Question 1 (15 marks) Commence each question on a SEPARATE page

a. Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \; dx$. **2**

b. Find $\int_0^3 \frac{5}{x^2 - 6x + 18} \; dx$ **3**

c. i. Find the real numbers a, b and c such that **3**

$$\frac{4x-6}{(x+1)(2x^2+3)} \equiv \frac{a}{x+1} + \frac{bx+c}{2x^2+3}$$

ii. Hence, or otherwise, find $\int \frac{4x-6}{(x+1)(2x^2+3)} \; dx$. **2**

d. Show that $\int_0^1 \frac{2y}{1+2y} \; dy = 1 - \frac{1}{2} \ln 3$. **2**

e. Using the substitution $t = \tan \frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{3}} \frac{dx}{13 + 5\sin x + 12\cos x}$ **3**

Question 2 (15 marks) Commence each question on a SEPARATE page

a. Simplify $(1 - 2i)^2$ **1**

b. Solve $z^2 = 5 - 12i$, giving your answer in the form $x + iy$, where x and y are real. **2**

c. i. Express $1 + i$ in modulus-argument form. **1**

ii. Hence evaluate $(1 + i)^{12}$ **2**

d. i. On an Argand diagram, shade the region where both $|z - 1| \le 1$ and $0 \le \arg z \le \frac{\pi}{6}$. **2**

ii. Find the perimeter of the shaded region. **2**

e. The points O, A, Z and C on the Argand diagram represent the complex numbers 0, 1, z and $z + 1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

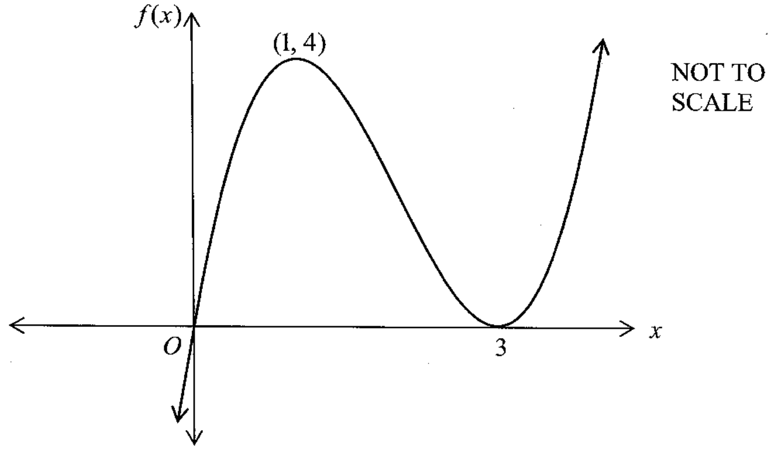
i. Explain why $OACZ$ is a rhombus. **1**

ii. Show that $\frac{z-1}{z+1}$ is purely imaginary. **2**

iii. Find the modulus and argument of $z + 1$. **2**

Question 3 (15 marks) Commence each question on a SEPARATE page

a. The function defined by $f(x) = x(x - 3)^2$ is drawn below.



Draw separate, one-third sketches, of the following:

i. $y = f(|x|)$ **1**

ii. $y = \dfrac{1}{f(x)}$ **1**

iii. $y^2 = f(x)$ **2**

iv. $y = \sin^{-1} f(x)$ **2**

v. $y = \log_e f(x)$ **2**

b. Consider the curve given by the equation $x^2 - y^2 + xy + 5 = 0$.

i. Show that $\dfrac{dy}{dx} = \dfrac{2x + y}{2y - x}$. **2**

ii. Hence find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = x$. **2**

c. Show that the line $y = -3x$ is a tangent to the curve $f(x) = x + \dfrac{1}{x-1}$, and find the co-ordinates of the point of contact. **3**

Question 4 (15 marks) Commence each question on a SEPARATE page

a. i. Given $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$. By writing $\cos^n x = \cos x.\cos^{n-1} x$, **3**

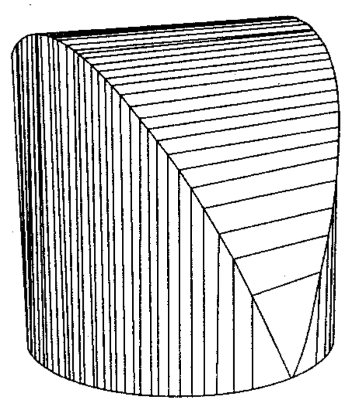
prove that $I_n = \frac{n-1}{n}.I_{n-2}$, where $n = 2, 3, 4, \ldots$

ii. Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^4 x\ dx$. **3**

b. For his visual arts major work, Lionel took a wooden cylinder and carved into it the shape shown in the diagram below.

The base of her shape is a circle with radius 8 cm.

Each vertical cross-section shown in the diagram is a square.



i. Show that the area A of the cross-section distance x cm from the centre of the base is $A = 4(64 - x^2)$. **2**

ii. Hence show that the volume V of the art project is given by **2**

$V = 8\int_0^8 (64 - x^2)\ dx$ and evaluate the integral.

c. i. If α is a double root of the polynomial equation $P(x) = 0$, show that α is a root of $P'(x) = 0$. **2**

ii. The polynomial equation $x^5 - ax^2 + b = 0$ has a multiple root. ($a \neq 0$, $b \neq 0$). **3**

Show that $108a^5 = 3125b^3$.

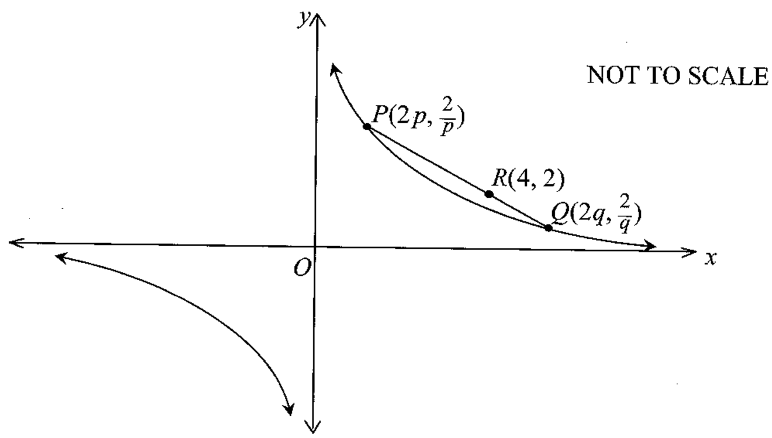
Question 5 (15 marks) Commence each question on a SEPARATE page

a. Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = 1$. **2**

b. $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are points on the rectangular hyperbola $xy = 4$.

