Name: ______________________ Class: ________

WHITEBRIDGE HIGH SCHOOL

2008

Trial HSC Examination

(Assessment 4)

MATHEMATICS EXTENSION 2

Time Allowed: 3 hours
(plus 5 minutes reading time)

Directions to Candidates

- All questions of equal value.
- Commence each question on a new page.
- Marks may be deducted for careless or badly arranged work.

Question 1 (15 marks) Commence each question on a SEPARATE page

a. Evaluate $\int_0^3 \frac{x}{\sqrt{16+x^2}}\,dx$. **3**

b. Find $\int \frac{dx}{x^2+6x+13}$ **2**

c. Find $\int xe^{-x}\,dx$. **2**

d. Find $\int \cos^3\theta \ \ d\theta$. **3**

e. (i) Find constants A, B and C such that **3**

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$$

(ii) Hence find $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx$. **2**

Question 2 (15 marks) Commence each question on a SEPARATE page

a. Given that $z = 1 + i$ and $w = -3$, find, in the form $x + iy$:

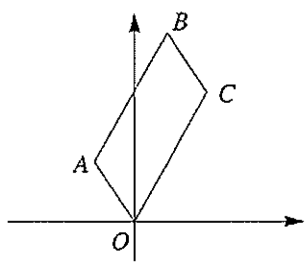
(i) wz^2 **1**

(ii) $\dfrac{z}{z+w}$ **2**

b. Using de Moivre's theorem, simplify $(-1 - i\sqrt{3})^{-10}$, expressing the answer in the form $x + iy$. **3**

c. Sketch the region described by the following $|z| < 2$ and $\dfrac{2\pi}{3} \le \arg z \le \dfrac{5\pi}{6}$. **2**

d.

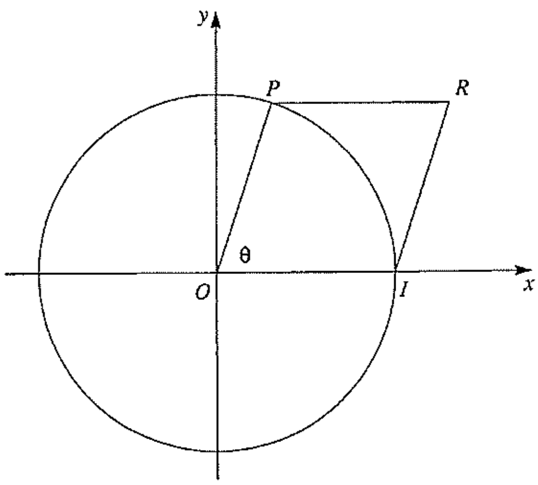


In the diagram above, $OABC$ is a parallelogram with $OA = \dfrac{1}{2}OC$. **3**

The point A represents the complex number $-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

If $\angle AOC = 60^\circ$, what complex number does C represent?

e. In the Argand diagram below, P represents $\cos\theta + i\sin\theta$, I represents the number $1 + 0i$, and R represents the number $z = 1 + \cos\theta + i\sin\theta$.



(i) Using the properties of the rhombus, or otherwise, show that z can be expressed as $z = 2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$. **2**

(ii) Hence show that $\frac{1}{z} = \frac{1}{2} - \frac{i}{2}\tan\frac{\theta}{2}$. **2**

Question 3 (15 marks) Commence each question on a SEPARATE page

a. Let 1, ω, ω^2 be the three cube roots of unity.

(i) Show that:

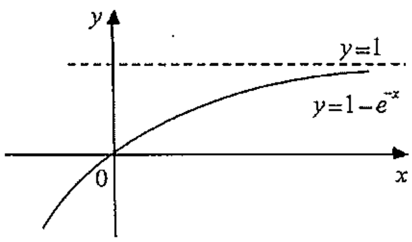
(α) $\omega^3 = 1$ **1**

(β) $1 + \omega + \omega^2 = 0$ **1**

(ii) If 1, ω, ω^2 are the roots of $x^3 + ax^2 + bx + c = 0$, find a, b and c. **3**

b. Find the acute angle between the tangent $x^3 + y^3 = 1$ at $x = 1$ and the line $y = x$. **4**

c. The graph shows the graph of $f(x) = 1 - e^{-x}$. On separate diagrams, sketch the graphs of the following functions, showing clearly the equations of any asymptotes:



(i) $y = [f(x)]^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

Question 4 (15 marks) Commence each question on a SEPARATE page

a. Show that the roots of the equation $z^{10} = 1$ are given by $z = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$ **2**

where $r = 0, 1, 2, 3, \ldots 9$

b. The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ.

(i) Find the monic cubic equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

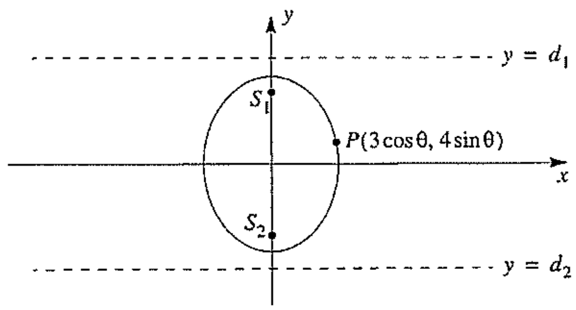
(ii) Find the monic cubic equation with roots $\frac{\beta+\gamma}{\alpha^2}$, $\frac{\gamma+\alpha}{\beta^2}$ and $\frac{\alpha+\beta}{\gamma^2}$. **3**

c. Given that $x = \theta + \frac{1}{2}\sin 2\theta$ and $y = \theta - \frac{1}{2}\sin 2\theta$:

(i) Show that $\frac{dy}{dx} = \tan^2\theta$ **2**

(ii) Show that $\frac{d^2y}{dx^2} = \tan\theta\sec^4\theta$ **2**

d.



