Name: ______________________ Class: ________

WHITEBRIDGE HIGH SCHOOL

2010

Trial Higher School Certificate

MATHEMATICS EXTENSION 2

Time Allowed: 3 hours
(plus 5 minutes reading time)

Directions to Candidates

- All questions to be completed on writing paper provided
- Commence each question on a new page.
- Marks may be deducted for careless or badly arranged work.

Question 1 (15 marks) Commence each question on a SEPARATE page

a. Evaluate $\int \left(e^{x} + e^{-\frac{x}{2}} \right)^{2} dx$. **2**

b. Use the substitution $u = 1 + \sin^2 x$ to find $\int \frac{\sin 2x}{\sqrt{1 + \sin^2 x}} dx$ **2**

c. Evaluate in simplest form $\int_0^{\frac{\pi}{4}} \frac{\sec x + \tan x}{\cos x} dx$. **3**

d. Evaluate in simplest form $\int_0^4 \frac{x - 9}{(x+1)(x^2+9)} dx$. **4**

e. Use the substitution $t = \tan \frac{x}{2}$ to evaluate, in simplest exact form, **4**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3 - \cos x - 2\sin x}.$$

Question 2 (15 marks) Commence each question on a SEPARATE page

a. If $z = 3 - i$ and $w = 1 + 2i$, find in the form $a + ib$, where a and b are real, the values of

i. $z - 2w$ **1**

ii. $z\overline{w}$ **1**

iii. $\frac{z}{w}$ **1**

b. i. Using the result for tan (A – B), show that $\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$. **1**

ii. Hence, express $z = (\sqrt{3} + 1) + (\sqrt{3} - 1)i$ in modulus argument form. **2**

iii. Express z^6 in the form $a + ib$, where a and b are real. **1**

c. i. On an Argand diagram, shade the region where both $|z - 1 - i| \leq \sqrt{2}$ and $0 \leq \arg z \leq \frac{\pi}{4}$. **2**

ii. Find in simplest exact form the area of the shaded region. **2**

d. i. If $y = \log_e(\cos\theta + i\sin\theta)$, show that $\frac{dy}{d\theta} = i$. **1**

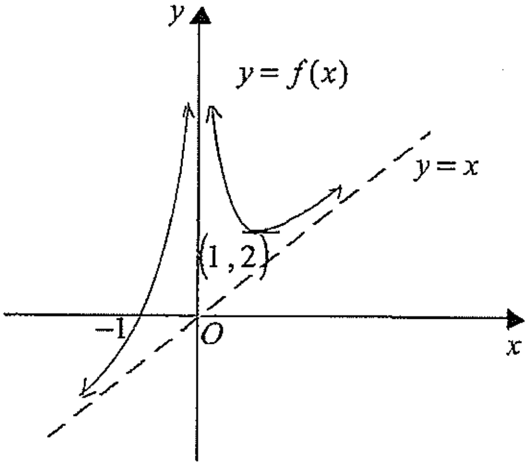
ii. Hence, by integration, show that $e^{i\theta} = \cos\theta + i\sin\theta$. **2**

iii. If $z = e^{i\theta}$, show that $z^4 + \frac{1}{z^4} = 2\cos 4\theta$. **1**

Question 3 (15 marks) Commence each question on a SEPARATE page

a. The polynomial $P(x) = x^3 - 6x^2 + 9x + c$ has a double zero. **3**
Find any possible values of the real number c.

b. The graph below shows the curve $y = f(x)$ with asymptotes $x = 0$ and $y = x$.



On separate diagrams, sketch the following graphs showing clearly any intercepts and asymptotes:

i. $y = |f(x)|$. **1**

ii. $y = f(|x|)$. **1**

iii. $y = f(x) - x$. **2**

iv. $y = \frac{1}{f(x)}$. **2**

c. $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$ and the equation $P(x) = 0$ has roots α, β, γ and δ.

i. Show that the equation $P(x) = 0$ has no integer roots. **1**

ii. Show that $P(x) = 0$ has a real root between 0 and 1. **1**

iii. Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$. **2**

iv. Hence find the number of real roots of the equation $P(x) = 0$, giving reasons. **2**

Question 4 (15 marks) Commence each question on a SEPARATE page

a. For the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, find

i. the eccentricity. **1**

ii. the coordinates of the foci. **1**

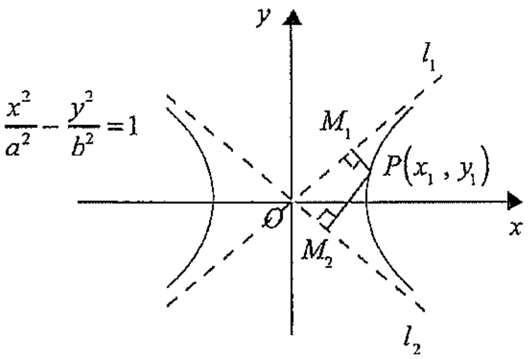
iii. the equations of the directrices. **1**

b. For the curve $y^3 + 2xy + x^2 + 2 = 0$,

i. show that $\frac{dy}{dx} = \frac{-2(y+x)}{3y^2+2x}$. **2**

ii. find the coordinates of any stationary points on the curve. **3**

c.