M is the midpoint of the chord PQ.

P and Q move on the hyperbola so that the chord PQ always passes through the point $R(4, 2)$.



i. Show that the equation of the chord PQ is $x + pqy = 2(p + q)$. **2**

ii. Show that $pq = p + q - 2$. **1**

iii. Hence sketch the locus of M, as P and Q move on the curve $xy = 4$. **4**

c. i. If $z = \cos\theta + i\sin\theta$, explain why $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$ for positive integers n. **2**

ii. By considering the binomial expansions of $(z + z^{-1})^3$ and $(z - z^{-1})^3$, show that $4(\cos^3\theta + \sin^3\theta) = (\cos 3\theta - \sin 3\theta) + 3(\cos\theta + \sin\theta)$. **2**

iii. Hence evaluate $\cos^3\frac{\pi}{12} + \sin^3\frac{\pi}{12}$. **2**

Question 6 (15 marks) Commence each question on a SEPARATE page

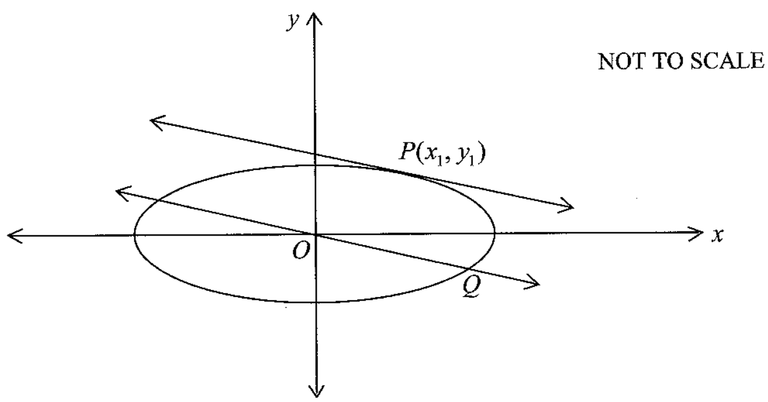
a. i. Verify that $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of $z^5 + z - 1 = 0$. **2**

ii. Find the monic cubic equation with real coefficients whose roots are also roots of $z^5 + z - 1 = 0$ but do not include α. **3**

b. $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O.

A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

The equation of the tangent to the ellipse at P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.



i. Show that the equation of the line OQ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0$. **2**

ii. Show that the coordinates of Q, in terms of x_1 and y_1 are $\left(\frac{ay_1}{b}, \frac{-bx_1}{a}\right)$. **3**

iii. Show that the distance between the tangent at P and the line OQ is $\frac{ab}{OQ}$. **4**

iv. Hence prove the area of the triangle OPQ is independent of the position P. **1**

Question 7 (15 marks) Commence each question on a SEPARATE page

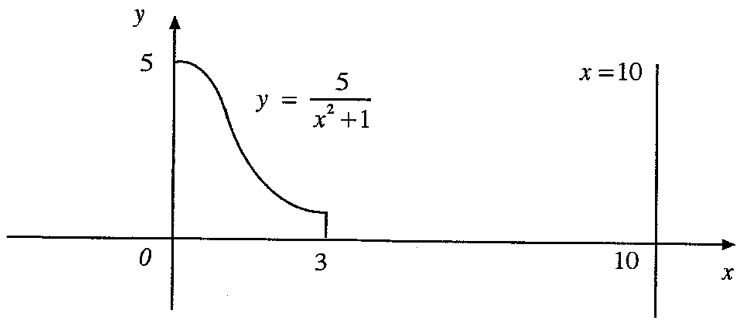
a. A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the particle has speed v ms^{-1}. The acceleration due to gravity is 10 ms^{-2}.

i. If the particle projected vertically upwards from rest with speed U ms^{-1}. Show that during its upwards motion, its acceleration a ms^{-2} is given by $a = -\frac{1}{10}(100 + v^2)$ **1**

ii. Hence show that its maximum height, H metres, is given by $H = 5\ln\left(\frac{U^2+100}{100}\right)$. **3**

iii. The particle falls vertically from rest. Show that during its downward motion its acceleration a ms^{-2} is given by $a = \frac{1}{10}(100 - v^2)$. **1**

iv. Hence show that it returns to its point of projection with speed V ms^{-1} given by $V = \frac{10U}{\sqrt{U^2+100}}$. **4** b.



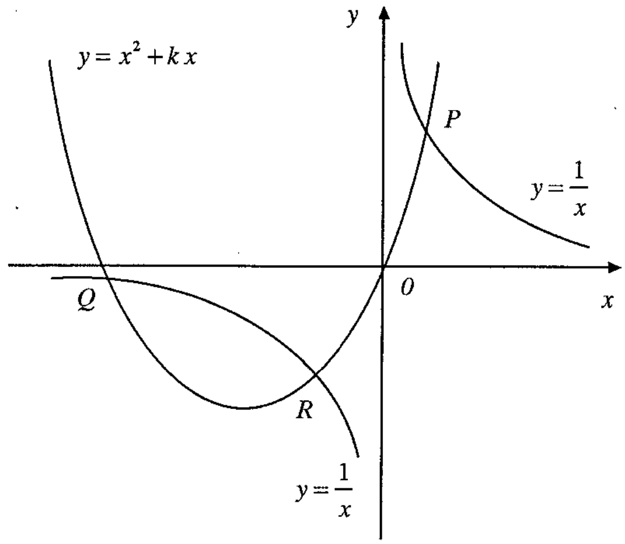
A circular flange is formed by rotating the region bounded by the curve $y = \frac{5}{x^2+1}$, the x axis and the lines $x = 0$ and $x = 3$, through one complete revolution about the line $x = 10$ (where all measurements are in cm).

i. Use the method of cylindrical shells to show that the volume V cm^3 of the flange is given by $V = \int_0^3 \frac{\pi(100-10x)}{x^2+1}\, dx$. **3**

ii. Hence find the volume of the flange correct to the nearest cm^3. **3**

Question 8 (15 marks) Commence each question on a SEPARATE page

a.



