **1**

(ii) $\dfrac{2}{iw}$ **1**

b. (i) Express $\sqrt{3} + i$ in mod-arg form. **2**

(ii) Find the exact value of $(\sqrt{3} + i)^{12}$ in the form $a + bi$ where a and b are real numbers. **2**

c. On separate diagrams, sketch the region where the inequalities hold:

(i) $|z - 3 + i| \leq 5$ and $|z + 2| \leq |z - 2|$. **2**

(ii) $2 < |z| < 3$ and $\frac{\pi}{6} < \arg z < \frac{\pi}{2}$. **2**

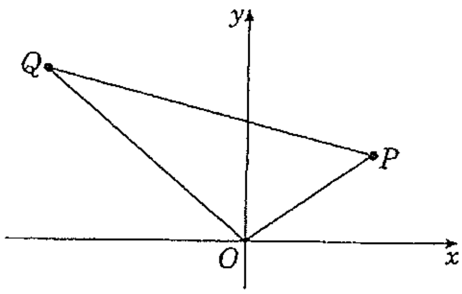
d. (i) If $w = \dfrac{1 + \sqrt{3}i}{2}$, show that $w^3 = -1$. **1**

(ii) Hence calculate w^{16}. **1**

Question 3 continues over page

Question 3 continued

e.



The diagram shows a complex plane with origin O.
The points P and Q represent arbitrary non-zero complex numbers z_1 and z_2 respectively.
Thus the length of PQ is $|z_1 - z_2|$.

(i) Use the diagram to show $|z_1 - z_2| \leq |z_1| + |z_2|$. **1**

(ii) Construct the point R representing $z_1 + z_2$. **1**
What type of quadrilateral is OPRQ?

(iii) If $|z_1 - z_2| = |z_1 + z_2|$, what can be said about the complex number $\frac{z_2}{z_1}$? **1**

Question 4 (15 marks) Commence each question on a SEPARATE page — Marks

a. The roots of $x^3 + 5x^2 + 11 = 0$ are α, β and γ.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

b. (i) Suppose the polynomial $P(x)$ has a double root at $x = a$. **1**
Prove that $P'(x)$ also has a root at $x = a$.

(ii) The polynomial $P(x) = x^4 + ax^3 + bx + 21$ has a double root at $x = 1$. **2**
Find the values of a and b.

(iii) Factorise the polynomial $P(x)$ of part (i) over the real numbers. **2**

c. Factorise $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over C if $2 - i$ is a zero. **3**

d. (i) Derive the five roots of the equation $z^5 - 1 = 0$. **2**

(ii) Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$. **2**

Question 5 (15 marks) Commence each question on a SEPARATE page Marks

a. The ellipse ***E*** has the equation $4x^2 + 9y^2 = 36$.

(i) Write down:

(α) its eccentricity **1**

(β) the coordinates of its foci S and S′. **1**

(γ) the equation of each directrix **1**

(δ) the length of the major axis. **1**

(ii) Sketch the ellipse *E*. **1**
Show the *x* and *y* intercepts as well as the features found in parts (β) and (γ) of part (i) above.

b. ***H*** is a rectangular hyperbola whose equation is given by $xy = \frac{1}{2}a^2$.

(i) Prove that, for all values of *t*, the point $P(\frac{at}{2}, \frac{a}{t})$ lies on ***H*** and that the equation of the tangent at P has the equation $2x + t^2y = 2at$. **2**

(ii) S is the point (a, a) and the perpendicular from S to the tangent described in (i) above meets the tangent at T. **2**
Prove that $t^2x - 2y = at^2 - 2a$ is the equation of the line ST.

(ii) Show that, as P moves on the hyperbola, the locus of T is a circle. **2**

Question 5 continues over page

Question 5 continued

c. The point P($a\sec\theta$, $b\tan\theta$) is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(i) Show that the equation of the tangent to the hyperbola at P is given by $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **3**

(ii) The tangent to the hyperbola at P meets the asymptotes at A and B. Prove that P is the midpoint of the interval AB. **3**

Question 6 (15 marks) Commence each question on a SEPARATE page

Marks

a. (i) Find the five fifth roots of $\sqrt{3} + i$. **2**

(ii) Show the roots on an Argand diagram. **2**

(iii) Find the area of the pentagon formed by the roots. **1**

b. (i) Given that $C_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, prove that $C_n = \frac{n-1}{n} C_{n-1}$, where $n = 2, 3, 4, 5, \ldots$ **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$. **1**