$P(x_1, y_1)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$,

with asymptotes l_1 and l_2.
M_1 and M_2 are the feet of the perpendiculars from P to l_1 and l_2 respectively.

i. Show that $PM_1 \times PM_2 = \frac{a^2b^2}{a^2+b^2}$. **3**

ii. Show that $\tan \angle M_1OM_2 = \frac{2ab}{a^2-b^2}$. **1**

iii. Hence find the area of ΔPM_1M_2 in terms of a and b. **3**

Question 5 (15 marks) Commence each question on a SEPARATE page

a. Let $I_m = \int x^m e^x \, dx$.

i. Show that $I_m = x^m e^x - mI_{m-1}$. **2**

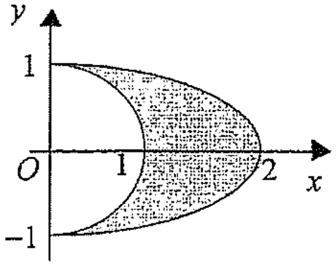
ii. Find the value of $\int_1^2 x^2 e^x \, dx$. **2**

b. A torus is generated by revolving $x^2 + y^2 \leq 4$ about the line $x = 5$.

i. By using the method of cylindrical shells show that the volume of one shell is given by $\Delta V = 4\pi(5 - x)\sqrt{4 - x^2}\,\Delta x$. **2**

ii. Hence find the volume of the torus. **2**

c. The base of a solid is the shaded region between the circle $x^2 + y^2 = 1$ and the ellipse $\frac{x^2}{4} + y^2 = 1$ for $x \geq 0$. Vertical cross-sections taken parallel to the x- axis are rectangles with heights equal to the squares of their bases.



i. Show that the volume V of the solid is given by $V = \int_{-1}^{1} \left(1 - y^2\right)^{\frac{3}{2}} \, dy$. **2**

ii. It can be shown that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. (Do NOT prove this).

Use the substitution $y = \sin u$ and the result from ii. to find the value of V. **3**

d. Consider the curve defined by the parametric equations $x = t^2 + t - 1$and $y = te^{2t}$. **2**

Show that $\frac{dy}{dx} = e^{2t}$.

Question 6 (15 marks) Commence each question on a SEPARATE page

a. Solve for x: $\tan^{-1} x + \tan^{-1} (1 - x) = \tan^{-1} \frac{9}{7}$. **4**

b. Consider the function $f(x) = \log_e(1 + \cos x)$, $-2\pi \leq x \leq 2\pi$, where $x \neq \pi$, $x \neq -\pi$.

Show that the function $f(x)$ is even and the curve $y = f(x)$ is concave down for all values of x in the domain. **3**

c. A particle of mass m kg falls from rest in a medium where the resistance to motion is proportional to the square of its speed and its terminal velocity is 20 ms^{-1}.
The value of g, the acceleration due to gravity is 10 ms^{-1}.
At time t seconds the particle has fallen x metres and acquired a velocity v ms^{-1}.

i. Explain why $\ddot{x} = \frac{1}{40}(400 - v^2)$. **2**

ii. Find t as a function of v by integration. **2**

iii. Hence, show $\frac{1}{40}\text{v} = \dfrac{\frac{1}{2}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)}{\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}}\right)}$. **2**

iv. Find x as a function of t. **2**

Question 7 (12 marks) Commence each question on a SEPARATE page

a. The roots of $x^3 - 7x + 6 = 0$ are α, β and γ. **2**

Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

b. Use mathematical induction to show that, for $n \geq 2$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + .. + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$ **4**

c. In a kitchen, the room temperature is 20°C.

Alison makes some coffee, pours a cup and adds milk.

The temperature of the coffee at this point is 80°C.

After p minutes, during which she answers the phone call, the temperature of the coffee has fallen to 35°C.

Then she is delayed for a further 4 minutes by the doorbell, after which the temperature is 27.5°C.

Assuming Newton's law of cooling which states that $T = A + Be^{-kt}$ where T is the temperature in °C, t is the time in minutes, A and B are constants,

i. find the values of A and B. **2**

ii. find p, the time that Alison spent on the phone. **3**

Question 8 (7 marks) Commence each question on a SEPARATE page

A car of mass m kg, with speed v metres/second travels around a circular track of radius R metres, inclined at an angle θ to the horizontal and g is the acceleration due to gravity.

i. Write down the vertical force and horizontal force equations. **1**

ii. Show that if there is no tendancy for the car to slip then $\tan\theta = \dfrac{v^2}{gR}$. **2**

iii. Express $\sin\theta$ and $\cos\theta$ in terms of v, g and R. **1**

iv. If the speed of the car is now halved, prove that the sideways frictional force F on the wheels exerted on the track is given by $F = \dfrac{3mgv^2}{4\sqrt{v^4 + g^2R^2}}$. **3**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1:

a. $\int (e^{x} + e^{-\frac{x}{2}})^{2}\,dx$

$= e^{2x} + 2e^{\frac{x}{2}} + e^{-x}\,dx$

$= \frac{1}{2}e^{2x} + 4e^{\frac{x}{2}} - e^{-x} + c$ 2

b. $u = 1 + \sin^{2}x$

$\frac{du}{dx} = 2\sin x\cos x$

$= \sin 2x$

$\therefore dx = \frac{du}{\sin 2x}$

$\therefore \int \frac{\sin 2x}{\sqrt{1+\sin^{2}x}}\,dx$

$= \int \frac{\cancel{\sin 2x}}{\sqrt{u}} \cdot \frac{du}{\cancel{\sin 2x}}$

$= \int u^{-\frac{1}{2}}\,du$

$= 2u^{\frac{1}{2}} + c$

$= 2\sqrt{1+\sin^{2}x} + c$ 2

c. $\int_0^{\frac{\pi}{4}} \frac{\sec x + \tan x}{\cos x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^{2}x} + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\,dx$