The curves $y = x^2 + kx$, where $k > 0$, and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha$, $x = \beta$ and $x = \gamma$.

i. Show that α, β and γ are the roots of the equation $x^3 + kx^2 - 1 = 0$. **1**

ii. Find the monic cubic equation with coefficients in terms of k whose roots are $-\alpha$, $-\beta$ and $-\gamma$. **1**

iii. Find the monic cubic equation with coefficients in terms of k whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **1**

iv. Show that the gradient of PQ, QR and RP have a sum of k and a product of -1. **2**

b. A light inextensible string OP is fixed at one end O. A particle P is attached to the other end and moves uniformly in a horizontal circle whose centre is vertically below and at a distance h centimetres from O.

i. By resolving the vertical and horizontal forces, show that $\omega = \sqrt{\frac{g}{h}}$ **2**

ii. Hence, show that the period of the motion is given by $2\pi\frac{\sqrt{h}}{g}$. **1**

iii. If the number of revolutions per second of the rotating particle is decreased from 2 rev/s to 1 rev/s, find the distance by which the level of the circle is lowered. **2**

c. i. By using de Moivres theorem and $(\cos\theta + i\sin\theta)^3$, or otherwise, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **2**

ii. Hence, show $\tan 3\theta = \frac{t(3-t^2)}{1-3t^2}$, where $t = \tan\theta$. **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a. $\int_0^{\pi/4} \sec^2 x \tan^2 x \, dx$

$= \frac{1}{3}\tan^3 x \Big]_0^{\pi/4}$ by inspection

$= \frac{1}{3}\left[\tan^3 \pi/4 - \tan 0\right]$

$= \frac{1}{3}$ 2

b. $\int_0^3 \frac{5}{x^2-6x+18}\,dx$

$= \int_0^3 \frac{5}{x^2-6x+9+9}\,dx$

$= 5\int_0^3 \frac{1}{9+(x-3)^2}\,dx$

$= \frac{5}{3}\tan^{-1}\frac{x-3}{3}\Big]_0^3$

$= \frac{5}{3}\left[\tan^{-1}0 - \tan^{-1}-1\right]$

$= \frac{5}{3}\left[0 - -\pi/4\right]$

$= \frac{5\pi}{12}$ 3

c i. $4x-6 = a(2x^2+3)+(bx+c)(x+1)$

Let $x=-1 \therefore -10 = 5a \quad \therefore a=-2$

Let $x=0 \therefore -6 = 3a+c$

$\therefore -6 = -6+c \quad \therefore c=0$

Let $x=2 \therefore 2 = 11a+6b+3c$

$\therefore 2 = -22+6b$

$6b = 24 \quad \therefore b=4$

$\therefore a=-2,\ b=4,\ c=0$ 3

ii. $\int \frac{4x-6}{(x+1)(2x^2+3)}\,dx = \int \frac{-2}{x+1} + \frac{4x}{2x^2+3}\,dx$

$= -2\ln(x+1) + \ln(2x^2+3) + c$

$= \ln\left[\frac{2x^2+3}{(x+1)^2}\right] + c$ 2

d. $\int_0^1 \frac{2y}{1+2y}\,dy = \int_0^1 \frac{1+2y-1}{1+2y}\,dy$ dx

$= \int_0^1 1 - \frac{1}{1+2y}\,dy$

$= y - \frac{1}{2}\ln(1+2y)\Big]_0^1$

$= 1 - \frac{1}{2}\ln 3 - (0)$

$= 1 - \frac{1}{2}\ln 3$ 2

e. $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]$

$= \frac{1}{2}(1+t^2)$

$\therefore \frac{dt}{dx} = \frac{1+t^2}{2} \quad \therefore dx = \frac{2\,dt}{1+t^2}$

$x = \pi/3 \ \therefore t = \frac{1}{\sqrt{3}}$ and $x=0, t=0$

$\int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2\,dt}{1+t^2}}{13 + \frac{5(2t)}{1+t^2} + \frac{12(1-t^2)}{1+t^2}}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{13+13t^2+10t+12-12t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{t^2+10t+25}$