Question 6 continues over page

Question 6 continued

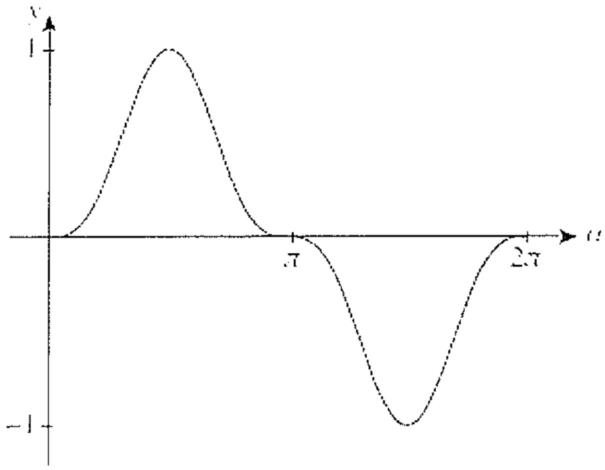
c. Consider the complex number $z = \cos\alpha + i\sin\alpha$

(i) Prove that $z^n - \dfrac{1}{z^n} = 2i\sin n\alpha$. **2**

(ii) Expand $(z - \dfrac{1}{z})^3$, and use the result from part (i) to show **2**

that $\sin^3\alpha = \dfrac{3}{4}\sin\alpha - \dfrac{1}{4}\sin 3\alpha$.

(iii) Below is a sketch of $y = \sin^3\alpha$ for $0 \le \alpha \le 2\pi$. **2**



Find the area of the region between the curve and the α-axis, for $0 \le \alpha \le 2\pi$.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1:

a. $\int \frac{\sin x}{\cos^3 x}\,dx$ Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$= \int \frac{\sin x}{u^3} \cdot \frac{du}{-\sin x}$

$= -\int u^{-3}\,du$

$= -\frac{u^{-2}}{-2}$

$= \frac{1}{2u^2}$

$= \frac{1}{2\cos^2 x} + c$ ②

b. $\int \frac{1}{x^2+2x+5}\,dx$

$= \int \frac{1}{x^2+2x+1+4}\,dx$

$= \int \frac{1}{(x+1)^2+2^2}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$ ②

c. i. $\frac{4x-6}{(x+1)(2x^2+3)} = \frac{a}{(x+1)} + \frac{bx+c}{(2x^2+3)}$

$= \frac{a(2x^2+3)+(bx+c)(x+1)}{(x+1)(2x^2+3)}$

$\therefore 4x-6 = a(2x^2+3)+(bx+c)(x+1)$

Subs $x=-1$

$\therefore -10 = 5a \quad \therefore a = -2$

Subs $x=0$

$\therefore -6 = 3a + c(1)$

$\therefore -6 = -6 + c \quad \therefore c = 0$ ③

Subs $x=1$

$-2 = 5a + (b+c).2$

$\therefore -2 = -10 + 2b$

$2b = 8 \quad \therefore b = 4$

$\therefore a=-2$, $b=4$, $c=0$

ii. $\int \frac{-2}{x+1}\,dx + \frac{4x}{2x^2+3}\,dx$ ②

$= -2\ln(x+1) + \ln(2x^2+3) + c$

d. $\int_0^1 x\tan^{-1}x\,dx$ $u = \tan^{-1}x$, $u' = \frac{1}{1+x^2}$, $v' = x$, $v = \frac{x^2}{2}$

$= \frac{x^2}{2}\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$

$= \left(\frac{1}{2}\cdot\frac{\pi}{4} - 0\right) - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$

$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx$

$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\,dx$

$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1$

$= \frac{\pi}{8} - \frac{1}{2}\left[1 - \tan^{-1}1 - 0\right]$

$= \frac{\pi}{8} - \frac{1}{2}\left[1 - \frac{\pi}{4}\right]$

$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$ ②

e. $\int_0^{\pi/3} \frac{d\theta}{13+5\sin\theta+12\cos\theta}$ $t = \tan\theta/2$

Now, $13+5\sin\theta+12\cos\theta$

$= 13 + 5\left[\frac{2t}{1+t^2}\right] + 12\left[\frac{1-t^2}{1+t^2}\right]$

$= 13 + \frac{10t}{1+t^2} + \frac{12-12t^2}{1+t^2}$

$= \frac{13+13t^2+10t+12-12t^2}{1+t^2}$

$= \frac{t^2+10t+25}{1+t^2}$ $d\theta = \frac{2\,dt}{1+t^2}$

$\therefore \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{t^2+10t+25}\cdot\frac{2\,dt}{1+t^2}$

$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{t^2+10t+25}\,dt$

$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t+5)^2}\,dt$

$= 2\int_0^{\frac{1}{\sqrt{3}}} (t+5)^{-2}\,dt$

$= 2\left[\dfrac{(t+5)^{-1}}{-1}\right]_0^{\frac{1}{\sqrt{3}}}$

$= \left[\dfrac{-2}{(t+5)}\right]_0^{\frac{1}{\sqrt{3}}}$

$= \left[\dfrac{-2}{\frac{1}{\sqrt{3}}+5} - \dfrac{-2}{5}\right] \times \dfrac{\sqrt{3}}{\sqrt{3}}$