$= \int_0^{\frac{\pi}{4}} \sec^{2}x + \underbrace{\sec x \tan x}_{\uparrow\text{ stand. int.}}\,dx$

$= \tan^{2}x + \sec x \Big]_0^{\frac{\pi}{4}}$

$= 1 + \sqrt{2} - (0 + 1)$

$= \sqrt{2}$ 3

d. $\int_0^{4} \frac{x-9}{(x+1)(x^{2}+9)}\,dx$

$\therefore \frac{a}{x+1} + \frac{bx+c}{x^{2}+9}$

$= \frac{a(x^{2}+9) + (x+1)(bx+c)}{(x+1)(x^{2}+9)}$

$\therefore x - 9 = a(x^{2}+9) + (x+1)(bx+c)$

let $x = -1 \therefore -10 = 10a \quad \therefore a = -1$

$x = 0 \therefore -9 = 9a + c$

$-9 = -9 + c$

$c = 0$

$x = 1 \therefore -8 = 10a + 2(b+c)$

$-8 = -10 + 2b$

$2 = 2b$

$b = 1$

$\therefore \int_0^{4} \frac{-1}{x+1} + \frac{x}{x^{2}+9}\,dx$

$= -\ln(x+1) + \frac{1}{2}\ln(x^{2}+9)\Big]_0^{4}$

$= -\ln 5 + \frac{1}{2}\ln 25 - \left[0 + \frac{1}{2}\ln 9\right]$

$= -\ln \cancel{5} + \ln \cancel{5} - \ln 3$

$= -\ln 3$ 4

e. $t = \tan\frac{x}{2} \quad \therefore \sin x = \frac{2t}{1+t^{2}}$

$\cos x = \frac{1-t^{2}}{1+t^{2}}$

$\frac{dt}{dx} = \frac{1}{2}\sec^{2}\frac{x}{2}$

$= \frac{1+\tan^{2}\frac{x}{2}}{2}$

$= \frac{1+t^{2}}{2}$

$\therefore dx = \frac{2\,dt}{1+t^{2}}$

$x = \frac{\pi}{2} \therefore t = 1$

$x = 0 \therefore t = 0$

$\therefore \int_0^{1} \frac{1}{3 - \frac{1-t^{2}}{1+t^{2}} - \frac{4t}{1+t^{2}}} \cdot \frac{2\,dt}{1+t^{2}}$

$= \int_0^{1} \frac{2\,dt}{3 + 3t^{2} - (1-t^{2}) - 4t}$

$= \int_0^{1} \frac{2\,dt}{4t^{2} - 4t + 2}$

$= \int_0^{1} \frac{dt}{2t^{2} - 2t + 1}$

$= \int_0^{1} \frac{dt}{1 + 2(t^{2} - t + \frac{1}{4}) - \frac{1}{2}}$

$= \int_0^{1} \frac{dt}{\frac{1}{2} + 2(t - \frac{1}{2})^{2}}$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\frac{1}{4} + (t-\frac{1}{2})^2}$$

$$= \frac{1}{2}\cdot 2 \tan^{-1}\left[\frac{t-\frac{1}{2}}{\frac{1}{2}}\right]\Big]_0^1$$

$$= \tan^{-1}\left[2(t-\tfrac{1}{2})\right]\Big]_0^1$$

$$= \tan^{-1}(2t-1)\Big]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1}(-1)$$

$$= \tan^{-1} 1 + \tan^{-1} 1$$

$$= 2\tan^{-1} 1$$

$$= \frac{\pi}{2}$$ 4

Question 2:

a. i. $z - 2w = 3 - i - 2(1+2i)$

$= 3 - i - 2 - 4i$

$= 1 - 5i$ 1

ii. $z\bar{w} = (3-i)(1-2i)$

$= 3 - 7i - 2$

$= 1 - 7i$ 1

iii. $\frac{z}{w} = \frac{3-i}{1+2i} \cdot \frac{1-2i}{1-2i}$

$= \frac{3-7i-2}{1+4}$

$= \frac{1-7i}{5}$ 1

b. i. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}.\tan\frac{\pi}{4}}$

$= \frac{\sqrt{3}-1}{1+\sqrt{3}.1}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$ 1

ii. $|z| = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$

$= \sqrt{3 + \cancel{2\sqrt{3}} + 1 + 3 - \cancel{2\sqrt{3}} + 1}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\arg z = \tan^{-1}\left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right] = \frac{\pi}{12}$ (from ii)

$\therefore |z| = 2\sqrt{2}$, $\arg z = \frac{\pi}{12}$

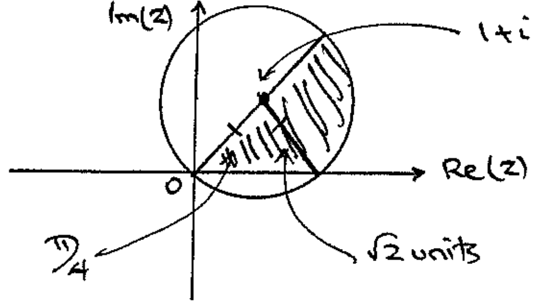
$\therefore z = 2\sqrt{2} \text{ cis } \frac{\pi}{12}$ 2

iii. $z^6 = [2\sqrt{2}]^6 \text{ cis } \frac{6\pi}{12}$

$= 64.8 \text{ cis } \frac{\pi}{2}$

$= 512 \text{ cis } \frac{\pi}{2} \longrightarrow 0 + 512i$ 1

c.