$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{(t+5)^2}$

$= 2\int_0^{\frac{1}{\sqrt{3}}} (t+5)^{-2}\,dt$

$= -2(t+5)^{-1}\Big]_0^{\frac{1}{\sqrt{3}}}$

$= \left[\frac{-2}{t+5}\right]_0^{\frac{1}{\sqrt{3}}}$

$= -2\left[\frac{1}{\frac{1}{\sqrt{3}}+5} - \frac{1}{5}\right]$

$= -2\left[\frac{\sqrt{3}}{1+5\sqrt{3}} - \frac{1}{5}\right]$ 3

$= -2\left[\frac{5\sqrt{3}-1-5\sqrt{3}}{5(1+5\sqrt{3})}\right]$

$= \frac{2}{5+25\sqrt{3}}$

Question 2

a. $(1-2i)^2 = 1-4i+4i^2$

$= 1-4i-4$ 1

$= -3-4i$

b. $z^2 = 5 - 12i$ Let $z = x + iy$

$\therefore x^2 - y^2 + 2xyi = 5 - 12i$

$\therefore x^2 - y^2 = 5$ $\quad 2xy = -12$

$\therefore y = -\frac{6}{x}$

$\therefore x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

$\therefore x = \pm 3 \quad \therefore y = \mp 2$

$\therefore z = 3 - 2i, \ -3 + 2i$ 2

c. i. $1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ 1

ii. $(1+i)^{12} = (2^{\frac{1}{2}})^{12} \operatorname{cis} 3\pi$

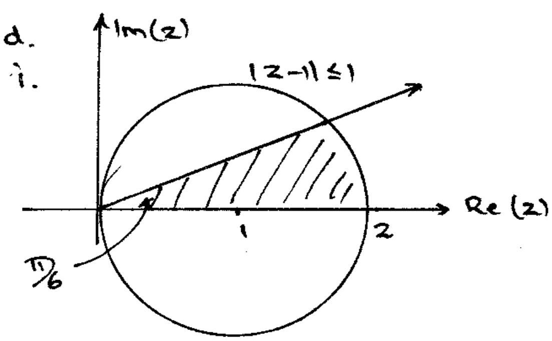
$= 2^6 \operatorname{cis} \pi$

$= 64[\cos\pi + i \sin\pi]$

$= -64$ 2

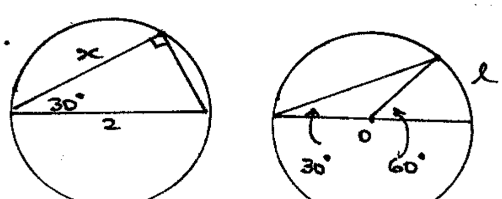
d.

i.



2

ii.



$\frac{x}{2} = \cos 30°$

$x = 2 \cdot \frac{\sqrt{3}}{2}$

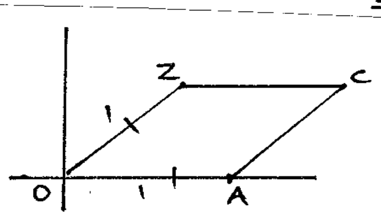
$= \sqrt{3}$

$\ell = \frac{1}{6} \times 2 \times \pi \times 1$

$= \frac{\pi}{3}$ 2

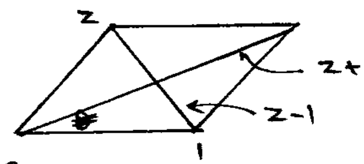
$\therefore$ Perimeter $= 2 + \sqrt{3} + \frac{\pi}{3}$ units

e.



i. OACZ is parallelogram by addition of vectors. As OA = OZ, then OACZ is rhombus (adj sides equal) 1

ii.

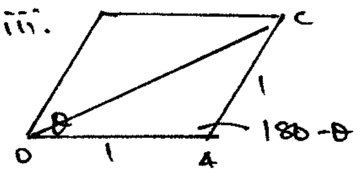


Diags of rhombus are perpendicular

$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

$\therefore \frac{z-1}{z+1}$ is purely imaginary 2

iii.



$\arg z = \theta$

$\therefore \arg(z+1) = \frac{\theta}{2}$

(as ∠ is bisected in rhomb.)

Now $|z+1| = \sqrt{1^2 + 1^2 - 2(1)(1)\cos(180 - \theta)}$

by cosine rule

$\therefore |z+1| = \sqrt{2 + 2\cos\theta}$

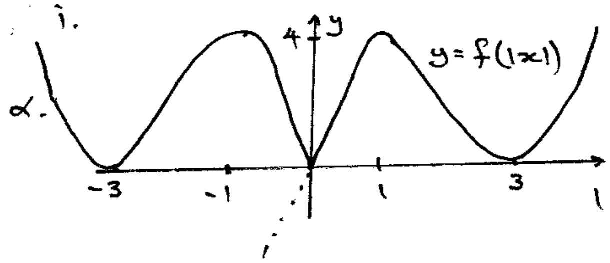
$\therefore$ modulus $= \sqrt{2 + 2\cos\theta}$

arg $= \frac{\theta}{2}$ 2

<u>Question 3</u>

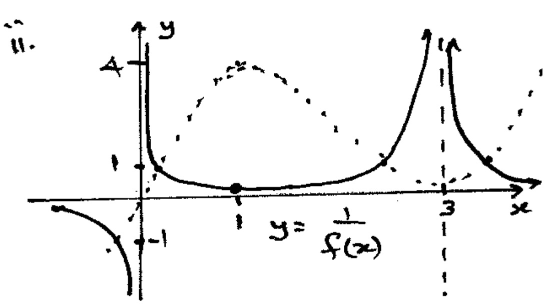
i.

α.



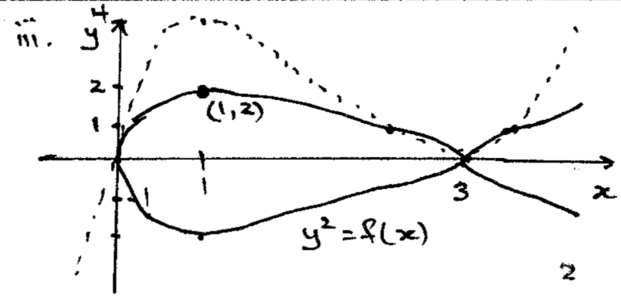
1

ii.



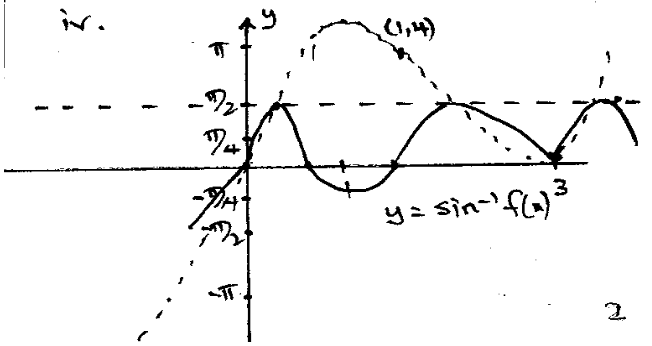
1

iii.



