WHITEBRIDGE HIGH SCHOOL
VIAM VITAE QUAERE

Student Name: ....

WHITEBRIDGE HIGH SCHOOL

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT 4

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 to 16

Total marks – 100

Section I

10 marks

- Attempt Questions 1 to 10
- Allow about 15 minutes for this section
- Use the multiple choice answer sheet

Section II

60 marks

- Attempt Questions 11 – 14
- Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

Section I

10 Marks

Attempt Questions 1 to 10

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice by colouring in the bubble that corresponds to your answer.

Marks

1 Which of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\, dx$? **1**

(A) $x + \frac{1}{4}\cos 2x + c$

(B) $x - \frac{1}{4}\cos 2x + c$

(C) $x + \frac{1}{2}\sin 2x + c$

(D) $x - \frac{1}{2}\sin 2x + c$

2 A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line and velocity v ms^{-1} given by $v^2 = 9(5 + 4x - x^2)$. Where is the centre of motion? **1**

(A) $x = -1$

(B) $x = 0$

(C) $x = 2$

(D) $x = 5$

3 If $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$, which of the following is an expression for $\frac{dy}{dx}$? **1**

(A) $\cot^2 \frac{\theta}{2}$

(B) $\cot \frac{\theta}{2}$

(C) $\tan \frac{\theta}{2}$

(D) $\tan^2 \frac{\theta}{2}$

Marks

4 In the Argand Diagram the locus of the point P representing the complex number z such that $|z - 1 + i| = 4$ is a circle. What are the centre and radius of this circle? **1**

(A) centre $(-1, 1)$ and radius 4
(B) centre $(-1, 1)$ and radius 2
(C) centre $(1, -1)$ and radius 4
(D) centre $(1, -1)$ and radius 2

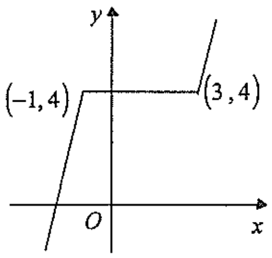
5 The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ. **1**

Which of the following equations has roots 2α, 2β and 2γ?

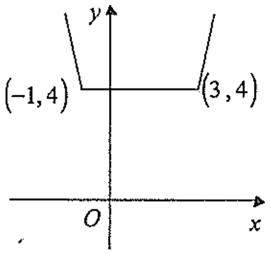
(A) $x^3 + 8x + 8 = 0$
(B) $x^3 + 16x + 8 = 0$
(C) $2x^3 + 4x + 2 = 0$
(D) $8x^3 + 4x + 1 = 0$

6 Which of the following is the graph of $y = |x + 1| + |x - 3|$? **1**

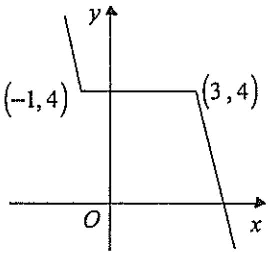
(A)



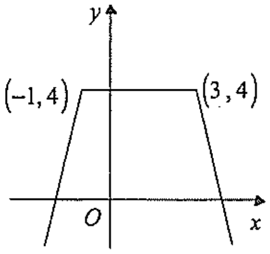
(B)



(C)



(D)



Marks

7 The normal at the point $P(cp, \frac{c}{p})$ on the rectangular hyperbola $xy = c^2$ has equation $p^3x - py = cp^4 - c$. This normal cuts the hyperbola at a second point $Q(cq, \frac{c}{q})$. What is the relationship between q and p? **1**

(A) $p^4q = -1$
(B) $p^3q = -1$
(C) $p^2q = -1$
(D) $pq = -1$

8 Which of the following is an expression for $\int x\, e^{\frac{x}{2}}\, dx$? **1**

(A) $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{4}e^{\frac{x}{2}} + c$

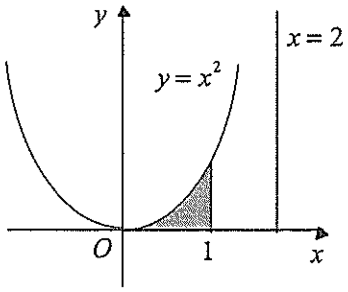
(B) $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{2}e^{\frac{x}{2}} + c$

(C) $2xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + c$

(D) $2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c$

Marks

9 **1**



The region bounded by the parabola $y = x^2$ and the x-axis between $x = 0$ and $x = 1$ is rotated through one revolution about the line $x = 2$ to form a solid of revolution about the line $x = 2$ to form a solid of volume V. Which of the following is an expression for V?