The diagram above shows an ellipse with parametric equation

$x = 3\cos\theta$

$y = 4\sin\theta$

(i) Write down the cartesian equations of the ellipse. **1**

(ii) Find the coordinates of the foci S_1 and S_2. **2**

(iii) Find the equation of the directrices $y = d_1$ and $y = d_2$. **2**

Question 5 (15 marks) Commence each question on a SEPARATE page

a. (i) Let P(x) be a polynomial of degree 4 with a zero of multiplicity 3. **2**
Show that $P'(x)$ has a zero of multiplicity 2.

(ii) Hence, or otherwise, find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, **2**
given that it has a zero of multiplicity 3.

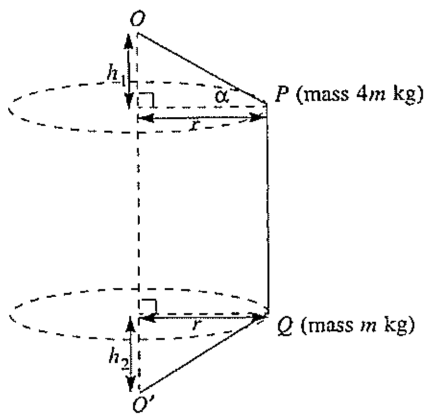
(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts **1**
on the coordinate axes.
Do NOT give the coordinates of turning points and inflections.

b. The roots of $a\tan^2\alpha + b\tan\alpha + c = 0$ are $\tan\alpha_1$ and $\tan\alpha_2$. **2**

Show that the value of $\tan(\alpha_1 + \alpha_2)$ is $\dfrac{b}{c - a}$.

c. A particle moves in a circle of radius r, with a constant speed rw. **1**
Write down the magnitude and direction of its acceleration.

d.



The diagram above shows two particles, P and Q, of masses 4m kg and m kg respectively, which are attached by a light inextensible string. The ends of the strings are attached to fixed points O and O′. O is vertically above O′. The particles P and Q move in horizontal circles, of equal radius r metres, about OO′, with the same constant angular velocity w, so that Q always remains vertically above P.

The depth of P below the level of O is h_1 and the height of Q above the level of O′ is h_2. The angle that OP makes with the horizontal is α.

(i) Let the tension in the string PQ be T newtons and the tension in the string *OP* be T_1 newtons. **2**

By drawing a force diagram and resolving forces acting on *P*, show that

$$T_1 \sin \alpha = 4mg + T$$
$$T_1 \cos\alpha = 4mw^2 r$$

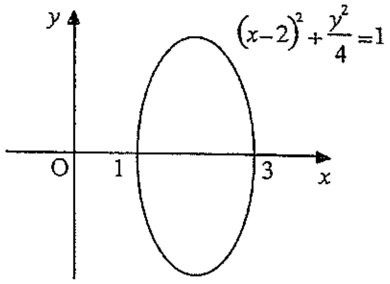
(ii) Hence show that $h_1 = \dfrac{4mg + T}{4mw^2}$. **2**

(iii) Hence show that $(4h_1 - h_2)w^2 = 5g$. **3**

Question 6 (15 marks) Commence each question on a SEPARATE page

a. When the polynomial $P(x)$ is divided by $(x + 2)(x - 3)$ the remainder is $4x + 1$. **2**
What is the remainder when $P(x)$ is divided by $(x + 2)$?

b.



The region enclosed by the ellipse $(x - 2)^2 + \frac{y^2}{4} = 1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by $V = 8\pi \int_1^3 x\sqrt{1 - (x - 2)^2}\, dx$ **2**

(ii) Hence find the volume of the solid of revolution in simplest exact form. **3**

c. From a point on the ground an object of mass m is projected vertically upwards with an initial speed of u. The object reaches a maximum height of H before falling back to the ground. The resistance due to air is equal to mkv^2, and g is the acceleration due to gravity.

(i) By using $\ddot{x} = v\frac{dv}{dx}$, show that $H = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$. **3**

(ii) P is the point of height h above the point of projection. **2**
Let V be the speed of the object at P on its upward path.
Show that $h = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kV^2}\right)$.

(iii) During the downward path of the object it passes through P with half the speed of when it was first at P. **3**
Show that $V = \sqrt{\frac{3g}{k}}$.

Question 7 (15 marks) Commence each question on a SEPARATE page

a. Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 + 5\bar{z} + 10i = 0$. **2**

b. Consider the rectangular hyperbola $xy = 4$.

(i) Show that the gradient of the tangent at the point $P(2p, \frac{2}{p})$ is $-\frac{1}{p^2}$. **2**

(ii) Show that the normal at P is given by $p^3x - py = 2(p^4 - 1)$. **2**

(iii) This normal meets the hyperbola again at $Q(2q, \frac{2}{q})$. **3**

By considering the product of the roots of the equation formed by the intersection of $xy = 4$ and $p^3x - py = 2(p^4 - 1)$, or otherwise, prove that $p^3q = -1$.

c. (i) Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$. **2**

(ii) Let $I_n = \int \frac{t^n}{1+t^2}\, dt$. **1**

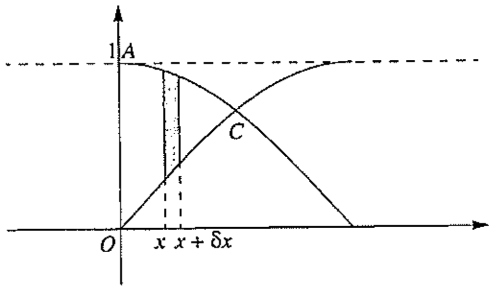
Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$ for $n \geq 2$.