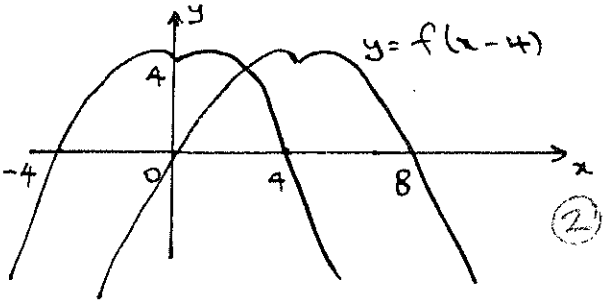
$= \dfrac{-2\sqrt{3}}{1+5\sqrt{3}} + \dfrac{2\sqrt{3}}{5\sqrt{3}}$

$= \dfrac{\cancel{-30} + 2\cancel{\sqrt{3}} + \cancel{30}}{5\cancel{\sqrt{3}}(5\sqrt{3}+1)} = \dfrac{2}{5(5\sqrt{3}+1)}$

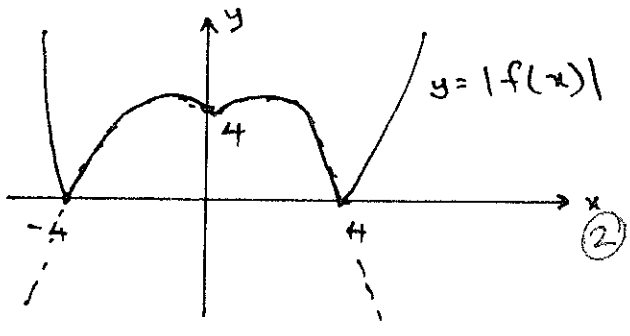
$= \dfrac{2}{25\sqrt{3}+5}$ (3)

Question 2

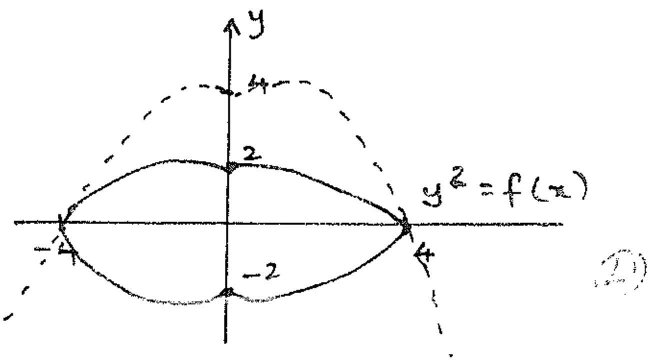
a. i.



ii.



iii.



b. $f(x) = \dfrac{\ln x}{x}$

i. $y = \dfrac{\ln x}{x}$

$y' = \dfrac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \dfrac{1 - \ln x}{x^2} = 0$

$1 - \ln x = 0$

$\ln x = 1$

$x = e$

$y(e) = \dfrac{\ln e}{e}$ $\quad\therefore$ Stat pt $\left(e, \frac{1}{e}\right)$

$= \dfrac{1}{e}$

$y' = \dfrac{1 - \ln x}{x^2}$

$y'' = \dfrac{x^2 \cdot -\frac{1}{x} - 2x(1 - \ln x)}{x^4}$

$= \dfrac{-x - 2x(1 - \ln x)}{x^4}$

$= \dfrac{-1 - 2(1 - \ln x)}{x^3}$

$= \dfrac{-3 + 2\ln x}{x^3}$

$y''(e) = \dfrac{-3 + 2\ln e}{e^3} < 0$ $\quad\therefore$ Max $\left(e, \frac{1}{e}\right)$

Now, $y'' = 0$ $\therefore \dfrac{-3 + 2\ln x}{x^3} = 0$

$2\ln x = 3$

$\ln x = \frac{3}{2}$

$x = e^{3/2}$

$\therefore$ Poss. pt of infl. at $x = e^{3/2}$

Check neighbourhood

$y''(e^{3/2} - \epsilon) < 0$

$y''(e^{3/2}) = 0$ $\quad\therefore$ Sign change

$y''(e^{3/2} + \epsilon) > 0$ $\quad\therefore y(e^{3/2}) = \dfrac{\ln e^{3/2}}{e^{\frac{3}{2}}}$

$= \frac{3}{2} \div e^{3/2}$

$= \dfrac{3}{2e^{3/2}}$

$\therefore$ Pt of infl $\left(e^{3/2}, \dfrac{3}{2e^{3/2}}\right)$

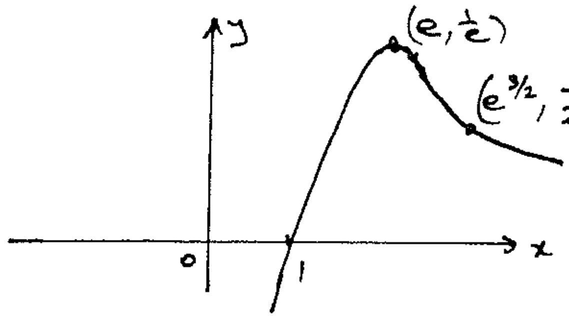
Now, $y = \frac{\ln x}{x}$ Disc at $x = 0$