(A) $\pi \int_0^1 (1 - x)^2 \, dy$

(B) $\pi \int_0^1 (1^2 - x^2) \, dy$

(C) $\pi \int_0^1 \left\{(2 - x)^2 - 1^2\right\} dy$

(D) $\pi \int_0^1 \left\{2^2 - (2 - x)^2\right\} dy$

10 If $z = \sqrt{3} - i$, then **1**

(A) $z = \sqrt{2}\,(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$

(B) $z = \sqrt{2}\,(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$

(C) $z = 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$

(D) $z = 2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$

Section II

90 Marks
Attempt Questions 11 to 16

Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

All necessary working should be shown in each question.

Question 11 **Begin a new booklet** **Marks**

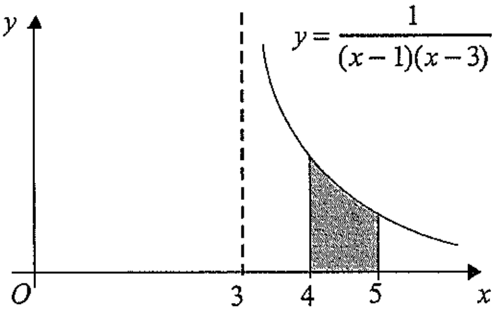
a. Evaluate $\int_0^{\frac{\pi}{4}} \cos\theta \sin^3\theta \; d\theta$. **2**

b. Find $\int \frac{\sin 2x + \sin x}{\cos^2 x}\, dx$. **2**

c. Use the substitution $u = 1 + x^2$ to evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\, dx$ in simplest exact form. **3**

d. Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos x}\, dx$ in simplest exact form. **3**

e.



The region bounded by the curve $y = \frac{1}{(x-1)(x-3)}$ and the x-axis between $x = 4$ and $x = 5$ is rotated though one revolution about the y-axis to form a solid of volume V.

(i) By considering strips of thickness δx perpendicular to the x-axis, use the method of cylindrical shells to show that $V = \pi \int_4^5 \frac{2x}{(x-1)(x-3)}\, dx$. **2**

(ii) Hence, find the value of V in simplest exact form. **3**

Question 12 **Begin a new booklet** **Marks**

a. Find the values of the real number k such that the equation $x^4 + kx^3 + x^2 + x + 1 = 0$ has an integer root. **2**

b. Find the complex number $z = a + bi$, where a and b are real, such that $2\bar{z} - iz = 1 + 4i$. **2**

c. Show that $z = \cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$ is a root of the equation $z^6 - z^3 + 1 = 0$. **2**

d. In an Argand Diagram, $ABCD$ is a quadrilateral such that vectors $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$ represent the complex numbers a, b, c, d respectively. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. M and N are the midpoints of PR and QS respectively.

(i) Show that vectors $\vec{OM}$ and $\vec{ON}$ both represent the complex number $\frac{1}{4}(a + b + c + d)$. **2**

(ii) Hence explain what type of quadrilateral $PQRS$ is. **1**

e. The equation $x^4 - kx + 1 = 0$, where k is a real number, has roots α, β, γ and δ.

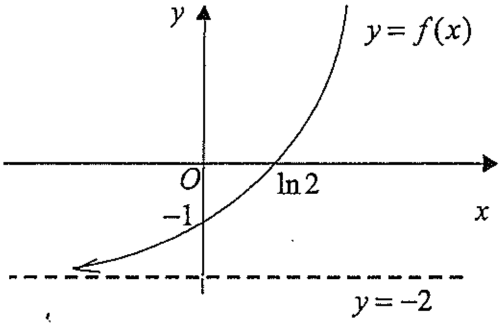
(i) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$. Hence explain why the equation has either two real and two non-real roots, or four non-real roots. **2**

(ii) If the equation has a double real root, show that its value is $3^{-\frac{1}{4}}$. **2**

(iii) Hence show that if the equation has a double real root then each of the two non-real roots has a real part $-3^{-\frac{1}{4}}$ and modulus $3^{-\frac{1}{4}}$. **2**

Question 13 **Begin a new booklet** **Marks**

a. The diagram below shows the graph of $f(x) = e^x - 2$.



On separate diagrams sketch the following graphs, in each case showing the intercepts on the axes and the equations of the asymptotes.