(iii) Show that $\int_0^1 \frac{t^6}{1+t^2}\, dt = \frac{13}{15} - \frac{\pi}{4}$. **3**

Question 8 (15 marks) Commence each question on a SEPARATE page

a. Find $\frac{dy}{dx}$ when $y = e^{xy}$. **2**

b. The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$.
The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



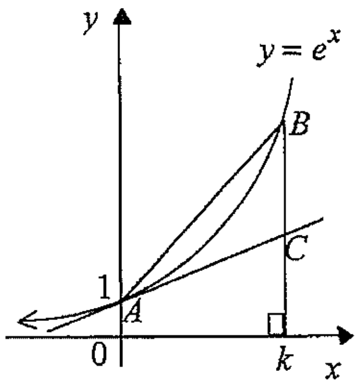
The region OAC is to be rotated about the line $y = 1$.

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resultant slice is given by **2**

$$\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\,\delta x.$$

(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. **4** c.



The curve $y = e^x$ cuts the y-axis at A.

B is a second point on the curve such that $x = k$ at B, where $k > 0$.

The tangent to the curve $y = e^x$ at A cuts the vertical line $x = k$ at the point C.

(i) By considering areas, show that $\frac{1}{2}k(k + 2) < e^k - 1 < \frac{1}{2}k(1 + e^k)$. **3**

Hence deduce that $2.5 < e < 3$.

(ii) Show that the curve $y = e^x$ bisects the area of ΔABC for some value of k such that $2 < k < 3$. Taking $k = 2.7$ as a first approximation, apply Newton's method once to obtain a second approximation. **3**

Give your answer to one decimal place.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a. $\int_0^3 \frac{x}{\sqrt{16+x^2}}\,dx$

Let $u = 16 + x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int_{16}^{25} \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du$

$= u^{\frac{1}{2}}\Big]_{16}^{25}$ 3

$= 5 - 4$

$= 1$

b. $\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{x^2+6x+9+4}$

$= \int \frac{dx}{4+(x+3)^2}$ 2

$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$

c. $\int \overset{u}{x}\overset{v'}{e^{-x}}\,dx = -xe^{-x} - \int 1 \cdot -e^{-x}\,dx$

$= -xe^{-x} + \int e^{-x}\,dx$ 2

$= -xe^{-x} - e^{-x} + c$

d. $\int \cos^3\theta\,d\theta = \int \cos^2\theta\cos\theta\,d\theta$

Let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$d\theta = \frac{du}{\cos\theta}$

$= \int (1-\sin^2\theta).\cos\theta\,d\theta$

$= \int (1-u^2)\cos\theta \cdot \frac{du}{\cos\theta}$

$= \int (1-u^2)\,du$

$= u - \frac{u^3}{3} + c$ 3

$= \sin\theta - \frac{\sin^3\theta}{3} + c$

e. i. $\frac{x^2-4x-1}{(1+2x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1+2x)}{(1+2x)(1+x^2)}$

$\therefore x^2-4x-1 = A(1+x^2)+(Bx+C)(1+2x)$

$= A + Ax^2 + Bx + 2Bx^2 + C + 2Cx$

$= (A+2B)x^2 + (B+2C)x + A+C$

$\therefore A + 2B = 1$ —①

$B + 2C = -4$ —②

$A + C = -1$ —③

From ① $A = 1 - 2B$

Subs in ③ $1 - 2B + C = -1$

$\therefore 2B - C = 2$ —④

2×④ $4B - 2C = 4$ —⑤

②+⑤ $5B = 0$

$\therefore B = 0$

Subs in ① $A = 1$

Subs in ③ $1 + C = -1$

$C = -2$

$\therefore A = 1, B = 0, C = -2$ 3

ii. $\int \frac{1}{1+2x} + \frac{-2}{1+x^2}\,dx$ 2

$= \frac{1}{2}\ln(1+2x) - 2\tan^{-1}x + c$

Question 2:

a. i. $wz^2 = -3(1+i)^2$

$= -3(1+2i-1)$

$= -6i$ 1

ii. $\frac{z}{z+w} = \frac{1+i}{-2+i} \times \frac{-2-i}{-2-i}$

$= \frac{-2-i-2i+1}{4+1}$

$= \frac{-1-3i}{5}$

$= -\frac{1}{5} - \frac{3i}{5}$ 2

b. $(-1-i\sqrt{3})^{-10}$

Let $z = -1 - i\sqrt{3}$

$|z| = 2$

$\arg z = \tan^{-1}\frac{-\sqrt{3}}{-1}$

$= -2\pi/3$

$\therefore \left[2\,\text{cis}\left(-\frac{2\pi}{3}\right)\right]^{-10}$

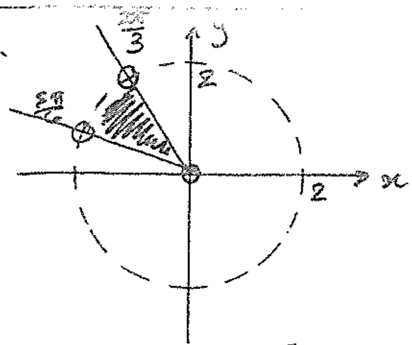
$= \frac{1}{1024}\left[\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3}\right]$

$= \frac{1}{1024}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= \frac{1}{1024}\left[-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

$= \frac{-1}{2048} + \frac{i\sqrt{3}}{2048}$

c.