2

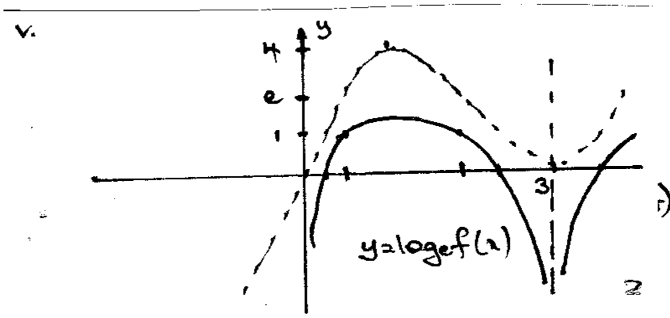
iv.



2

[Note: $\frac{\pi}{2}=1.57$, $\pi=3.14$, $4>\pi$]

v.



2

b. $x^2-y^2+xy+5=0$

i. $2x-2y\frac{dy}{dx}+y+x\frac{dy}{dx}=0$

$(2y-x)\frac{dy}{dx}=2x+y$

$\therefore \frac{dy}{dx}=\frac{2x+y}{2y-x}$ 2

ii. $y=x$ $\therefore$ grad $=1$

$\frac{2x+y}{2y-x}=1$

$2x+y=2y-x$

$y=3x$ ——①

Also, $x^2-y^2+xy+5=0$ —②

Subs ① in ②

$x^2-9x^2+x(3x)+5=0$

$x^2-9x^2+3x^2+5=0$

$5x^2=5$ $\therefore x=\pm 1$

Subs in ① $y=\pm 3$

$\therefore (1,3)$ and $(-1,-3)$ 2

c. Solve simultaneously,

$y=-3x$ and $y=x+\frac{1}{x-1}$

$\therefore x+\frac{1}{x-1}=-3x$

$\therefore x(x-1)+1=-3x(x-1)$

$x^2-x+1=-3x^2+3x$

$4x^2-4x+1=0$

$(2x-1)^2=0$

$\therefore$ $x=\frac{1}{2}$ is only soln

$\therefore (\frac{1}{2},-\frac{3}{2})$ is the point of contact $\therefore$ tangent 3

Question 4

a. i. $I_n=\int_0^{\pi/2}\cos^n x\,dx$

$=\int_0^{\pi/2}\cos^{n-1}x.\cos x\,dx$

$\uparrow u \quad \uparrow v'$

$=\cos^{n-1}x.\sin x\Big]_0^{\pi/2}$

$-\int_0^{\pi/2}(n-1)\cos^{n-2}x.-\sin x.\sin x\,dx$

$=0+(n-1)\int_0^{\pi/2}\cos^{n-2}x\sin^2 x\,dx$

$=(n-1)\int_0^{\pi/2}\cos^{n-2}x(1-\cos^2 x)\,dx$

$=(n-1)\int_0^{\pi/2}\cos^{n-2}x-(n-1)\int_0^{\pi/2}\cos^n x\,dx$

$=(n-1)I_{n-2}-(n-1)I_n$

$\therefore I_n+(n-1)I_n=(n-1)I_{n-2}$

$nI_n=(n-1)I_{n-2}$

$\therefore I_n=\frac{n-1}{n}I_{n-2}$ 3

ii. $\int_0^{\pi/2}\cos^4 x\,dx=I_4$

$\therefore I_4=\frac{3}{4}I_2$

and $I_2=\frac{1}{2}I_0$

Now, $I_0=\int_0^{\pi/2}\cos^0 x\,dx$

$=\int_0^{\pi/2}1\,dx$

$=x\Big]_0^{\pi/2}=\frac{\pi}{2}$

$\therefore I_0 = \pi/2$

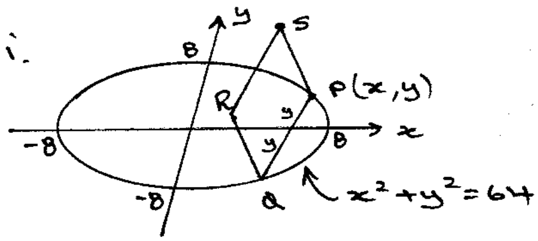
$\therefore I_2 = \frac{1}{2}(\pi/2)$

$= \pi/4$

$I_4 = \frac{3}{4}(\pi/4)$

$= 3\pi/16$ 3

b. i.



$PQ = 2y$

$\therefore$ Area $PQRS = 4y^2$

$= 4[64 - x^2]$ 2

ii. Volume = Sum of areas

$\therefore V = \int_{-8}^{8} 4(64 - x^2)\,dx$

$= 8\int_0^8 (64 - x^2)\,dx$

$= 8\left[64x - \frac{x^3}{3}\right]_0^8$

$= 8\left[512 - \frac{512}{3}\right]$

$= 2730\frac{2}{3}\text{ cm}^3$ 2

c. i. If α is double root,

$P(x) = (x-\alpha)^2 Q(x)$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$

$\therefore P'(\alpha) = 0$ 2

ii. Let $P(x) = x^5 - ax^2 + b$

$P'(x) = 5x^4 - 2ax = 0$

$x(5x^3 - 2a) = 0$

$\therefore x = 0, \quad x = \sqrt[3]{\frac{2a}{5}}$

But $P(0) \neq 0$

$\therefore x = \sqrt[3]{\frac{2a}{5}}$ is root

Now subs:

$P\left(\left(\frac{2a}{5}\right)^{\frac{1}{3}}\right) = \left(\frac{2a}{5}\right)^{5/3} - a\left(\frac{2a}{5}\right)^{2/3} - b = 0$

$a^{5/3}\left[\left(\frac{2}{5}\right)^{5/3} - \left(\frac{2}{5}\right)^{2/3}\right] = -b$

$a^{5/3}\cdot\left(\frac{2}{5}\right)^{2/3}\left[\frac{2}{5} - 1\right] = -b$

cube both sides:

$a^5\cdot\left(\frac{2}{5}\right)^2\left[-\frac{3}{5}\right]^3 = -b^3$

$\frac{-108a^5}{3125} = -b^3$

$\therefore 108a^5 = 3125b^3$ 3

Question 5

a. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$

$\therefore \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$

$\frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$

$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$ 2

b. i. grad $PQ = \dfrac{\frac{2}{p} - \frac{2}{q}}{2p - 2q}$

$= \frac{\cancel{2}q - \cancel{2}p}{pq} \times \frac{1}{\cancel{2}p - \cancel{2}q}$