(i) $y = \{f(x)\}^2$. **1**

(ii) $y = \log_e f(x)$. **1**

(iii) $y = \dfrac{1}{f(x)}$. **2**

(iv) $y^2 = |f(x)|$. **2**

b. Prove that $\dfrac{d}{dx}\left[\sqrt{bx - x^2} + \dfrac{b}{2}\cos^{-1}\left(\dfrac{2x - b}{b}\right)\right] = -\sqrt{\dfrac{x}{b - x}}$, for $x \geq 0$. **3**

c. For $n = 0, 1, 2, 3, \ldots$, let $I_n = \displaystyle\int_{e^{-1}}^{1} (1 + \log_e x)^n \, dx$ and $J_n = \displaystyle\int_{e^{-1}}^{1} (\log_e x)(1 + \log_e x)^n \, dx$.

(i) Show that $I_n = 1 - nI_{n-1}$ for $n = 1, 2, 3, \ldots$ **2**

(ii) Show that $J_n = 1 - (n + 2)I_n$ for $n = 0, 1, 2, 3, \ldots$ **2**

(iii) Hence, find the value of J_3 in simplest exact form. **2**

Question 14 **Begin a new booklet** **Marks**

a. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$, has eccentricity e.

$P(a\cos\theta, b\sin\theta)$ is a point on the ellipse in the first quadrant, S is the focus of the ellipse nearer to P and $Q(a\cos\phi, b\sin\phi)$, $-\pi < \phi \leq \pi$, is a second point on the ellipse so that the normal to the ellipse at Q is parallel to the normal at P.

Point T is the intersection of the tangent at P with the normal at Q.

V lies on the tangent at P so that SV is parallel to QT.

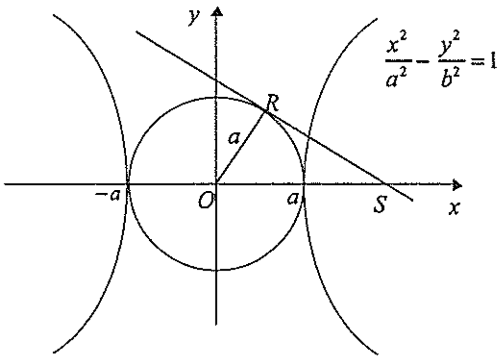
(i) Show this information on a sketch. **1**

(ii) Find the gradient of the normal at P by differentiation and deduce that $\phi = \theta - \pi$. **3**

(iii) Show that the normal at P has x-intercept $ae^2\cos\theta$. **2**

(iv) Show that $\dfrac{VP}{VT} = \dfrac{1 - e\cos\theta}{1 + e\cos\theta}$ **2**

b.



S is the focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a \neq b$, which lies on the positive x-axis.
R is a point on the auxiliary circle of the hyperbola such that R lies in the first quadrant and SR is a tangent to the auxiliary circle.
The eccentricity of the hyperbola is e.

(i) Show that R lies on a directrix of the hyperbola. **2**

(ii) Show that SR has equation $y = -\dfrac{1}{\sqrt{e^2 - 1}}(x - ae)$. **2**

(iii) If SR meets the hyperbola at the point $(a\sec\theta, b\tan\theta)$, show that **3**

$$e^2(2 - e^2)\sec^2\theta - 2e\sec\theta + \{e^2 + (e^2 - 1)^2\} = 0.$$

Question 15 **Begin a new booklet** **Marks**

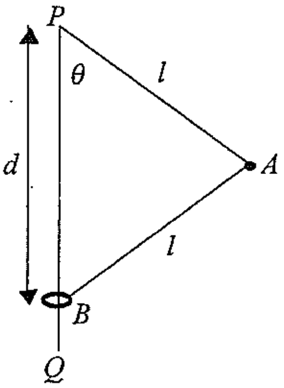
a. A bullet is fired vertically into the air with a speed of 800 ms^{-1}. In the air the bullet experiences air resistance equal to $\frac{mv}{5}$, as well as gravity g. Gravity can be assumed to equal 10 ms^{-2}.

(i) Find the height reached to the nearest metre. **3**

(ii) Find the time taken to achieve this height. **2**

(iii) As the bullet returns to the ground it is subject to the same forces. Find the terminal velocity. **2**

b.



PQ is a smooth vertical rod. Particle A of mass m is attached to point P by a string of length l and A is also attached by a second string of length l to a smooth ring B of mass M which is free to slide on the rod PQ without friction. A is set in motion in a horizontal circle about PQ with angular velocity ω. B is in equilibrium.