2

Area = Area of Δ + Area of Sector

$= \frac{1}{2} \times \sqrt{2} \times \sqrt{2} + \frac{1}{4}.\pi.(\sqrt{2})^2$

$= 1 + \frac{\pi}{2}$

$\therefore$ area is $\left(1 + \frac{\pi}{2}\right)u^2$ 2

d. i. $y = \log_e(\cos\theta + i\sin\theta)$

$\frac{dy}{d\theta} = \frac{-\sin\theta + i\cos\theta}{\cos\theta + i\sin\theta}$

$= \frac{i(\cancel{\cos\theta + i\sin\theta})}{\cancel{\cos\theta + i\sin\theta}} = i$ 1

ii. From i.

$y = \int i \, d\theta = i\theta + c$

But as $\theta = 0$, then

$y = \log_e(\overset{1}{\cancel{\cos}}0 + i\overset{0}{\cancel{\sin}}0)$

$= \log_e 1$

$= 0$

$\therefore y = 0, \theta = 0 \therefore c = 0$

$\therefore y = i\theta$

$\therefore i\theta = \log_e(\cos\theta + i\sin\theta)$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$ 2

iii. $z^4 + z^{-4} = e^{4i\theta} + e^{-4i\theta}$

$= (\cos 4\theta + i\sin 4\theta) + (\cos(-4\theta) + i\sin(-4\theta))$

$= \cos 4\theta + i\sin \cancel{4\theta} + \cos 4\theta - i\sin \cancel{4\theta}$

$= 2\cos 4\theta$ 1

Question 3:

a. $P(x) = x^3 - 6x^2 + 9x + c$

$P'(x) = 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, 1$

$\therefore P(3) = 3^3 - 6(3)^2 + 9(3) + c = 0$

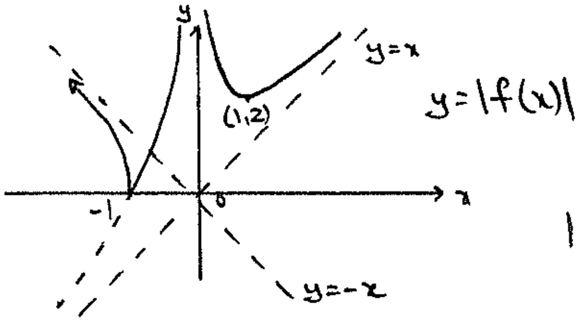
$\therefore c = 0$

$P(1) = 1^3 - 6(1)^2 + 9(1) + c = 0$

$c = -4$

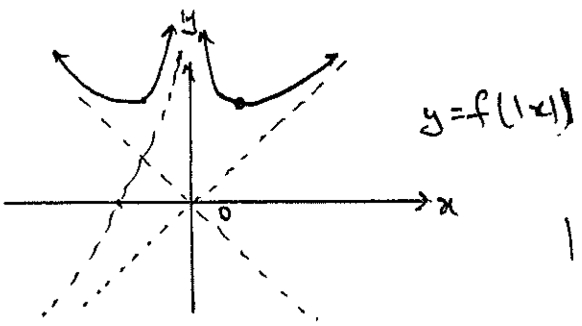
$\therefore c = 0 \text{ or } -4$ 3

b. i.



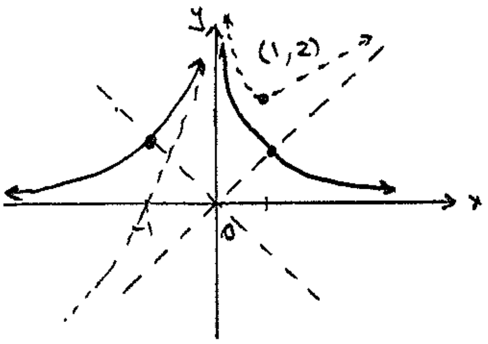
1

ii.



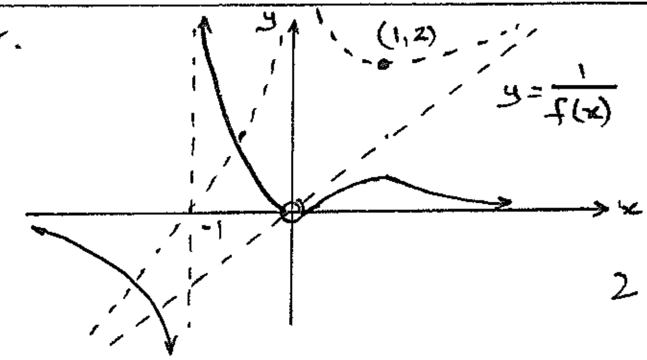
1

iii.



2

iv.



2

b. $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$

i. only poss. roots are ± 1 1

$P(1) \neq 0$, $P(-1) \neq 0$ $\therefore$ no integer roots

ii. $P(0) = 1$

$P(1) = \cancel{1} - \cancel{2} + 3 - 4 + \cancel{1} = -1$ 1

$\therefore$ root lies $0 < x < 1$

iii. As $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= 2^2 - 2(3)$

$= 4 - 6$ 2

$= -2$

iv. As sum of squares of roots < 0, then at least one root is imaginary. But as co.eff of $P(x)$ are real, and complex roots are in conjugate pairs, then at least 2 roots are complex. But from ii, real root 2 between 1 & 2 $\therefore$ 2 real roots exist.