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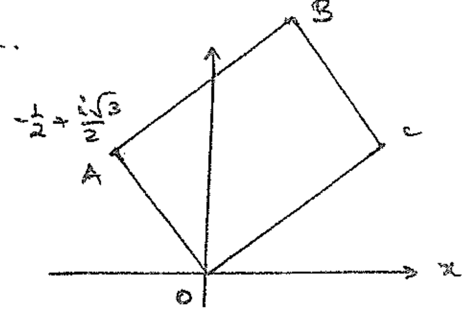
d.



Now $AO = \frac{1}{2} OC$ and we rotate OA anticlockwise 60°

ie multiply OA by 2 · cis −60°

$\therefore OC = 2 \times OA \times cis\left(-\frac{\pi}{3}\right)$

Now, for OA mod $= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= 1$

$\arg = \tan^{-1} \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$

$= \frac{2\pi}{3}$

$\therefore OC = 2 \times cis \frac{2\pi}{3} \times cis\left(-\frac{\pi}{3}\right)$

$= 2\, cis \frac{\pi}{3}$ $\left[\text{as } \frac{2\pi}{3} + -\frac{\pi}{3} = \frac{\pi}{3}\right]$

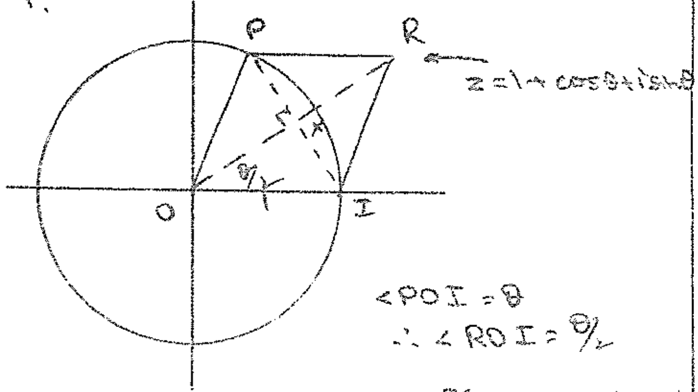
$= 2\left[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right]$

$= 2\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

$= 1 + i\sqrt{3}$

3

e. i.



$\angle POI = \theta$

$\therefore \angle ROI = \theta/2$

In $\triangle OXI$, by trig $\cos \theta/2 = \frac{OR}{1} = |OR|$

$\therefore z = |OR|\, cis\, \theta/2$

But $OR = 2 \times OX$

$\therefore OR = 2\cos \theta/2$

$\therefore z = 2\cos\theta/2\,(\cos\theta/2 + i\sin\theta/2)$ 2

ii. $\frac{1}{z} = z^{-1} = \left[2\cos\theta/2\,(\cos\theta/2 + i\sin\theta/2)\right]^{-1}$

$= \frac{1}{2\cos\theta/2}\left[\cos -\theta/2 + i\sin -\theta/2\right]$

by de Moivre

$= \frac{1}{2\cos\theta/2}\left[\cos\theta/2 - i\sin\theta/2\right]$

$= \frac{1}{2}\left[\frac{\cos\theta/2 - i\sin\theta/2}{\cos\theta/2}\right]$

$= \frac{1}{2}\left[1 - i\tan\theta/2\right]$

$= \frac{1}{2} - \frac{i}{2}\tan\theta/2$ 2

Question 3

a. i. since ω is a root of $z^3 = 1$

α. $\therefore \omega^3 = 1$ 1

β. $\therefore z^3 - 1 = 0$

$\therefore$ sum of roots $= -\frac{b}{a} = 0$

$\therefore 1 + \omega + \omega^2 = 0$

ii. $x^3 + ax^2 + bx + c = 0$

$\therefore 1 + \omega + \omega^2 = -a$

$\therefore 0 = -a$ (from β above)

$\therefore a = 0$

Sum of roots in pairs:

$\omega + \omega^2 + \omega^3 = b$

$\omega(1 + \omega + \omega^2) = b$

$\therefore b = 0$

Product of roots:

$1(\omega)(\omega^2) = -c$

$\omega^3 = -c$

But $\omega^3 = 1$ $\therefore c = -1$ 3

$\therefore a = 0, b = 0, c = -1$

b. Diff wrt x:

$3x^2 + 3y^2 \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^2}{y^2}$

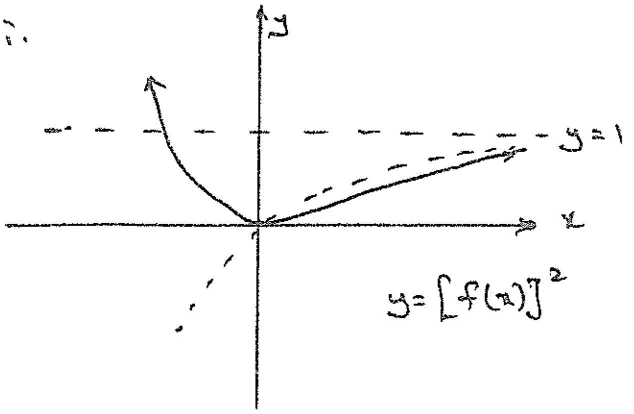
As $x=1$, $y=0$

$\therefore \frac{dy}{dx} =$ undef. $\therefore$ tangent vertical

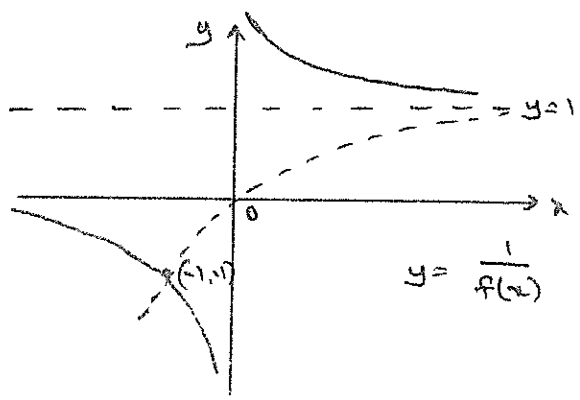
and for $y=x$, gradient $=1$ $\therefore \theta = 45^\circ$