(i) Draw diagrams showing the forces on each of A and B, and hence show that if T_1 and T_2 are the tensions in the strings AP and AB respectively when AP makes an angle θ with the vertical, then $T_1 - T_2 = \frac{mg}{\cos\theta}$, $T_1 + T_2 = ml\omega^2$ and $T_2 = \frac{Mg}{\cos\theta}$. **4**

(ii) Hence, express the distance d of B below P in terms of g, ω^2 and $\frac{M}{m}$. **3**

(iii) Deduce that $\omega^2 \geq \frac{g}{l}\left(1 + \frac{2M}{m}\right)$. **1**

Question 16 **Begin a new booklet** **Marks**

a. Use Mathematical Induction to show that $3^n - 1 \geq 2n$ for all positive integers $n \geq 1$. **3**

b. Find the equation of the tangent to the curve $3x^2 - 2xy - y^2 - 20 = 0$ at the point (3, 1). **2**

c. With respect to the x and y axes, the line $x = 1$ is a directrix and the point (2, 0) is a focus of eccentricity $\sqrt{2}$. Find the equation of the conic, and sketch the curve indicating its asymptotes, foci and directrices. **3**

d. (i) Prove that $\cot^{-1}(2x - 1) - \cot^{-1}(2x + 1) = \tan^{-1}\left(\frac{1}{2x^2}\right)$, where $x \geq 1$. **3**

(ii) Find the sum of $K = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \ldots + \tan^{-1}\left(\frac{1}{2n^2}\right)$, where n is a positive integer. **3**

(iii) Show that $\lim\limits_{n \to +\infty} K = \frac{\pi}{4}$. **1**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Multiple Choice Answer Sheet

Student Name: ________________________________ **Class:** _______________

1. Ⓐ Ⓑ Ⓒ Ⓓ

2. Ⓐ Ⓑ Ⓒ Ⓓ

3. Ⓐ Ⓑ Ⓒ Ⓓ

4. Ⓐ Ⓑ Ⓒ Ⓓ

5. Ⓐ Ⓑ Ⓒ Ⓓ

6. Ⓐ Ⓑ Ⓒ Ⓓ

7. Ⓐ Ⓑ Ⓒ Ⓓ

8. Ⓐ Ⓑ Ⓒ Ⓓ

9. Ⓐ Ⓑ Ⓒ Ⓓ

10. Ⓐ Ⓑ Ⓒ Ⓓ

Section 1

1. $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} \, dx$

$= \int \frac{\cancel{\cos x + \sin x}(\sin^2 x - \sin x \cos x + \cos^2 x)}{\cancel{\cos x + \sin x}} \, dx$

$= \int 1 - \sin x \cos x \, dx$

$= \int 1 - \frac{1}{2} \sin 2x \, dx$

$= x + \frac{1}{4} \cos 2x + c$ $\quad \therefore A$

2. $\ddot{x} = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$

$= \frac{d}{dx}\left[\frac{9}{2}(5 + 4x - x^2)\right]$

$= \frac{9}{2} \cdot (4 - 2x)$

$= 9(2 - x)$

$= -9(x - 2)$

As $\ddot{x} = -n^2(x - c)$ then

centre of motion at $x = 2$ $\quad \therefore C$

3. $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$\frac{dy}{d\theta} = \sin\theta$

$\frac{dx}{d\theta} = 1 - \cos\theta$

$\therefore \frac{d\theta}{dx} = \frac{1}{1 - \cos\theta}$

$\therefore \frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}$

$= \frac{\sin 2(\theta/2)}{1 - \cos 2(\theta/2)}$

$= \frac{2\sin\theta/2 \cos\theta/2}{1 - [1 - 2\sin^2\theta/2]}$

$= \frac{\cancel{2\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$

$= \cot\theta/2$ $\quad \therefore B$

4. $|z - (1 - i)| = 4$

$\therefore$ centre $(1, -1)$, radius 4

$\therefore$ C

5. If $x = \alpha$ is root $\alpha^3 + 2\alpha + 1 = 0$

Now, if $x = 2\alpha$ $\therefore \alpha = \frac{x}{2}$ is soln.

$\therefore \left(\frac{x}{2}\right)^3 + 2\left(\frac{x}{2}\right) + 1 = 0$

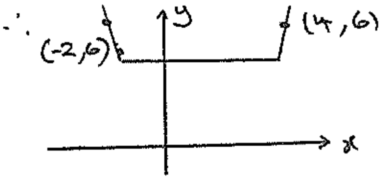
$\therefore \frac{x^3}{8} + x + 1 = 0$

$x^3 + 8x + 8 = 0$ $\quad \therefore A$

6. Subs in $x = -2$ $\therefore y = 6$

$x = 4$ $\therefore y = 6$

$\therefore$



$\therefore B$

7. Subs Q in eqn:

$p^3(cq) - p\left(\frac{c}{q}\right) = cp^4 - c$

$cp^3q - \frac{cp}{q} = cp^4 - c$

$p^3q^2 - p = p^4q - q$

$p^3q(q \overset{-1}{\cancel{-}} p) = p \cancel{-} q)$

$-p^3q = 1$

$\therefore p^3q = -1$ $\quad \therefore B$

8. $\int x e^{x/2} \, dx$. Using int by parts:

$u = x$ $\quad v' = e^{x/2}$

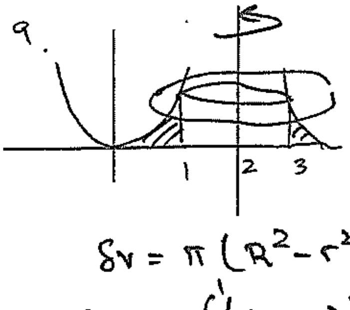
$u' = 1$ $\quad v = 2e^{x/2}$

$\therefore 2xe^{x/2} - \int 2e^{x/2} \, dx$

$= 2xe^{x/2} - 2 \cdot 2e^{x/2} + c$

$= 2xe^{x/2} - 4e^{x/2} + c$ $\quad \therefore D$

9.



$R: 2 - x$

$r = 1$

$\delta V = \pi(R^2 - r^2)\delta y$

$\therefore V = \pi\int_0^1 ((2 - x)^2 - 1^2 \, dy$

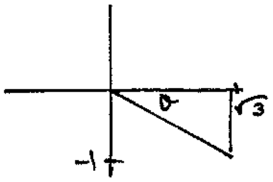
$\therefore C$

10. $z=\sqrt{3}-i$

$|z|=2$

$\arg z = \tan\left(\frac{-1}{\sqrt{3}}\right)$

$= -\pi/6$



$\therefore z=2\left(\cos(-\pi/6)+i\sin(-\pi/6)\right)$

$= 2\left(\cos\pi/6 - i\sin\pi/6\right) \quad \therefore D$

Question 11

a. $\int_0^{\pi/4} \cos\theta \sin^3\theta \, d\theta$

$= \frac{1}{4}\sin^4\theta \Big]_0^{\pi/4}$

$= \frac{1}{4}\left[(\sin\pi/4)^4 - (\sin 0)^4\right]$

$= \frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right]$

$= \frac{1}{4}\left[\frac{1}{4}\right]$

$= \frac{1}{16}$ 2

b. $\int \frac{\sin 2x + \sin x}{\cos^2 x} \, dx$

$= \int \frac{2\sin x\cos x + \sin x}{\cos^2 x} \, dx$

$= \int 2 \frac{\sin x}{\cos x} + \sec x \tan x \, dx$

$= -2\log(\cos x) + \sec x + c$ 2

c. $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}} \, dx$

$u = 1+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x=\sqrt{3} \therefore u=4$

$x=0 \therefore u=1$

$\int_1^4 \frac{x^{\cancel{3}2}}{(1+x^2)^{3/2}} \cdot \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int_1^4 \frac{u-1}{u^{3/2}} \, du$

$= \frac{1}{2}\int_1^4 u^{-\frac{1}{2}} - u^{-3/2} \, du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$

$= \left[u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right]_1^4$

$= \left(2+\frac{1}{2}\right) - (1+1)$ 3

$= \frac{1}{2}$

d. $\int_0^{\pi/2} \frac{1}{2+\cos x} \, dx$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = \frac{2 \, . \, dt}{1+t^2}$

$x=\pi/2, \; t=1$

$x=0, \; t=0$

$= \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$

$= 2\int_0^1 \frac{1}{2+2t^2+1-t^2} \, .dt$

$= 2\int_0^1 \frac{1}{3+t^2} \, dt$

$= 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}} \Big]_0^1$

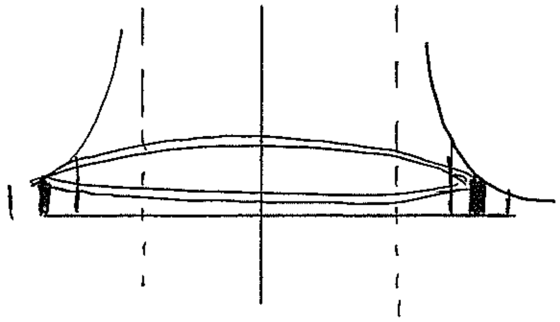
$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \cancel{\tan^{-1}} 0\right]$

$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$

$= \frac{\pi}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi\sqrt{3}}{9}$ 3

e.