$\therefore$ Asymp. at $x = 0$

Also, $x \to \infty$ $\therefore$ $\frac{\ln \infty}{\infty} \to 0$

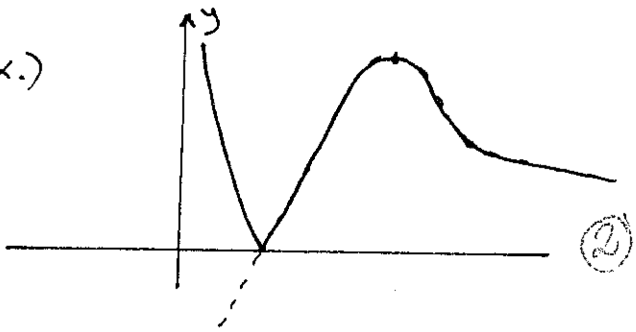
$\therefore$ Asymp at $y = 0$

ii.

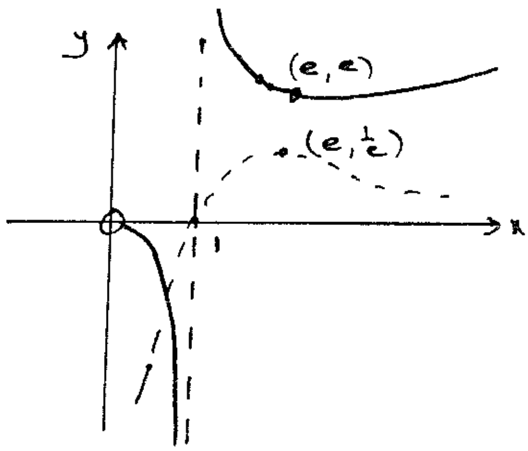


iii.

(α.)



(β)



Question 3

a. i. $z\bar{w} = (5+3i)(-3-2i)$

$= -15 - 10i - 9i + 6$

$= -9 - 19i$

ii. $\frac{2}{iw} = \frac{2}{i(-3+2i)}$

$= \frac{2}{-2-3i} \times \frac{-2+3i}{-2+3i}$

$= \frac{-4+6i}{4+9}$

$= \frac{-4}{13} + \frac{6i}{13}$

b. i. $\sqrt{3} + i$

Let $z = \sqrt{3} + i$ $|z| = \sqrt{3+1}$

$= 2$

$\arg z = \tan^{-1} \frac{1}{\sqrt{3}}$

$= \frac{\pi}{6}$

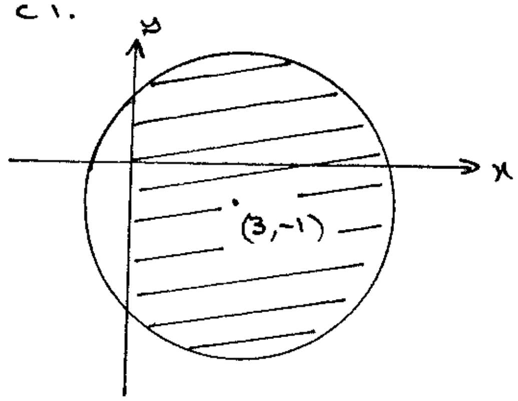
$\therefore \sqrt{3} + i = 2 \text{ cis } \pi/6$

ii. $(\sqrt{3}+i)^{12} = 2^{12} \text{ cis } \frac{12\pi}{6}$

$= 4096 [\cos 2\pi + i \sin 2\pi]$

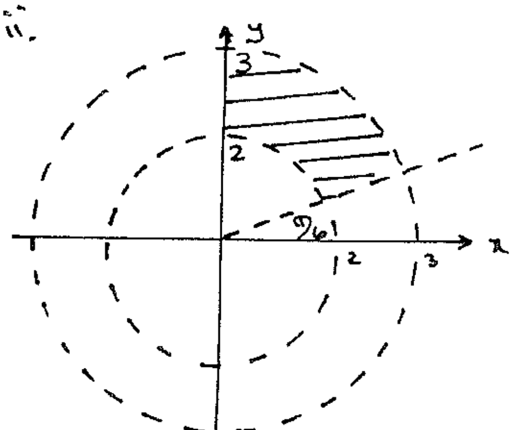
$= 4096$

c i.