$\therefore \angle$ between is 45°

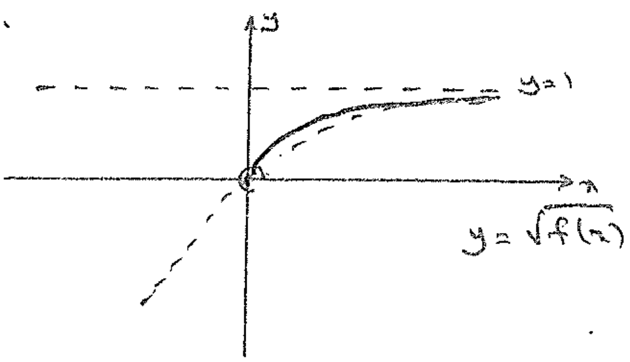
c i.



ii



iii.



Question 4

a. $z^{10}=1$ $\therefore z=1$ is one root

Now, roots are equally spaced around unit circle

$\therefore$ spaced $\frac{2\pi}{10}$ apart ie $\frac{\pi}{5}$

$\therefore$ roots are $z = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$ where $r = 0, 1, 2, \ldots, 9$ $[r=0 \rightarrow z=1]$

b. Roots of form $x = \frac{1}{\alpha}$ ie $\alpha = \frac{1}{x}$

$\therefore$ Subs $\frac{1}{x}$

$\therefore \left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right) + 1 = 0$

$\frac{1}{x^3} + \frac{2}{x} + 1 = 0$

Mult thru by x^3:

$1 + 2x^2 + x^3 = 0$

ie $x^3 + 2x^2 + 1 = 0$

ii. As $x^3 + 2x + 1 = 0$ has roots α, β, γ $\therefore \alpha+\beta+\gamma = 0$

$\therefore \beta + \gamma = -\alpha$

But if root is $\frac{\beta+\gamma}{\alpha^2} = \frac{-\alpha}{\alpha^2} = \frac{-1}{\alpha}$

$\therefore$ question is roots are $\frac{-1}{\alpha}, \frac{-1}{\beta}, \frac{-1}{\gamma}$

$\therefore$ equation is $\left(\frac{-1}{x}\right)^3 + \left(-\frac{2}{x}\right) + 1 = 0$

$-1 - 2x^2 + x^3 = 0$

$\therefore x^3 - 2x^2 - 1 = 0$

c. i. $\frac{dx}{d\theta} = 1 + \cos 2\theta$

$\frac{dy}{d\theta} = 1 - \cos 2\theta$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= 1 - \cos 2\theta \times \frac{1}{1+\cos 2\theta}$

$= \frac{1-\cos 2\theta}{1+\cos 2\theta}$

$= \frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta - 1)}$

$= \frac{2\sin^2\theta}{2\cos^2\theta}$

$= \tan^2\theta$

ii. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\tan^2\theta\right]$

$= \frac{d}{dx}\left[(\tan\theta)^2\right]$

$= 2\tan\theta\sec^2\theta \cdot \frac{d\theta}{dx}$

$= 2\tan\theta\sec^2\theta \frac{1}{1+\cos 2\theta}$

$= 2\tan\theta\sec^2\theta \frac{1}{2\cos^2\theta}$

$= \tan\theta\sec^4\theta$

d i. $x = 3\cos\theta$

$\frac{x}{3} = \cos\theta$

$\frac{x^2}{9} = \cos^2\theta$ ——— ①

Similarly, $\frac{y^2}{16} = \sin^2\theta$ — ②

② + ① $\frac{x^2}{9} + \frac{y^2}{16} = 1$

ii. As $a > b$ always $\therefore a = 4, b = 3$

$\therefore$ for ellipse: $b^2 = a^2(1-e^2)$

$9 = 16(1-e^2)$

$1 - e^2 = \frac{9}{16}$

$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

$\therefore$ foci $(0, \pm\sqrt{7})$

$\therefore S_1(0, \sqrt{7}),\ S_2 = (0, -\sqrt{7})$

iii. Directrices $y = \pm\frac{a}{e}$

$\therefore y = \pm 4 \div \frac{\sqrt{7}}{4}$

$= \pm\frac{16}{\sqrt{7}}$

$\therefore$ directrices $y = \frac{16}{\sqrt{7}},\ y = \frac{-16}{\sqrt{7}}$

Question 5

a i. Let $x = a$ be the root of $P(x)$

$\therefore P(x) = (x-a)^3 Q(x)$

$\therefore P'(x) = (x-a)^3 Q'(x) + Q(x).3(x-a)^2$

$= (x-a)^2[(x-a)Q'(x) + 3Q(x)]$

which has a root of $x = a$ with multiplicity 2

ii. $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$

$P'(x) = 32x^3 - 75x^2 + 54x - 11$

$P''(x) = 96x^2 - 150x + 54 = 0$

$6(16x^2 - 25x + 9) = 0$

$6(16x - 9)(x - 1) = 0$

$\therefore$ zeros of $P''(x)$ are $\frac{9}{16}, 1$

Now test in $P(x)$

$P(1) = 8(1)^4 - 25(1)^3 + 27(1)^2 - 11(1) + 1 = 0$

$\therefore x = 1$ is the triple root

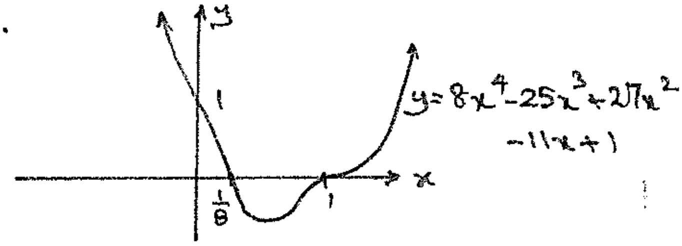
$\therefore$ Sum of roots are $1 + 1 + 1 + \beta = \frac{25}{8}$