$\delta V = 2\pi r h \, \delta x$

$\therefore V = 2\pi \int \text{rad. height} \, dx$

$= V = 2\pi \int_4^5 x \cdot \frac{1}{(x-1)(x-3)} \, dx$

$= 2\pi \int_4^5 \frac{x}{(x-1)(x-3)} \, dx$ 2

ii. $\frac{a}{x-1} + \frac{b}{x-3} = \frac{x}{(x-1)(x-3)}$

$\therefore a(x-3) + b(x-1) = x$

$x = 3 \quad \therefore 2b = 3 \quad \therefore b = \frac{3}{2}$

$x = 1 \quad \therefore -2a = 1 \quad \therefore a = -\frac{1}{2}$

$\therefore 2\pi \int_4^5 \frac{-\frac{1}{2}}{x-1} + \frac{\frac{3}{2}}{x-3} \, dx$

$= \pi \int_4^5 \frac{-1}{x-1} + \frac{3}{x-3} \, dx$

$= \pi \left[3\ln(x-3) - \ln(x-1) \right]_4^5$

$= \pi \left[3\ln 2 - \ln 4 - (3\cancel{\ln 1} - \ln 3) \right]$

$= \pi \left[3\ln 2 - \ln 4 + \ln 3 \right]$

$= \pi \left[\ln 8 - \ln 4 + \ln 3 \right]$

$= \pi \left[\ln 2 + \ln 3 \right]$ 3

$= \pi \ln 6$

Question 12

a. As constant term is 1 then the root (α) is a factor of 1

ie two cases:

$\alpha = 1 \quad \therefore x = 1 : 1 + k + 1 + 1 + 1 = 0$

$\therefore k = -4$

$\alpha = -1 \quad \therefore x = -1 \therefore 1 - k + 1 - \cancel{1} + \cancel{1} = 0$

$\therefore k = 2$

$\therefore k = 2, -4$ 2

b. $z = a + bi \quad \therefore \bar{z} = a - bi$

$\therefore 2(a - bi) - i(a + bi) = 1 + 4i$

$\therefore 2a + b = 1$

$-2b - a = 4$

ie $2a + b = 1$ ①

$a + 2b = -4$ ②

$2 \times$ ① $4a + 2b = 2$ ③

③ − ② $3a = 6$

$a = 2$

Subs in ① $A + b = 1$

$b = -3$

2

$\therefore 2 - 3i$

c. $z = \cos \pi/9 + i \sin \pi/9$

$z^3 = \cos \pi/3 + i \sin \pi/3$

$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$z^6 = \cos 2\pi/3 + i \sin 2\pi/3$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

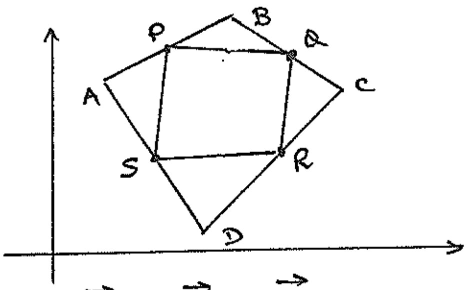
$\therefore z^6 - z^3 + 1 = -\cancel{\frac{1}{2}} + i\cancel{\frac{\sqrt{3}}{2}} - \left(\cancel{\frac{1}{2}} + i\cancel{\frac{\sqrt{3}}{2}}\right) + 1$

$= -1 + 0 + 1$

$= 0$ 2

$\therefore z^6 - z^3 + 1 = 0$

d.
i.



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$\therefore \overrightarrow{OP} = a + \frac{1}{2}(b - a)$

$= a + \frac{1}{2}b - \frac{1}{2}a$

$= \frac{1}{2}(a + b)$

Similarly $OR = \frac{1}{2}(c + d)$.

Now, M is midpoint of PR,

$\therefore \overrightarrow{OM} = \frac{1}{2}\left[\frac{1}{2}(a+b) + \frac{1}{2}(c+d)\right]$

$= \frac{1}{4}\left[a + b + c + d\right]$

Similarly, since N is midpoint of QS, then $\overrightarrow{ON} = \frac{1}{2}\left[\frac{1}{2}(b+c) + \frac{1}{2}(d+a)\right]$

$= \frac{1}{4}\left[a + b + c + d\right]$ 2

ii. M and N are the same point.

∴ diagonals of quadrilaterals bisect each other ∴ parallelogram 1

e. i. $x^4 - kx + 1 = 0$.