- $|z - 3 + i| = 5$ is circle with centre $3 - i$ and radius 5
- $|z+2| = |z-2|$ is locus of points equidistant from $x = -2$ & $x = 2$
  ie $x = 0$ (y axis)

c ii.



d. $(1+\sqrt{3}i)(1+\sqrt{3}i)$

$= 1 + 2\sqrt{3}i - 3$

$= -2 + 2\sqrt{3}i$

$\therefore (1+\sqrt{3}i)^3 = (1+\sqrt{3}i)(-2+2\sqrt{3}i)$

$= -2 + 2\sqrt{3}i - 2\sqrt{3}i - 6$

$= -8$

$\therefore \omega^3 = \left[\frac{1+\sqrt{3}i}{2}\right]^3$

$= \frac{-8}{8}$

$= -1$ (1)

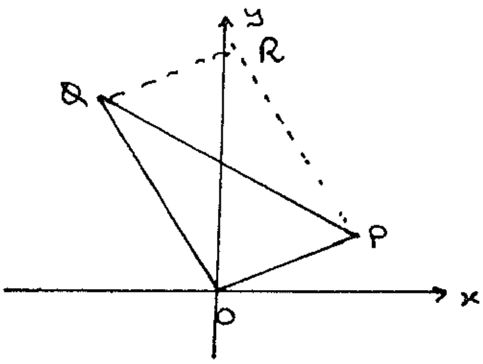
ii. $\omega^{16} = (\omega^3)^5 \times \omega$

$= (-1)^5 \cdot \omega$

$= -\omega$

$= \frac{-1-\sqrt{3}i}{2}$ (1)

e.



i.

$OP = |z_1|$, $OQ = |z_2|$

$PQ = |z_1 - z_2|$

Now PQ is longest side of ΔPOQ

$\therefore PQ \leq OP + OQ$

$\therefore |z_1 - z_2| \leq |z_1| + |z_2|$ (1)

ii. OPRQ is parallelogram (1)

iii. $|z_1 - z_2| = |z_1 + z_2|$

$\therefore$ diagonals of parallelogram equal

$\therefore$ OPRQ is rectangle

$\therefore \angle POQ = \pi/2$

$\therefore \overrightarrow{OQ} = i\,OP$

$\therefore z_2 = i z_1$ (1)

$\therefore \frac{z_2}{z_1} = i.$

Question 4

a. i. Roots are $x = \alpha^2, \beta^2, \gamma^2$

If $x = \alpha^2 \therefore \alpha = \sqrt{x}$

Now as α is root of $x^3 + 5x^2 + 11 = 0$

$\therefore \sqrt{x}$ is also soln (4)

$\therefore P(\sqrt{x}) = (\sqrt{x})^3 + 5(\sqrt{x})^2 + 11 = 0$

$x\sqrt{x} = -5x - 11$

square both sides:

$x^3 = (-5x-11)^2$

$x^3 = 25x^2 + 110x + 121$

$\therefore x^3 - 25x^2 - 110x - 121 = 0$ (2)

ii. $\alpha^2 + \beta^2 + \gamma^2$: sum of roots

$\therefore 25$ (1)

b. i. $P(x) = (x-\alpha)^2 Q(x)$

$\therefore P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$ (1)

$\therefore P'(\alpha) = 0 \therefore x = \alpha$ is root of $P'(x)$

ii. $P(x) = x^4 + ax^3 + bx + 21$

$P'(x) = 4x^3 + 3ax^2 + b$

$P'(1) = 4 + 3a + b = 0$

$\therefore 3a + b = -4$ —— (1)

Also $P(1) = 1 + a + b + 21 = 0$

$a + b = -22$ —— (2)

(1) − (2) $\quad 2a = 18$

$a = 9$

subs in (2) $\quad 9 + b = -22$

$b = -31$ (2)

iii. $P(x) = x^4 + 9x^3 - 31x + 21$

$= (x-1)^2 \cdot Q(x)$

Now, $(x-1)^2 = x^2 - 2x + 1$

$$\begin{array}{r} x^2 + 11x + 21 \\ x^2 - 2x + 1 \overline{\smash{)}\, x^4 + 9x^3 - 31x + 21} \\ \underline{x^4 - 2x^3 + x^2} \\ 11x^3 - x^2 - 31x \\ \underline{11x^3 - 22x^2 + 11x} \\ 21x^2 - 42x + 21 \\ \underline{21x^2 - 42x + 21} \\ 0 \end{array}$$

$\therefore (x-1)^2(x^2 + 11x + 21)$

$= (x-1)^2\left(x^2 + 11x + \frac{121}{4} - \frac{121}{4} + 21\right)$

$= (x-1)^2\left(\left(x + \frac{11}{2}\right)^2 - \frac{37}{4}\right)$

$= (x-1)^2\left(x+\frac{11}{2}+\frac{\sqrt{37}}{2}\right)\left(x+\frac{11}{2}-\frac{\sqrt{37}}{2}\right)$

$= (x-1)\left(x+\frac{11+\sqrt{37}}{2}\right)\left(x+\frac{11-\sqrt{37}}{2}\right)$ (2)

c. As $P(x)$ has real coeff, and $2-i$ is zero $\therefore$ $2+i$ is also zero

$\therefore (x-2+i)(x-2-i)$

$\therefore$ using diff. of 2 squares

$= [(x-2)+i][(x-2)-i]$

$= x^2-4x+4+1$

$= x^2-4x+5$

$$\begin{array}{r} x^2-x-2 \\ x^2-4x+5\,\overline{)\,x^4-5x^3+7x^2+3x-10} \\ \underline{x^4-4x^3+5x^2} \\ -x^3+2x^2+3x \\ \underline{-x^3+4x^2-5x} \\ -2x^2+8x-10 \\ \underline{-2x^2+8x-10} \\ 0 \end{array}$$