$\beta = 3\frac{1}{8} - 3$

$= \frac{1}{8}$

$\therefore$ zeros are $1, 1, 1, \frac{1}{8}$

iii.



b. $a\tan^2\alpha + b\tan\alpha + c = 0$

Roots are $\tan\alpha_1, \tan\alpha_2$

$\therefore \tan\alpha_1 + \tan\alpha_2 = -\frac{b}{a}$

$\tan\alpha_1 \tan\alpha_2 = \frac{c}{a}$

$\therefore \tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1\tan\alpha_2}$

$= -\frac{b}{a} \div \left(1 - \frac{c}{a}\right)$

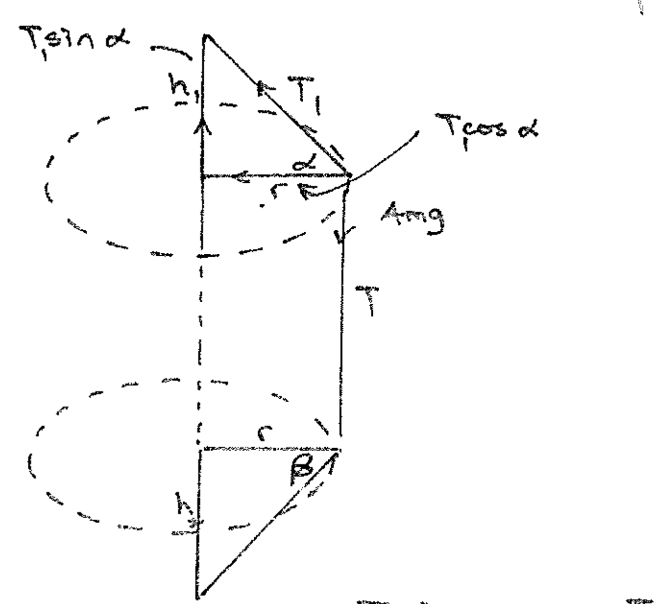
$= -\frac{b}{a} \div \frac{a-c}{a}$

$= -\frac{b}{a} \times \frac{a}{a-c}$

$= \frac{b}{c-a}$

c. acceleration is $r\omega^2$ and directed towards centre of the circle

d.



Resolving vertically: $T_1\sin\alpha - 4mg - T = 0$

$\therefore T_1\sin\alpha = 4mg + T$ ——— ①

Resolving horizontally:

$T_1 \cos\alpha = (4m)r\omega^2$

$\therefore T_1 \cos\alpha = 4mr\omega^2$ —— ②

ii. ① ÷ ②

$$\frac{\cancel{T_1}\sin\alpha}{\cancel{T_1}\cos\alpha} = \frac{4mg+T}{4mr\omega^2}$$

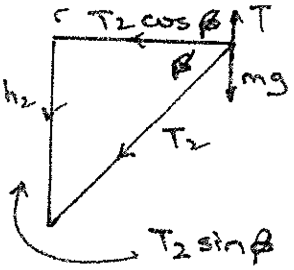
$$\tan\alpha = \frac{4mg+T}{4mr\omega^2}$$

Now, by trig: $\tan\alpha = \frac{h_1}{r}$

$$\therefore \frac{h_1}{r} = \frac{4mg+T}{4mr\omega^2}$$

$$\therefore h_1 = \frac{4mg+T}{4m\omega^2}$$

iii. Consider forces at Q



Let tension be T_2

Vertically: $T - T_2\sin\beta - mg = 0$

$\therefore T_2 \sin\beta = T - mg$ —— ①

Horizontally:

$T_2\cos\beta = mr\omega^2$ —— ②

① ÷ ② $\tan\beta = \frac{T-mg}{mr\omega^2}$

From trig: $\tan\beta = \frac{h_2}{r}$

$$\therefore \frac{h_2}{r} = \frac{T-mg}{mr\omega^2}$$

$$\therefore h_2 = \frac{T-mg}{m\omega^2}$$

Now $(4h_1 - h_2)\omega^2$

$$= \left(\frac{4mg+T}{m\omega^2} - \frac{T-mg}{m\omega^2}\right)\omega^2$$

$$= \frac{4mg+\cancel{T}-\cancel{T}+mg}{m}$$

$$= 5g$$

Question 6

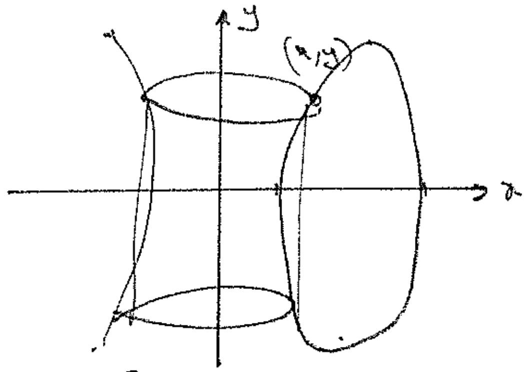
a. $P(x) = Q(x)(x+2)(x-3) + (4x+1)$

$\therefore P(-2) = 0 + (4(-2)+1)$

$= -7$

$\therefore$ remainder is -7

b.