$= \frac{-1}{pq}$

$\therefore y - \frac{2}{p} = \frac{-1}{pq}(x - 2p)$

$pqy - 2q = -x + 2p$

$\therefore x + pqy = 2(p+q)$ —— (1) 2

ii. subs R into (1)

$\therefore 4 + 2pq = 2p + 2q$

$pq = p + q - 2$ —— (2) 1

iii. Midpoint: M

$\left(\frac{2p+2q}{2}, \frac{\frac{2}{p} + \frac{2}{q}}{2}\right)$

$= \left(p+q, \frac{2q+2p}{pq} \times \frac{1}{2}\right)$

$= \left(p+q, \frac{p+q}{pq}\right)$

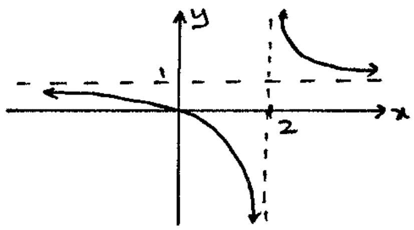
$\therefore$ Let $x = p + q \quad y = \frac{p+q}{pq}$

From ② $pq = p + q - 2$

$\therefore pq = x - 2$

Now, from $y = \dfrac{p+q}{pq}$

$$\therefore y = \frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}$$



4

c. i. $z = \cos\theta + i\sin\theta$

By deMoivres theorem:

$z^n = \cos n\theta + i \sin n\theta$ —①

$z^{-n} = \cos -n\theta + i\sin -n\theta$

$= \cos n\theta - i \sin n\theta$ —②

①+② $z^n + z^{-n} = 2\cos n\theta$

①−② $z^n - z^{-n} = 2i\sin n\theta$ 2

ii. $(z+z^{-1})^3 = z^3 + 3z + 3z^{-1} + z^{-3}$

$= (z^3 + z^{-3}) + 3(z + z^{-1})$

$(2\cos\theta)^3 = 2\cos 3\theta + 6\cos\theta$

$8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$

$\therefore 4\cos^3\theta = \cos 3\theta + 3\cos\theta$ —③

Similarly,

$(z - z^{-1})^3 = z^3 - 3z + 3z^{-1} - z^{-3}$

$= (z^3 - z^{-3}) - 3(z - z^{-1})$

$\therefore (2i\sin\theta)^3 = 2i\sin 3\theta - 6i\sin\theta$

$8i^3\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$-4i\sin^3\theta = i\sin 3\theta - 6i\sin\theta$

$\therefore -4\sin^3\theta = \sin 3\theta - 6\sin\theta$ —④

③−④ $4(\cos^3\theta + \sin^3\theta)$

$= (\cos 3\theta + \sin 3\theta) + 3(\cos\theta + \sin\theta)$

2

iii. From ii,

$\cos^3 \pi/12 + \sin^3 \pi/12$

$= \frac{1}{4}\left\{\cancel{\cos \pi/4 - \sin \pi/4} + 3(\cos \pi/12 + \sin \pi/12)\right\}$

$= \frac{3}{4}\left\{\cos \pi/12 + \sin \pi/12\right\}$

$= \frac{3}{4}\cdot\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos \pi/12 + \frac{1}{\sqrt{2}}\sin \pi/12\right\}$

$= \frac{3\sqrt{2}}{4}\left[\sin \pi/4 \cos \pi/12 + \cos \pi/4 \sin \pi/12\right]$

$= \frac{3\sqrt{2}}{4}\left[\sin(\pi/4 + \pi/12)\right]$

$= \frac{3\sqrt{2}}{4}\left[\sin \pi/3\right]$

$= \frac{3\sqrt{2}}{4}\cdot\frac{\sqrt{3}}{2}$ 2

$= \frac{3\sqrt{6}}{8}$

Question 6

a. i. $\alpha = \cos \pi/3 + i\sin \pi/3$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$ —①

Now, $\alpha^5 = \cos 5\pi/3 + i\sin 5\pi/3$

$= \cos \pi/3 - i\sin \pi/3$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$ —②

$\therefore P(z) = z^5 + z - 1$

$= \alpha^5 + \alpha - 1$

$= \frac{1}{2} - \cancel{\frac{\sqrt{3}}{2}i} + \frac{1}{2} + \cancel{\frac{\sqrt{3}}{2}i} - 1 = 0$

$\therefore \alpha$ is a root 2

ii. Roots in conjugate pairs:

$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ and $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\therefore$ Quadratic is

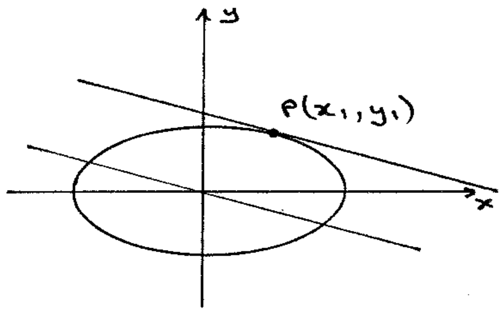
$z^2 - (\text{sum of roots})z + (\text{prod of roots}) = 0$

$z^2 - z + 1$

$$\begin{array}{r}
z^3 + z^2 - 1 \\
z^2 - z + 1 \;\big)\; z^5 + 0z^4 + 0z^3 + 0z^2 + z - 1 \\
\underline{z^5 - z^4 + z^3} \\
z^4 - z^3 + 0z^2 \\
\underline{z^4 - z^3 + z^2} \\
-z^2 + z - 1 \\
\underline{-z^2 + z - 1} \\
0
\end{array}$$

$\therefore z^3 + z^2 - 1$ 3

b.



i. Tangent at P: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Now, if OQ is parallel, ∴ of the form

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = c$$

Now, OQ passes through (0,0)