$\therefore (x^2-4x+5)(x^2-x-2)$ (3)

$= (x-2+i)(x-2-i)(x-2)(x+1)$

d. i. $z^5-1=0$

$z^5=1$

Let $z=\cos\theta+i\sin\theta$

$\therefore \cos 5\theta + i\sin 5\theta = 1+0i$

$\therefore \cos 5\theta = 1 \qquad \sin 5\theta = 0$

$\therefore 5\theta = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

$\therefore \theta = \frac{2\pi k}{5}$ where $k=0,\pm1,\pm2$

ie $z_1 = \text{cis}\, 0 = 1$

$z_2 = \text{cis}\, \frac{2\pi}{5}$

$z_3 = \text{cis}\, -\frac{2\pi}{5}$ (2)

$z_4 = \text{cis}\, \frac{4\pi}{5}$

$z_5 = \text{cis}\, -\frac{4\pi}{5}$

ii. Now $z_3 = \cos -\frac{2\pi}{5} + i\sin -\frac{2\pi}{5}$

$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

Similarly for z_5

Now $z_1+z_2+z_3+z_4+z_5=0$ using sum of roots $= \frac{-b}{a} = 0$

$\therefore 1+\cos\frac{2\pi}{5}+\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\sin\frac{4\pi}{5}+\cos\frac{4\pi}{5}-\sin\frac{4\pi}{5}=0$

$\therefore 1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$

$2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=-1$ (2)

$\therefore \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$

Question 5

a. i. α. $4x^2+9y^2=36$

$\therefore \frac{x^2}{9}+\frac{y^2}{4}=1$

ie $a=3, b=2$

Using $b^2=a^2(1-e^2)$

$4=9(1-e^2)$

$\frac{4}{9}=1-e^2$

$e^2=\frac{5}{9}$

$\therefore e=\frac{\sqrt{5}}{3}, \; e>0$ (1)

β. foci: $(\pm ae, 0)$ ie $(\pm\sqrt{5}, 0)$ (1)

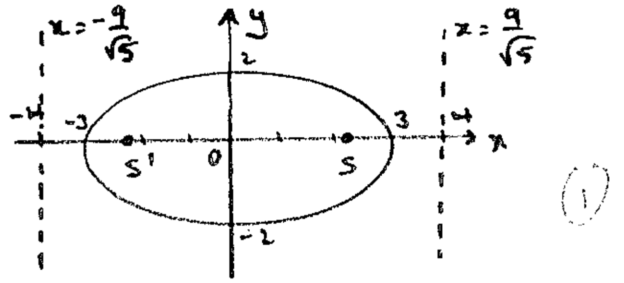
γ. Directrices: $x=\pm\frac{a}{e}$

$=\pm 3 \div \frac{\sqrt{5}}{3}$

$x=\pm\frac{9}{\sqrt{5}}$ (1)

δ. Major axis $= 2a = 2\times 3 = 6$ units (1)

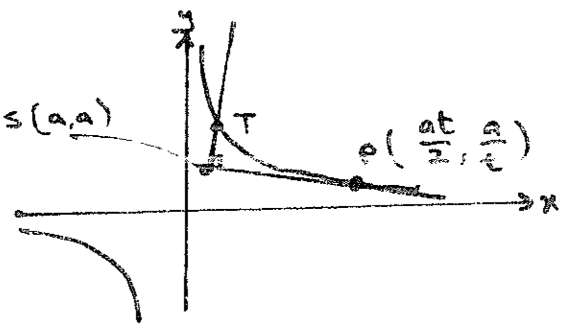
ii.



(1)

b. $xy=\frac{1}{2}a^2$

i.



Subs $\left(\frac{at}{2}, \frac{a}{t}\right)$ in $xy = \frac{1}{2}a^2$

ie $\frac{at}{2} \cdot \frac{a}{t} = \frac{1}{2}a^2$

$\frac{a^2}{2} = \frac{1}{2}a^2$ ✓

$\therefore$ P lies on H

Now, as $xy = \frac{1}{2}a^2$

$\therefore y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$\therefore$ At P: $y' = -\frac{a}{t} \div \frac{at}{2}$

$= -\frac{a}{t} \times \frac{2}{at}$

$= -\frac{2}{t^2}$

$\therefore$ Eqn of tangent:

$y - \frac{a}{t} = -\frac{2}{t^2}\left(x - \frac{at}{2}\right)$

$t^2y - at = -2x + at$ (2)

$\therefore 2x + t^2y = 2at$ —— ①

ii. grad of normal: $\frac{t^2}{2}$

$\therefore$ eqn of ST: $y - a = \frac{t^2}{2}(x - a)$

$2y - 2a = t^2x - at^2$ (2)