i. $V = 2\pi\int r\,h\,dx$

$= 2\pi\int_1^3 x \cdot 2y\,dx$

$= 4\pi\int_1^3 xy\,dx$

Now, $(x-2)^2 + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1-(x-2)^2$

$y^2 = 4\left[1-(x-2)^2\right]$

$y = 2\sqrt{1-(x-2)^2}$

$\therefore V = 8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$

ii. let $u = x-2$

$\therefore \frac{du}{dx} = 1$

$dx = du$

$\therefore V = 8\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}\,du$

$= 8\pi\left[\int_{-1}^{1} u\sqrt{1-u^2}\,du + \int_{-1}^{1} 2\sqrt{1-u^2}\,du\right]$

Now, $u\sqrt{1-u^2}$ is an odd function

$\therefore \int u\sqrt{1-u^2}\,dx = 0$

and $\int_{-1}^{1}\sqrt{1-u^2}\,du$ is area of semi-circle, radius 1

$\therefore 8\pi\left[0 + 2\left(\frac{1}{2}\pi\times 1^2\right)\right] = 8\pi^2$

c. ↑ object $\downarrow mkv^2$ $\downarrow mg$

i. $m\ddot{x} = -mkv^2 - mg$

$\therefore \ddot{x} = -kv^2 - g$

$\therefore v\frac{dv}{dx} = -(g+kv^2)$

$\frac{dv}{dx} = \frac{-(g+kv^2)}{v}$

$\frac{dx}{dv} = \frac{-v}{g+kv^2}$

$x = -\frac{1}{2k}\ln(g+kv^2)+c$

$x=0$, $v=U$: $\therefore 0 = -\frac{1}{2k}\ln(g+ku^2)+c$

$c = \frac{1}{2k}\ln(g+ku^2)$

$\therefore x = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right)$ — ①

Now, $x=H$, $v=0$: $H = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$

3

ii. At P, $x=h$, $v=V$

Subs in ①

$h = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kV^2}\right)$

2

iii. ↓ object $\downarrow mg$ $\uparrow mkv^2$

$m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

$v\frac{dv}{dx} = g - kv^2$

$\frac{dv}{dx} = \frac{g-kv^2}{v}$

$\frac{dx}{dv} = \frac{v}{g-kv^2}$

$x = -\frac{1}{2k}\ln(g-kv^2)+c$

Let starting point by $x=0$.

$\therefore x=0$, $v=0$: $0 = -\frac{1}{2k}\ln(g-0)+c$

$c = \frac{1}{2k}\ln g$

$\therefore x = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$

Let h be distance from ground

$\therefore$ fallen $H-h$

$\therefore x = H-h$, $v = \frac{1}{2}V$

$\therefore H-h = \frac{1}{2k}\ln\left(\frac{g}{g-\frac{kV^2}{4}}\right)$

$= \frac{1}{2k}\ln\left[\frac{4g}{4g-kV^2}\right]$

Now, using H from i and h from ii

$\frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right) - \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right)$

$= \frac{1}{2k}\ln\left(\frac{4g}{4g-kV^2}\right)$

mult thru by 2k:

$\ln\left(\frac{g+\cancel{ku^2}}{g} \times \frac{g+kv^2}{g+\cancel{ku^2}}\right)$

$= \ln\frac{4g}{4g-kV^2}$

$\therefore \frac{g+kv^2}{g} = \frac{4g}{4g-kv^2}$

$\cancel{4g^2} - kgv^2 + 4kgv^2 - k^2v^4 = \cancel{4g^2}$

$k^2v^4 = 3kgv^2$

$v^2 = \frac{3g}{k}$

$\therefore v = \sqrt{\frac{3g}{k}}$

3

Question 7

a. $z = a+ib$

$\therefore a^2+b^2+5(a-ib)+10i = 0+0i$

$\therefore a^2+b^2+5a = 0$ — ①

$-5b+10 = 0$ — ②

$\therefore b = 2$

Subs in ① $a^2 + 4 + 5a = 0$

$a^2 + 5a + 4 = 0$

$(a+4)(a+1) = 0$

$a = -4, -1$

$\therefore -4+2i, -1+2i$ 2

b. i. $xy = 4$

$\therefore$ Diff wrt x:

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

At P: grad $= -\frac{2}{p} \times \frac{1}{2p}$

$= -\frac{1}{p^2}$ 2

ii. grad of normal $= p^2$

$\therefore y - \frac{2}{p} = p^2(x - 2p)$

$py - 2 = p^3(x - 2p)$

$py - 2 = p^3x - 2p^4$ 2

$\therefore p^3x - py = 2(p^4 - 1)$ —①

iii. As $xy = 4$ $\therefore y = \frac{4}{x}$

Subs in ①

$p^3x - \frac{4p}{x} = 2(p^4 - 1)$

$p^3x^2 - 4p = 2x(p^4 - 1)$

$p^3x^2 - 4p - 2p^4x + 2x = 0$

$\therefore p^3x^2 - 2(p^4-1)x - 4p = 0$

Now, product of roots $= \frac{-4p}{p^3} = \frac{-4}{p^2}$

But roots are $2p$ and $2q$

$\therefore 4pq = -\frac{4}{p^2}$ 3

$\therefore p^3q = -1$

c. RHS $= t^{n-2} - \frac{t^{n-2}}{1+t^2}$

$= \frac{(1+t^2)t^{n-2} - t^{n-2}}{1+t^2}$

$= \frac{t^{n-2} + t^n - t^{n-2}}{1+t^2}$

$= \frac{t^n}{1+t^2}$ 2

$=$ LHS

ii. $I_n = \int \frac{t^n}{1+t^2}\,dt$

$= \int t^{n-2} - \frac{t^{n-2}}{1+t^2}\,dt$

$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2}\,dt$

$= \frac{t^{n-1}}{n-1} - I_{n-2}$ 1

iii. Let $J_n = \int_0^1 \frac{t^n}{1+t^2}\,dt$

$= \left.\frac{t^{n-1}}{n-1}\right]_0^1 - J_{n-2}$

$= \frac{1}{n-1} - J_{n-2}$

$\therefore J_6 = \frac{1}{5} - J_4$

$= \frac{1}{5} - \left[\frac{1}{3} - J_2\right]$

$= \frac{1}{5} - \frac{1}{3} + 1 - J_0$

But $J_0 = \int_0^1 \frac{1}{1+t^2}\,dt$

$= \tan^{-1}\Big]_0^1$

$= \pi/4 - 0$

$= \pi/4$

$\therefore J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \pi/4$

$= \frac{13}{15} - \frac{\pi}{4}$ 3

Question 8

a. $y = e^{xy}$

$\frac{dy}{dx} = \left(y + x\frac{dy}{dx}\right)e^{xy}$

$\therefore \frac{dy}{dx} = ye^{xy} + x\frac{dy}{dx}e^{xy}$

$\frac{dy}{dx}(1-xe^{xy}) = ye^{xy}$

$\therefore \frac{dy}{dx} = \frac{ye^{xy}}{1-xe^{xy}}$

b. i. $\sin x = \cos x$

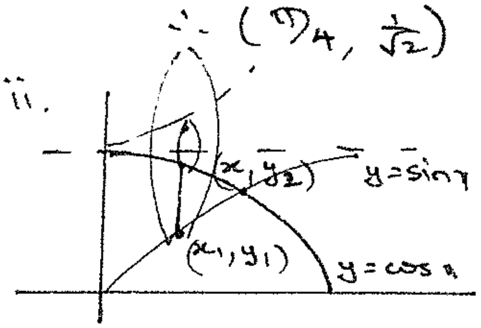
$\therefore \frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$\tan x = 1$

$x = \pi/4$

$\therefore (\pi/4, \frac{1}{\sqrt{2}})$

ii.



$\delta V = \pi$ (radii of annulus squared) δx

$= \pi\left([1-\sin x]^2 - [1-\cos x]^2\right)\delta x$

$= \pi(1 - 2\sin x + \sin^2 x - 1 + 2\cos x - \cos^2 x)\,\delta x$

$= \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\,\delta x$

iii. $V = \pi\int_0^{\pi/4}(2\cos x - 2\sin x - \cos 2x)\,dx$

$= \pi\left[2\sin x + 2\cos x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \pi\left[2\cdot\frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}} - \frac{1}{2} - (0 + 2 - 0)\right]$

$= \pi\left[2\sqrt{2} - \frac{5}{2}\right]$

$= \frac{\pi}{2}\left[4\sqrt{2} - 5\right]$ units^3

c. i. Now tangent AC:

$y = e^x \quad \therefore y' = e^x$

$y'(0) = e^0$

$= 1$

$\therefore$ eqn: $y - 1 = 1(x - 0)$

$y - 1 = x$

$\therefore y = x + 1$ —— (1)

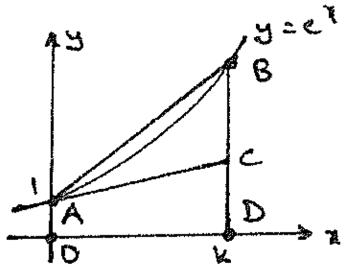
Now subs $x = k$ in (1) $\therefore y = k+1$

$\therefore C(k, k+1)$

Also, $B(k, e^k)$

Now, from diagram:

Area AODC $< \int_0^k e^x\,dx <$ Area AODB



$\therefore \frac{1}{2}k(1 + k + 1) < e^x\Big]_0^k < \frac{1}{2}k(1 + e^k)$

$\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1 + e^k)$

Now, let $k = 1$

$\therefore 1.5 < e - 1 < \frac{1+e}{2}$

$\therefore e - 1 > 1.5 \qquad 2e - 2 < 1 + e$

$e > 2.5 \qquad e < 3$

$\therefore 2.5 < e < 3$

ii. If area $\triangle ABC$ is bisected

$\therefore e^k - 1 - \frac{1}{2}k(k+2) = \frac{1}{2}k(1+e^k) - (e^k - 1)$

$2e^k - 2 - k^2 - 2k = k + ke^k - 2e^k + 2$

$4e^k - ke^k - k^2 - 3k - 4 = 0$

$(4-k)e^k - k^2 - 3k - 4 = 0$

Let $f(k) = (4-k)e^k - k^2 - 3k - 4$

Now $f(2) = 0.78 > 0$

$f(3) = -1.9 < 0$

$\therefore$ as $f(k)$ is continuous,

$\therefore$ k lies between 2 & 3 if

$f(k) = 0 \quad \therefore 2 < k < 3$

Now $f'(k) = -e^k + (4-k)e^k - 2k - 3$

$= (3-k)e^k - 2k - 3$

$f'(2.7) = 3.9361 \quad f(2.7) = -0.4635$

$\therefore k_1 = 2.7 - \frac{-0.4635}{3.9361}$

$= 2.688$

$\therefore$ 2nd approx is 2.7 (1 dpl)