$\therefore\ 0 + 0 = c \quad \therefore c = 0$

$$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0 \quad \text{— ①}$$

2

ii. From ① $\frac{yy_1}{b^2} = -\frac{xx_1}{a^2}$

$$y = -\frac{b^2 x_1 x}{a^2 y_1} \quad \text{— ②}$$

and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ — ③

Subs ② in ③

$$\frac{x^2}{a^2} + \frac{b^2 x_1^2 x^2}{a^4 y_1^2} = 1$$

$$x^2\left[\frac{1}{a^2} + \frac{b^2 x_1^2}{a^4 y_1^2}\right] = 1$$

$$x^2\left[\frac{a^2 y_1^2 + b^2 x_1^2}{a^4 y_1^2}\right] = 1 \quad \text{— ④}$$

Now, from ③ $b^2x^2 + a^2y^2 = a^2b^2$

and subs in (x_1, y_1):

$$b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$$

Now subs in ④

$$x^2\left[\frac{a^2b^2}{a^4 y_1^2}\right] = 1$$

$$x^2 = \frac{a^4 y_1^2}{a^2 b^2}$$

$$\therefore x = \frac{a^2 y_1}{ab}$$

$$\therefore x = \frac{a y_1}{b}$$

Subs in ② $y = \frac{-\cancel{b}^{\,2} x_1}{a^{\cancel{2}} \cancel{y_1}} \cdot \frac{\cancel{a} \cancel{y_1}}{\cancel{b}}$

$$= -\frac{b x_1}{a}$$

$$\therefore \left(\frac{a y_1}{b}, -\frac{b x_1}{a}\right)$$

3

iii. Use perpendicular dist. formula:

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$= \frac{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + 0}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}} \qquad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$= 1 \div \sqrt{\frac{b^4 x_1^2 + a^4 y_1^2}{a^4 b^4}}$$

$$= \frac{a^2 b^2}{\sqrt{x_1^2 b^4 + y_1^2 a^4}} \quad \text{— Ⓐ}$$

Now, length OQ:

$$\sqrt{\left(\frac{ay_1}{b} - 0\right)^2 + \left(\frac{-bx_1}{a} - 0\right)^2}$$

$$= \sqrt{\frac{a^2 y_1^2}{b^2} + \frac{b^2 x_1^2}{a^2}}$$

$$= \sqrt{\frac{a^4 y_1^2 + b^4 x_1^2}{a^2 b^2}}$$

$$= \frac{\sqrt{a^4 y_1^2 + b^4 x_1^2}}{ab}$$

Now Ⓐ $= \frac{ab}{Ⓑ}$

∴ distance is $\frac{ab}{OQ}$

4

iv. Area $\Delta OPQ = \frac{1}{2} \times d \times OQ$

1

$$= \frac{1}{2} \times \frac{ab}{\cancel{OQ}} \times \cancel{OQ}$$

$= \frac{1}{2}ab$ which is independent of P. (ie of x_1, y_1)

Question 7:

a.

i.

$\uparrow x$ $\qquad \downarrow \frac{1}{10}mv^2$, $\downarrow 10g$

$m\ddot{x} = -10m - \frac{1}{10}mv^2$

$= -\frac{1}{10}(100 + v^2)$ 1

ii. $v\frac{dv}{dx} = -\frac{1}{10}(100+v^2)$

$\frac{dv}{dx} = -\frac{(100+v^2)}{10v}$

$\frac{dx}{dv} = \frac{-10v}{100+v^2}$

$x = -5\ln(100+v^2) + c$

$x=0, \; v=U$

$\therefore 0 = -5\ln(100+U^2)+c$

$c = 5\ln(100+U^2)$

$\therefore x = 5\ln\left(\frac{100+U^2}{100+v^2}\right)$

Max height when $v=0$

$\therefore x = 5\ln\left(\frac{100+U^2}{100}\right)$ — ① 3

iii.

$\downarrow x$ $\qquad \uparrow \frac{1}{10}mv^2$, $\downarrow 10m$

$m\ddot{x} = 10m - \frac{1}{10}mv^2$

$\ddot{x} = \frac{1}{10}(100 - v^2)$ 1

iv. $v\frac{dv}{dx} = \frac{1}{10}(100-v^2)$

$\frac{dv}{dx} = \frac{100-v^2}{10v}$

$\frac{dx}{dv} = \frac{10v}{100-v^2}$

$x = -5\ln(100-v^2)+c$

Let particle fall from $x=0$ ***

$\therefore x=0, \; v=0$

$\therefore 0 = -5\ln 100 + c$

$\therefore c = 5\ln 100$

$\therefore x = 5\ln\left(\frac{100}{100-v^2}\right)$

Let $v = V$

$x = 5\ln\frac{100}{100-V^2}$ — ②

Now ① = ②

$\therefore 5\ln\left(\frac{100+U^2}{100}\right) = 5\ln\left(\frac{100}{100-V^2}\right)$

$\therefore \frac{100+U^2}{100} = \frac{100}{100-V^2}$

$100 - V^2 = \frac{10000}{100+U^2}$

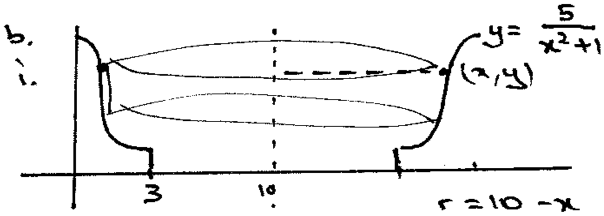
$V^2 = 100 - \frac{10\,000}{100+U^2}$

$= \frac{\cancel{10000} + 100U^2 - \cancel{10000}}{U^2+100}$

$V = \frac{10U}{\sqrt{U^2+100}}$ 4

b.

i.