$\therefore t^2x - 2y = at^2 - 2a$ —— ②

iii. If we are to prove a circle, then we need x^2 and y^2

$\therefore 2x + t^2y = 2at$ —— ①

$t^2x - 2y = at^2 - 2a$ —— ②

Square both sides of ① & ②

$\therefore 4x^2 + 4t^2xy + t^4y^2 = 4a^2t^2$ —— ③

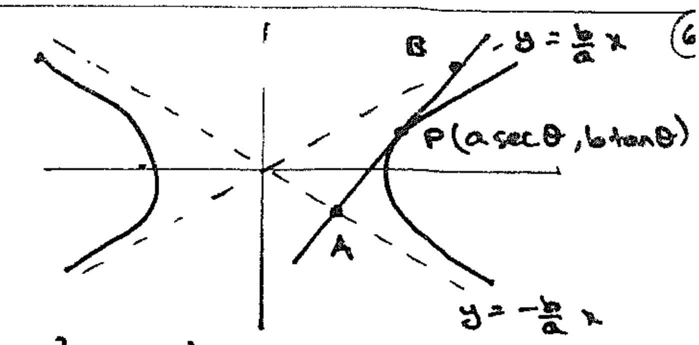
$t^4x^2 - 4t^2xy + 4y^2 = a^2t^4 - 4a^2t^2 + 4a^2$ —— ④

Now ③ + ④

$(t^4+4)x^2 + (t^4+4)y^2 = a^2(t^4+4)$

$\therefore x^2 + y^2 = a^2$ $\therefore$ circle (2)

c. (6)



i. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{2x}{a^2} \div \frac{2y}{b^2}$

$= \frac{2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{b^2x}{a^2y}$

At P: $y' = \frac{b^2}{a^2} \cdot \frac{a\sec\theta}{b\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ Eqn of tangent:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$ [= 1]

$\therefore \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ —— ① (3)

ii. Co-ords of B:

Subs $y = \frac{b}{a}x$ in ①

$\therefore \frac{x\sec\theta}{a} - \frac{b}{a} \cdot \frac{x\tan\theta}{b} = 1$

$x\sec\theta - x\tan\theta = a$

$x(\sec\theta - \tan\theta) = a$

$x = \frac{a}{\sec\theta - \tan\theta}$

$\therefore y = \frac{b}{a} \cdot \frac{a}{\sec\theta - \tan\theta}$ [ie subs in $y = \frac{b}{a}x$]

$= \frac{b}{\sec\theta - \tan\theta}$

$\therefore B\left[\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right]$

Now, coords of A:

Subs $y = -\frac{b}{a}x$ in ①

$$x \sec\theta + x\tan\theta = a$$

$$\therefore \quad x = \frac{a}{\sec\theta + \tan\theta}$$

$$\therefore \quad y = -\frac{b}{a} \cdot \frac{a}{\sec\theta + \tan\theta}$$

$$= \frac{-b}{\sec\theta + \tan\theta}$$

$$\therefore A\left[\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right]$$

Now, coords of midpoint:

$$x_{mp} : \frac{1}{2}\left[\frac{a}{\sec\theta - \tan\theta} + \frac{a}{\sec\theta + \tan\theta}\right]$$

$$= \frac{1}{2}\left[\frac{a\sec\theta + a\tan\theta + a\sec\theta - a\tan\theta}{\sec^2\theta - \tan^2\theta}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sec\theta}{1}\right]$$

$$= a\sec\theta$$

Similarly

$$y_{mp} : \frac{1}{2}\left[\frac{b}{\sec\theta - \tan\theta} + \frac{-b}{\sec\theta + \tan\theta}\right]$$

$$= \frac{1}{2}\left[\frac{b\sec\theta + b\tan\theta - b\sec\theta + b\tan\theta}{\sec^2\theta - \tan^2\theta}\right]$$

$$= a\tan\theta \qquad (3)$$

$\therefore P(a\sec\theta, a\tan\theta)$ is midpt.

Question 6

a i. Let $z^5 = \sqrt{3} + i$

$$\therefore z^5 = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= 2\left[\cos \pi/6 + i\sin \pi/6\right]$$

$$= 2\left[\text{cis}(\pi/6 + 2\pi k)\right]$$

$$\therefore z = 2^{1/5}\,\text{cis}\left(\pi/30 + \frac{2\pi k}{5}\right)$$

where $k = 0, \pm 1, \pm 2$ (2)

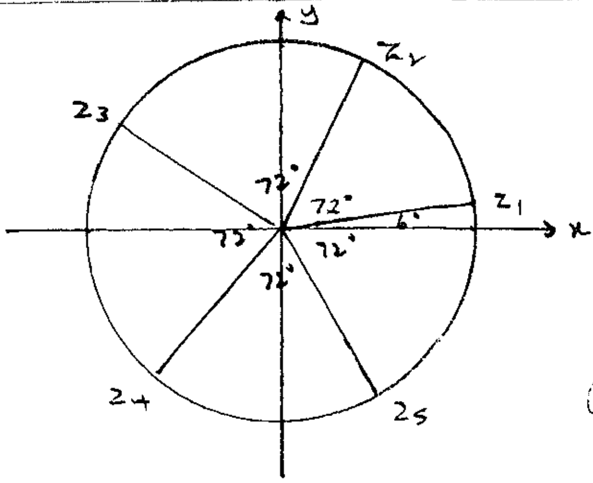
ii. Using $\frac{2\pi}{5} = 72°$

$\therefore$ roots differ by $72°$.