$\therefore \alpha+\beta+\gamma+\delta = -\frac{b}{a} = 0$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a} = 0$

Now, $\alpha^2+\beta^2+\gamma^2+\delta^2$

$= (\alpha+\underset{0}{\cancel{\beta+\gamma+\delta}})^2 - 2[\alpha\beta+\alpha\gamma+\underset{0}{\cancel{\alpha\delta+\beta\gamma}}+\beta\delta+\gamma\delta]$

$= 0$

Now, 0 is not root and roots cannot all be real if Σ of squares is 0. As coefficients real, then roots appear in conjugate pairs. This means 2 non-real or 4 non-real. 2

ii. Let roots be $\alpha, \alpha, \beta, \bar{\beta}$

$P(x) = x^4 - kx + 1$

$P'(x) = 4x^3 - k$

$\therefore \quad \alpha^4 - k\alpha + 1 = 0$ —— ①

$4\alpha^3 - k = 0$ —— ②

$\alpha \times$ ② $\quad 4\alpha^4 - k\alpha = 0$ —— ③

③ − ① $\quad 3\alpha^4 - 1 = 0$

$\alpha^4 = \frac{1}{3}$

$\alpha = 3^{-\frac{1}{4}}$ 2

iii. $2\alpha+\beta+\bar{\beta} = 0$

$\therefore 2\,\mathrm{Re}(\beta) = -2\alpha$

$\mathrm{Re}(\beta) = -\alpha$

$\therefore \mathrm{Re}(\beta) = -3^{-\frac{1}{4}}$

Hence, $\mathrm{Re}(\bar{\beta}) = -3^{-\frac{1}{4}}$

Also, $\alpha . \alpha . \beta . \bar{\beta} = 1$

$\therefore |\beta|^2 = \frac{1}{\alpha^2}$

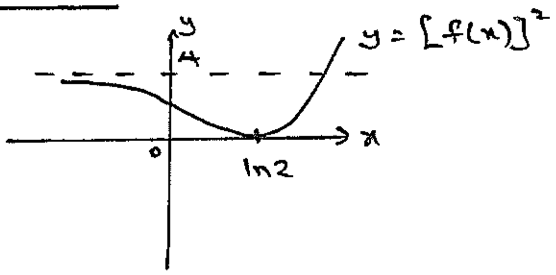
$|\beta|^2 = 3^{\frac{1}{16}}$

$\therefore |\beta| = 3^{\frac{1}{4}}$ 2

Also, $|\bar{\beta}| = 3^{\frac{1}{4}}$

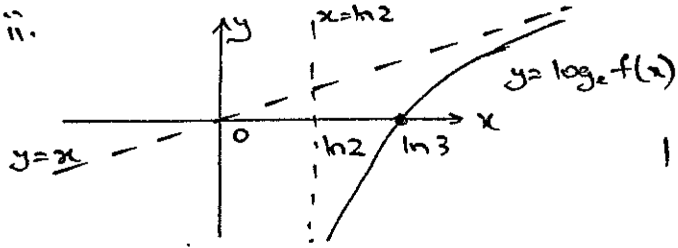
Question 15

a i.



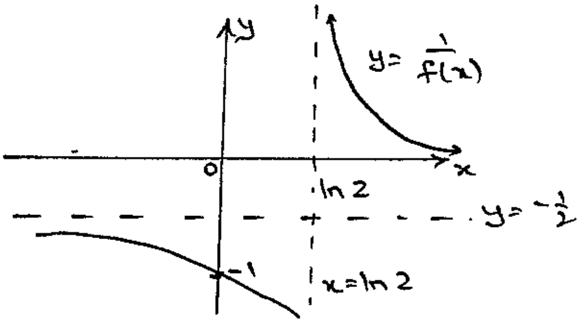
1

ii.



1

iii.



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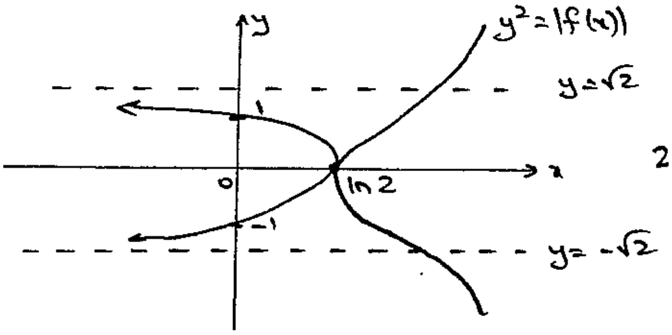
iv.