$V = \int 2\pi r h \, dx$

$= 2\pi\int_0^3 (10-x)\cdot y \, dx$

$= 2\pi\int_0^3 (10-x)\cdot\frac{5}{x^2+1}\,dx$

$= \int_0^3 \frac{\pi(100-10x)}{x^2+1}\,dx$ 3

ii. $V = 100\pi\int_0^3 \frac{1}{x^2+1}\,dx - 5\pi\int_0^3 \frac{2x}{x^2+1}\,dx$

$= 100\pi \tan^{-1}x\Big]_0^3 - 5\pi\ln(x^2+1)\Big]_0^3$

$= 100\pi\tan^{-1}3 - 5\pi\ln 10$

$= 356\text{cm}^3$ 3

Question 8:

a i. $y = x^2 + kx$ —— ①

$y = \frac{1}{x}$ —— ②

① = ② $x^2 + kx = \frac{1}{x}$

$\therefore x^3 + kx^2 = 1$

$\therefore x^3 + kx^2 - 1 = 0$ 1

$\therefore \alpha, \beta, \gamma$ are roots of equation.

ii. subs $-x$ into $x^3 + kx^2 - 1 = 0$

$\therefore (-x)^3 + k(-x)^2 - 1 = 0$

$-x^3 + kx^2 - 1 = 0$

$\therefore x^3 - kx^2 + 1 = 0$ 1

iii. subs $\frac{1}{x}$ into $x^3 + kx^2 - 1 = 0$

$\therefore \left(\frac{1}{x}\right)^3 + k\left(\frac{1}{x}\right)^2 - 1 = 0$

$\frac{1}{x^3} + \frac{k}{x^2} - 1 = 0$

$1 + kx - x^3 = 0$ 1

$\therefore x^3 - kx - 1 = 0$

iv. $P(\alpha, \frac{1}{\alpha}), Q(\beta, \frac{1}{\beta}), R(\gamma, \frac{1}{\gamma})$

grad PQ : $\dfrac{\frac{1}{\alpha} - \frac{1}{\beta}}{\alpha - \beta} = \dfrac{\beta - \alpha}{\alpha\beta} \times \dfrac{1}{\alpha - \beta}$

$= \dfrac{-1}{\alpha\beta}$

Similarly, other gradients

$\dfrac{-1}{\alpha\gamma}$ and $\dfrac{-1}{\beta\gamma}$

$\therefore$ Sum: $\dfrac{-1}{\alpha\beta} - \dfrac{1}{\alpha\gamma} - \dfrac{1}{\beta\gamma}$

$= \dfrac{-\gamma - \beta - \alpha}{\alpha\beta\gamma}$

$= \dfrac{-(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$

$= \dfrac{-(-k)}{1}$

$= k$

Product $= \dfrac{-1}{\alpha\beta} \cdot \dfrac{-1}{\alpha\gamma} \cdot \dfrac{-1}{\beta\gamma}$

$= \dfrac{-1}{\alpha^2\beta^2\gamma^2}$

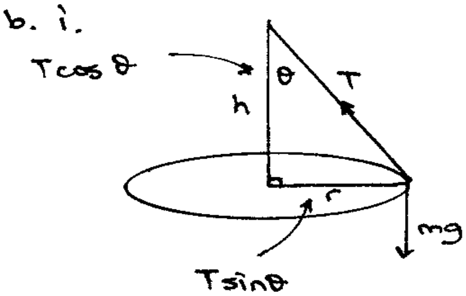
$= \dfrac{-1}{(\alpha\beta\gamma)^2}$

$= \dfrac{-1}{1}$

$= -1$

$\therefore$ sum $= k$, product $= -1$ 2

b. i.



Vertically: $\Sigma = 0$

$\therefore T\cos\theta - mg = 0$

$\therefore T\cos\theta = mg$ —— ①

Horizontally: $\Sigma = mr\omega^2$

$\therefore T\sin\theta = mr\omega^2$ —— ②

② ÷ ① $\tan\theta = \dfrac{\cancel{m}r\omega^2}{\cancel{m}g}$

$\therefore \tan\theta = \dfrac{r\omega^2}{g}$

But $\tan\theta = \dfrac{r}{h}$

$\therefore \dfrac{\cancel{r}\omega^2}{g} = \dfrac{\cancel{r}}{h}$

$\omega^2 = \dfrac{g}{h}$ 2

$\therefore \omega = \sqrt{\dfrac{g}{h}}$

ii. Period $= \dfrac{2\pi}{\omega}$

$= 2\pi \div \sqrt{\dfrac{g}{h}}$

$= 2\pi\sqrt{\dfrac{h}{g}}$ 1

iii. ω = angular velocity

$\therefore$ 2 rev/sec $= 2 \times 2\pi$

$= 4\pi$

and 1 rev/sec $= 2\pi$

From i. $\omega^2 = \dfrac{g}{h}$

$\therefore h = \dfrac{g}{\omega^2}$

$\omega = 4\pi, \quad h = \frac{g}{16\pi^2}$

$\omega = 2\pi, \quad h = \frac{g}{4\pi^2}$

$\therefore$ Difference $= \frac{g}{16\pi^2} - \frac{g}{4\pi^2}$

$= \frac{g}{16\pi^2} - \frac{4g}{16\pi^2}$

$= \frac{-3g}{16\pi^2}$

$\therefore$ lowered by $\frac{3g}{16\pi^2}$ cm 2

c. i. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ — ①

or $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta . i\sin\theta + 3\cos\theta . i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ — ②

As ① = ② and equating real & imag.

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta + 3\cos^3\theta - 3\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ 2

ii. $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$

$= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

$= \frac{\sin\theta(3 - 4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)}$

Divide top & bottom by $\cos^2\theta$

$= \tan\theta\left[\frac{\frac{3}{\cos^2\theta} - 4\tan^2\theta}{4 - \frac{3}{\cos^2\theta}}\right]$

$= \tan\theta\left[\frac{3\sec^2\theta - 4\tan^2\theta}{4 - 3\sec^2\theta}\right]$

$= \tan\theta\left[\frac{3(1+\tan^2\theta) - 4\tan^2\theta}{4 - 3(1+\tan^2\theta)}\right]$

$= \tan\theta\left[\frac{3 - \tan^2\theta}{1 - 3\tan^2\theta}\right]$

$= t\left[\frac{3-t^2}{1-3t^2}\right]$

$= \frac{t(3-t^2)}{1-3t^2}$ 3