Now, for $z_1 = 2^{\frac{1}{5}}\,\text{cis}\,6°$

$$\left(= 2^{\frac{1}{5}}\,\text{cis}\,\pi/30\right)$$

(7)



(2)

iii. $5 \times \frac{1}{2}ab\sin C$

ie $5 \times \frac{1}{2} \times 2^{\frac{1}{5}}\, 2^{\frac{1}{5}} \sin 72°$

$= 3.14\ u^2$ (1)

b. i. $C_n = \int_0^{\pi/2} \cos^n dx$

$$= \int_0^{\pi/2} \cos x \cdot \cos^{n-1} x\, dx$$

$u = \cos^{n-1} x \qquad v' = \cos x$

$u' = (n-1)\cos^{n-2} x \cdot -\sin x$

$\quad = -(n-1)\sin x \cos^{n-2} x$

$v = \sin x$

$$\therefore = \sin x \cos^{n-1} x\Big]_0^{\pi/2}$$

$$- \int_0^{\pi/2} -(n-1)\sin x \cos^{n-2} x \sin x\, dx$$

$$= 0 + \int_0^{\pi/2} (n-1)\sin^2 x \cos^{n-2} x\, dx$$

$$= (n-1)\int_0^{\pi/2} (1 - \cos^2 x)\cos^{n-2} x\, dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2} x - (n-1)\int_0^{\pi/2} \cos^n x\, dx$$

$$= (n-1)C_{n-2} - (n-1)C_n$$

$$\therefore C_n + (n-1)C_n = (n-1)C_{n-2}$$

$$nC_n = (n-1)C_{n-2}$$

$$\therefore C_n = \frac{n-1}{n} C_{n-2} \qquad (3)$$

ii. $\int_0^{\pi/2} \cos^4 x\, dx = C_4$

$$\therefore C_4 = \frac{3}{4}C_2 = \frac{3}{4}\left[\frac{1}{2}C_0\right]$$

$$= \frac{3}{4} \cdot \frac{1}{2}\int_0^{\pi/2} 1\, dx$$

$$= \frac{3}{8}\ x\Big]_0^{\pi/2}$$

$= \frac{3}{8}\left[\frac{\pi}{2} - 0\right]$

$= \frac{3\pi}{16}$ (1)

c. i. $z = \cos\alpha + i\sin\alpha$

$z^n = \cos n\alpha + i\sin n\alpha$

$\frac{1}{z^n} = z^{-n} = \cos(-n)\alpha + i\sin(-n)\alpha$

$= \cos n\alpha - i\sin n\alpha$

$\therefore z^n - \frac{1}{z^n} = \cos n\alpha + i\sin n\alpha$
$- (\cos n\alpha - i\sin n\alpha)$

$= 2i\sin n\alpha$ (2)

ii. Using $n = 1$

$\therefore z - \frac{1}{z} = 2i\sin\alpha$

$\therefore (z - \frac{1}{z})^3 = 8i^3\sin^3\alpha$

$= -8i\sin^3\alpha$ —— ①

Also, $(z - \frac{1}{z})^3 = (z - \frac{1}{z})(z - \frac{1}{z})^2$

$= (z - \frac{1}{z})(z^2 - 2 + \frac{1}{z^2})$

$= z^3 - 2z + \frac{1}{z} - z - \frac{2}{z} - \frac{1}{z^3}$

$= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$

$= (z^3 - \frac{1}{z^3}) - 3(z - \frac{1}{z})$

$= 2i\sin 3\alpha - 6i\sin\alpha$

$= (2\sin 3\alpha - 6\sin\alpha)\, i$ —— ②

Now ① = ②

$\therefore -8\sin^3\alpha = 2\sin 3\alpha - 6\sin\alpha$

$\therefore \sin^3\alpha = \frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha$ (2)

iii. Area $= 2\int_0^{\pi} \sin^3\alpha \, d\alpha$

$= 2\int_0^{\pi} \frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha \, d\alpha$

$= \frac{1}{2}\int_0^{\pi} 3\sin\alpha - \sin 3\alpha \, d\alpha$

$= \frac{1}{2}\left[-3\cos\alpha + \frac{1}{3}\cos 3\alpha\right]_0^{\pi}$

$= \frac{1}{2}\left[3 - \frac{1}{3} - (-3 + \frac{1}{3})\right]$

$= \frac{1}{2}\left[2\frac{2}{3} + 2\frac{2}{3}\right]$

$= 2\frac{2}{3}\ u^2$