Question 4:

a. i. For ellipse, $b^2 = a^2(1 - e^2)$

$a^2 = 8$, $b^2 = 4$

$\therefore 4 = 8(1 - e^2)$

$1 - e^2 = \frac{1}{2}$

$e^2 = \frac{1}{2}$ $\therefore e = \frac{1}{\sqrt{2}}$ 1

ii. foci : $(\pm ae, 0)$

$= (\pm 2\sqrt{2} \cdot \frac{1}{\sqrt{2}}, 0)$ 1

$= (\pm 2, 0)$

iii. direct: $x = \pm \frac{a}{e}$

$= \pm 2\sqrt{2} \div \frac{1}{\sqrt{2}}$

$\therefore x = \pm 4$ 1

b. $y^3 + 2xy + x^2 + 2 = 0$ ——— ①

i. $3y^2 \frac{dy}{dx} + 2\{ y + x.\frac{dy}{dx}] + 2x = 0$

$3y^2 \frac{dy}{dx} + 2y + 2x\frac{dy}{dx} + 2x = 0$

$(3y^2 + 2x)\frac{dy}{dx} = -2y - 2x$

$\therefore \frac{dy}{dx} = \frac{-2(x+y)}{3y^2+2x}$ 2

ii. $\frac{dy}{dx} = 0 \quad \therefore -2(x+y) = 0$

$x + y = 0$

$y = -x$ ——— ②

Subs in ① $-x^3 - 2x^2 + x^2 + 2 = 0$

$x^3 + x^2 - 2 = 0$

Let $P(x) = x^3 + x^2 - 2$

$P(1) = 0 \quad \therefore x - 1$ is factor

$$
\begin{array}{rl}
 & x^2+2x+2 \\
x-1 \,\big) & \overline{x^3+x^2+0x-2} \\
 & \underline{x^3-x^2} \\
 & 2x^2+0x \\
 & \underline{2x^2-2x} \\
 & 2x-2 \\
 & \underline{2x-2} \\
 & 0
\end{array}
$$

$\therefore \frac{dy}{dx} = (x-1)(x^2+2x+2)$

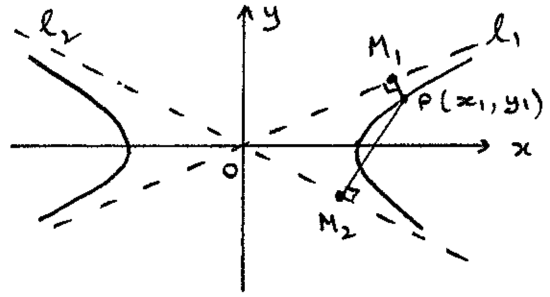
No soln as $\Delta < 0$

$\therefore x = 1$

$y(1) = -1 \quad \therefore$ Stat pt $(1,-1)$ 3

c.

i.



for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

asymptotes are $y = \pm \frac{b}{a}x$

ie $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$

$\therefore l_1: ay = bx$ ie $bx - ay = 0$

$l_2: ay = -bx$ ie $bx + ay = 0$

Using ⊥ar dist. formula:

$PM_1 \times PM_2 = \left| \frac{bx_1 - ay_1}{\sqrt{b^2+a^2}} \right| . \left| \frac{bx_1 + ay_1}{\sqrt{b^2+a^2}} \right|$

$= \frac{|b^2x_1^2 - a^2y_1^2|}{b^2+a^2}$ ——— ①

But $\frac{x_1^2}{a^2} - \frac{y^2}{b^2} = 1$

ie $b^2x_1^2 - a^2y_1^2 = a^2b^2$ ——— ②

Subs ② into ①:

$PM_1 \times PM_2 = \frac{a^2b^2}{b^2+a^2}$ 3

ii. Using $\tan\theta = \left| \frac{M_1 - M_2}{1 + M_1M_2} \right|$

and $M_1 = \frac{b}{a}$, $m_2 = -\frac{b}{a}$

$\therefore \tan\theta = \left| \frac{\frac{b}{a} + \frac{b}{a}}{1 + \frac{b}{a} . -\frac{b}{a}} \right|$

$= \left| \frac{2b}{a} \div \left(1 - \frac{b^2}{a^2}\right) \right|$

$= \left| \frac{2b}{a} \div \left(\frac{a^2-b^2}{a^2}\right) \right|$

$= \left| \frac{2b}{\cancel{a}} \times \frac{a^{\cancel{2}}}{a^2-b^2} \right|$ 1

$\therefore \tan \angle M_1OM_2 = \frac{2ab}{a^2-b^2}$

iii. Now, as $\angle OM_1P = \angle OM_2P = 90°$,

$\therefore OM_1PM_2$ is cyclic quad.

$\therefore \angle M_1OM_2$ and $\angle M_1PM_2$ are supp.

$\therefore \angle M_1PM_2 = 180 - \angle M_1OM_2$

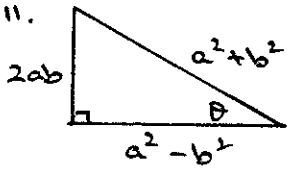
Now, area of $\Delta = \frac{1}{2} ab \sin C$

$= \frac{1}{2} . PM_1 \times PM_2 \times \sin(180° - \angle M_1OM_2)$

$= \frac{1}{2} . \frac{a^2b^2}{a^2+b^2} . \sin \angle M_1OM_2$

(as $\sin(180-A) = \sin A$)

From ii.



$\therefore \sin\theta = \frac{2ab}{a^2+b^2}$

$\therefore$ Area $= \frac{1}{\cancel{2}} . \frac{a^2b^2}{a^2+b^2} . \frac{\cancel{2}ab}{a^2+b^2}$

$= \frac{a^3b^3}{(a^2+b^2)^2} \, u^2$ 3

Question 5:

a. i. $I_m = \int x^m e^x dx$

let $u = x^m$ $v' = e^x$

$u' = mx^{m-1}$ $v = e^x$

$\therefore I_m = x^m e^x - \int mx^{m-1} . e^x dx$

$= x^m e^x - m I_{m-1}$ 2

ii. $\int_1^2 x^2 e^x dx$

$= x^2 e^x \Big]_1^2 - 2 I_1$

$= 4e^2 - e - 2\left[\int_1^2 x e^x dx\right]$

$= 4e^2 - e - 2\left[xe^x\Big]_1^2 - I_0\right]$

$= 4e^2 - e - 2.(2e^2 - e) + 2\int_1^2 x \, dx$

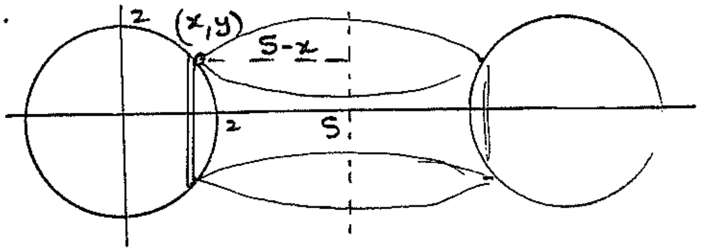
$= 4e^2 - e - 2(2e^2 - e) + 2\int_1^2 e^x dx$

$= 4e^2 - e - 4e^2 + 2e + 2(e^2 - e^1)$

$= e + 2e^2 - 2e$

$= 2e^2 - e$ 2

b.



$x^2 + y^2 = 4$

$\therefore y = \pm\sqrt{4 - x^2}$

radius of $5 - x$

$\Delta V = 2\pi$ (radius)(height)

$= 2\pi(5 - x) . 2\sqrt{4 - x^2} . \Delta x$

$\therefore \Delta V = 4\pi(5 - x)\sqrt{4 - x^2} . \Delta x$ 2

ii. $\int_{-2}^{2} 4\pi(5 - x)\sqrt{4 - x^2} dx$

$= 4\pi \int_{-2}^{2} 5\sqrt{4 - x^2} - x\sqrt{4 - x^2} dx$

semi-circle

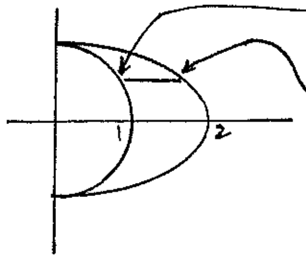
odd function $\therefore \int = 0$

$= 20\pi . \frac{1}{2} . \pi . 2^2$

$= 40\pi^2$ $\therefore$ volume is $40\pi^2 u^3$ 2

c.

i.



$x^2 = 1 - y^2$

$\therefore x = \sqrt{1 - y^2}$

$\frac{x^2}{4} = 1 - y^2$

$\therefore \frac{x}{2} = \sqrt{1 - y^2}$

$x = 2\sqrt{1 - y^2}$

$\therefore$ length: $2\sqrt{1-y^2} - \sqrt{1-y^2}$

$= \sqrt{1-y^2}$

Also, height $= (\sqrt{1-y^2})^2$

$= 1 - y^2$

$\therefore$ Area of rectangle

$= (1-y^2)^{\frac{1}{2}} . (1-y^2)$

$= (1-y^2)^{\frac{3}{2}}$

$\therefore V = \int_{-1}^{1} (1-y^2)^{\frac{3}{2}} dy$ 2

ii. let $y = \sin u$

$\therefore \frac{dy}{du} = \cos u$ $\therefore dy = \cos u \, du$

$v = 2\int_0^1 (1-y^2)^{\frac{3}{2}}\,dy$ (as f'n is even)

$= 2\int_0^{\pi/2} (1-\sin^2 u)^{\frac{3}{2}} .\ \cos u \,.\, du$

$= 2\int_0^{\pi/2} \cos^3 u \,.\, \cos u \ du$

$= 2\int_0^{\pi/2} \cos^4 u \ du$

$= \frac{2}{8}\int_0^{\pi/2} \cos 4u + 4\cos 2u + 3 \ du$

$= \frac{1}{4}\left[\frac{1}{4}\sin 4u + 2\sin 2u + 3u\right]_0^{\pi/2}$

$= \frac{1}{4}\left\{\frac{1}{4}\cancel{\sin 2\pi} + 2\cancel{\sin\pi} + 3\pi/2 - (0)\right]$

$= \frac{3\pi}{8}$ 3

d. $\frac{dx}{dt} = 2t+1$

$\frac{dy}{dt} = 1.e^{2t} + t.\,2e^{2t}$

$= e^{2t}(1+2t)$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$= e^{2t}(1+\cancel{2t})\cdot\frac{1}{1+\cancel{2t}}$ 2

$= e^{2t}$

Question 6:

a. Let $\alpha = \tan^{-1}x$ $\therefore \tan\alpha = x$

$\beta = \tan^{-1}(1-x)$ $\therefore \tan\beta = 1-x$

$\therefore \alpha+\beta = \tan^{-1}\frac{9}{7}$

$\therefore \tan(\alpha+\beta) = \frac{9}{7}$

$\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{9}{7}$

$\frac{\cancel{x}+1-\cancel{x}}{1-x(1-x)} = \frac{9}{7}$

$7 = 9\left[1-x+x^2\right]$

$7 = 9-9x+9x^2$

$9x^2-9x+2=0$

$(3x-1)(3x-2)=0$

$x = \frac{1}{3},\ \frac{2}{3}$ 4

b. i. $f(-x) = \log_e(1+\cos(-x))$

$= \log_e(1+\cos x) = f(x)$

$\therefore$ even

$f'(x) = \frac{-\sin x}{1+\cos x}$

$f''(x) = \frac{(1+\cos x).-\cos x + \sin x.-\sin x}{(1+\cos x)^2}$

$= \frac{-\cos x - \cos^2 x - \sin^2 x}{(1+\cos x)^2}$

$= \frac{-\cos x - (\overset{1}{\cancel{\sin^2 x + \cos^2 x}})}{(1+\cos x)^2}$

$= \frac{-(\cancel{1+\cos x})}{(1+\cos x)^{\cancel{2}}}$

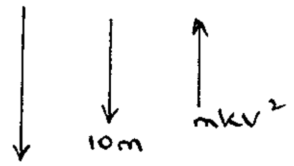
$= \frac{-1}{1+\cos x}$

But $-1 \le \cos x \le 1$, and as

$x \ne \pi,\ x \ne -\pi$ $\therefore \cos x \ne -1$ 3

$\therefore f''(x) < 0$ for all x in domain.

c.



$m\ddot{x} = 10m - mkv^2$

$\ddot{x} = 10 - kv^2$

Terminal velocity is $\ddot{x} = 0$

$\therefore 10 - kv^2 = 0$

But $v = 20$ $\therefore 10 - 400k = 0$

$k = \frac{1}{40}$

$\therefore \ddot{x} = 10 - \frac{v^2}{40}$

$= \frac{1}{40}(400 - v^2)$ 2

ii. $\frac{dv}{dt} = \frac{1}{40}(400-v^2)$

$\frac{dt}{dv} = \frac{40}{400-v^2}$

$= \frac{40}{(20-v)(20+v)}$

$\therefore \frac{a}{20-v}+\frac{b}{20+v}=\frac{a(20+v)+b(20-v)}{(20-v)(20+v)}$

$\therefore a(20+v)+b(20-v)=40$

Let $v=20 \therefore 40a=40 \therefore a=1$

$v=-20 \therefore 40b=40 \therefore b=1$

$\therefore \frac{dt}{dv}=\frac{1}{20-v}+\frac{1}{20+v}$

$\therefore t=-\ln(20-v)+\ln(20+v)+c$

$\therefore t=\ln\left(\frac{20+v}{20-v}\right)+c$ 2

Now, $t=0, v=0 \therefore c=0$

$\therefore t=\ln\left(\frac{20+v}{20-v}\right)$

iii. $\therefore e^{t}=\frac{20+v}{20-v}$

$(20-v)e^{t}=20+v$

$20e^{t}-ve^{t}=20+v$

$20e^{t}-20=v(1+e^{t})$

$\therefore 20(e^{t}-1)=v(e^{t}+1)$

$20e^{\frac{1}{2}t}(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})=ve^{\frac{1}{2}t}(e^{\frac{1}{2}t}+e^{-\frac{1}{2}t})$

$\therefore v=\frac{20(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})}{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}$

$\therefore \frac{1}{40}v=\frac{\frac{1}{2}(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})}{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}$ 2

iv. $\frac{1}{40}\frac{dx}{dt}=\frac{\frac{1}{2}(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})}{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}$

$\frac{1}{40}\cdot x=\ln(e^{\frac{1}{2}t}+e^{-\frac{1}{2}t})+c$

$t=0, x=0$

$\therefore 0=\ln(e^{0}+e^{0})+c$

$\therefore c=-\frac{1}{2}\ln 2$

$\therefore \frac{1}{40}x=\ln(e^{\frac{1}{2}t}+e^{-\frac{1}{2}t})-\frac{1}{2}\ln 2$

$\frac{1}{40}x=\ln\left[\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{2}\right]$

$\therefore x=40\ln\left[\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{2}\right]$ 2

Question 7:

a. If α is root,

then $\alpha^3-7\alpha+6=0$

ie $\alpha^3=7\alpha-6$ — ①

Similarly, $\beta^3=7\beta-6$ — ②

$\gamma^3=7\gamma-6$ — ③

①+②+③:

$\alpha^3+\beta^3+\gamma^3=7\alpha-6+7\beta-6+7\gamma-6$

$=7(\alpha+\beta+\gamma)-18$

$=7(0)-18$ 2

$=-18$ ↖ as $\alpha+\beta+\gamma=0$

b. Step 1: Prove true for $n=2$

$\therefore \text{LHS}=\frac{1}{1^2}+\frac{1}{2^2}$

$=1+\frac{1}{4}$

$=1\frac{1}{4}$

$\text{RHS}=2-\frac{1}{2}$

$=1\frac{1}{2}$

as $1\frac{1}{4}\le 1\frac{1}{2} \therefore \text{LHS}\le\text{RHS}$

$\therefore$ true for $n=2$

Step 2: Assume true for $n=k$

$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}\le 2-\frac{1}{k}$

Now, prove true for $n=k+1$

$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\le 2-\frac{1}{k+1}$

$\text{LHS}=\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$

$\le 2-\frac{1}{k}+\frac{1}{(k+1)^2}$

$=2-\left[\frac{1}{k}-\frac{1}{(k+1)^2}\right]$

$=2-\left[\frac{(k+1)^2-1}{k(k+1)^2}\right]$

$=2-\left[\frac{k^2+2k+1-1}{k(k+1)^2}\right]$

$$= 2 - \frac{1}{k+1}\left[\frac{k^2+2k}{k(k+1)}\right]$$

$$= 2 - \frac{1}{k+1}\left[\frac{\cancel{k}(k+2)}{\cancel{k}(k+1)}\right]$$

$$\leq 2 - \frac{1}{k+1}, \text{ as } \frac{k+2}{k+1} > 1$$

$\therefore$ true for $n = k+1$

Step 3: As true for $n=1$, then true for $n = 2, 3, \ldots$ for $n \geqslant 2$ 4

b. i. $T = A + Be^{-kt}$

When $t = -\infty$, $T = 20$

$\therefore 20 = A + B/0 \quad \therefore A = 20$

$\therefore T = 20 + Be^{-kt}$

When $t = 0$, $T = 80$

$\therefore 80 = 20 + Be^{\cancel{0}} \quad \therefore B = 60$ 2

$\therefore T = 20 + 60e^{-kt} \quad \therefore A = 20, B = 60$

ii. When $t = p$, $T = 35$

$35 = 20 + 60e^{-kp}$

$60e^{-kp} = 15$

$e^{-kp} = 0.25$ —— ①

Now, $t = p+4$, $T = 27.5$

$\therefore 27.5 = 20 + 60e^{-k(p+4)}$

$60e^{-k(p+4)} = 7.5$

$\therefore e^{-kp-4k} = 0.125$

$\therefore \dfrac{e^{-kp}}{e^{4k}} = 0.125$ —— ②

$\therefore \dfrac{0.25}{e^{4k}} = 0.125$

$\therefore e^{4k} = 2$

$4k = \ln 2$

$k = \dfrac{\ln 2}{4}$

Subs in ① $e^{-p\left(\frac{\ln 2}{4}\right)} = 0.25$

$-p\left(\frac{\ln 2}{4}\right) = \ln 0.25$

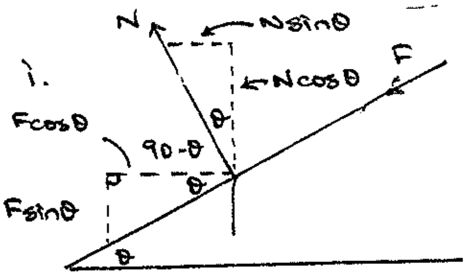
$p = -\ln 0.25 \div \left(\frac{\ln 2}{4}\right)$

$= 8$ 3

$\therefore$ Alison spent 8 min on phone.

Question 8:

i.



Vertical: $N\cos\theta - F\sin\theta - mg = 0$ —③

Horiz: $N\sin\theta + F\cos\theta = \dfrac{mv^2}{R}$ —④ 1

ii. If no slip, then $F = 0$

$\therefore N\cos\theta = mg$ —— ①

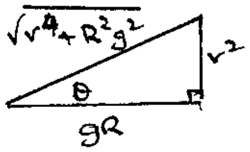
$N\sin\theta = \dfrac{mv^2}{R}$ —— ②

$\therefore$ ② $\div$ ① $\tan\theta = \dfrac{mv^2}{R} \div mg$

$= \dfrac{\cancel{m}v^2}{R} \times \dfrac{1}{\cancel{m}g}$

$= \dfrac{v^2}{Rg}$ 2

iii.



As $\tan\theta = \dfrac{v^2}{gR}$

Ⓧ $\therefore \sin\theta = \dfrac{v^2}{\sqrt{v^4+R^2g^2}} \quad \cos\theta = \dfrac{gR}{\sqrt{v^4+R^2g^2}}$ 1

iv. Now, mult. ③ by $\sin\theta$:

$N\sin\theta\cos\theta - F\sin^2\theta - mg\sin\theta = 0$ —⑤

and mult ④ by $\cos\theta$:

$N\sin\theta\cos\theta + F\cos^2\theta - \dfrac{mv^2}{R}.\cos\theta = 0$ —⑥

⑥ - ⑤ $F(\sin^2\theta \cancel{+} \cos^2\theta) + mg\sin\theta$

$$- \frac{mv^2}{R} \cdot \cos\theta = 0$$

But vel is halved ∴ subs $\frac{v}{2}$

$$\therefore F = \frac{mv^2}{4R}\cos\theta - mg\sin\theta = 0$$

Now, using ⊛

$$F = \frac{mv^2}{4\cancel{R}} \cdot \frac{g\cancel{R}}{\sqrt{v^4+R^2g^2}} - mg \cdot \frac{v^2}{\sqrt{v^4+R^2g^2}}$$

$$\therefore F = \left| \frac{mgv^2}{4\sqrt{v^4+R^2g^2}} - \frac{mgv^2}{\sqrt{v^4+R^2g^2}} \right|$$

$$= \frac{3mv^2g}{4\sqrt{v^4+g^2R^2}}$$

Note: F must be opp. direction

3