2

b. $\frac{d}{dx}\left[(bx-x^2)^{\frac{1}{2}} + \frac{b}{2}\cos^{-1}\frac{2x-b}{b}\right]$

$= \frac{1}{2}(bx-x^2)^{-\frac{1}{2}}(b-2x) + \frac{b}{2}\frac{-\frac{2}{b}}{\sqrt{1-\left(\frac{2x-b}{b}\right)^2}}$

$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{1}{\sqrt{\frac{b^2-(2x-b)^2}{b^2}}}$

$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{b}{\sqrt{\cancel{b^2}-4x^2+4xb-\cancel{b^2}}}$

$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{b}{\sqrt{4bx-4x^2}}$

$= \frac{b-2x}{2\sqrt{bx-x^2}} - \frac{b}{2\sqrt{bx-x^2}}$

$= \frac{-x}{\sqrt{bx-x^2}}$

$= -\sqrt{\frac{x^2}{bx - x^2}}$ for $x \geq 0$

$= -\sqrt{\frac{x}{b-x}}$ 3

c. i. Let $(1+\log_e x)^n = \underset{u}{(1+\log_e x)^n} . \underset{v'}{1}$

$\therefore I_n = \int_{e^{-1}}^{1} (1+\log_e x)^n$

$= x(1+\log_e x)^n \Big]_{e^{-1}}^{1} - \int_{e^{-1}}^{1} \cancel{x}.n(1+\log_e x)^{n-1}.\frac{1}{\cancel{x}}dx$

$= 1 - \frac{1}{e}(\underbrace{\cancel{1+\log\frac{1}{e}}}_{0})^n - \int_{e^{-1}}^{1} n.(1+\log_e x)^{n-1}dx$

$= 1 - nI_{n-1}$ for $n = 1, 2, 3, \ldots$ 2

ii. Let $(\log_e x)(1+\log_e x)^n$

$= ((1+\log_e x) - 1)(1+\log_e x)^n$

$= (1+\log_e x)^{n+1} - (1+\log_e x)^n$

$\therefore J_n = I_{n+1} - I_n$

$= [1-(n+1)I_n] - I_n$

$= 1 - (n+2)I_n$ for $n = 0, 1, 2, \ldots$ 2

iii. $J_3 = 1 - 5I_3$

$= 1 - 5(1 - 3I_2)$

$= -4 + 15I_2$

$= -4 + 15(1 - 2I_1)$

$= 11 - 30I_1$

$= 11 - 30(1 - I_0)$

$= -19 + 30I_0$

Now $I_0 = \int_{e^{-1}}^{1} 1\,dx$

$= x\Big]_{e^{-1}}^{1}$

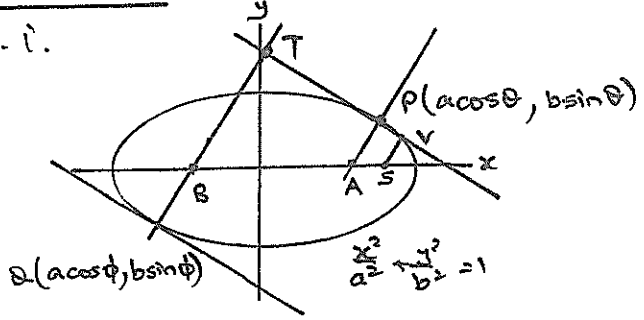
$= 1 - \frac{1}{e}$

$\therefore J_3 = -19 + 30[1 - \frac{1}{e}]$

$= 11 - \frac{30}{e}$ 2

Question 14

a. i.



1

ii. $x = a\cos\theta$ $\frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta$ $\frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} . \frac{d\theta}{dx}$

$= b\cos\theta . \frac{1}{-a\sin\theta}$

$= \frac{b\cos\theta}{-a\sin\theta}$

$\therefore$ grad of normal: $\frac{a\sin\theta}{b\cos\theta}$

Now, grads of normals at P, Q equal:

$\therefore \frac{a\sin\theta}{b\cos\theta} = \frac{a\sin\phi}{b\cos\phi}$

$\therefore \tan\theta = \tan\phi$

Now, $\theta = \phi$, or $\theta = \pi + \phi$

But $\theta \neq \phi$ $\therefore$ $\theta = \pi + \phi$

ie $\phi = \theta - \pi$ 3

iii. Normal at P:

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}[x - a\cos\theta]$

Let $y = 0$ $\therefore -b^2\sin\theta\cos\theta = a\sin\theta . x - a^2\sin\theta\cos\theta$

$\therefore ax = \frac{\cancel{\sin\theta}\cos\theta(a^2-b^2)}{\cancel{\sin\theta}}$

$\therefore ax = \cos\theta(a^2-b^2)$

Now, $b^2 = a^2(1-e^2)$

ie $b^2 = a^2 - a^2e^2$

$a^2e^2 = a^2 - b^2$

$\therefore \cancel{a}x = \cos\theta . a^{\cancel{2}}e^2$

$\therefore x = ae^2\cos\theta$ 2

iv. Similarly, x-int where normal at Q meets x axis : $-ae^2\cos\theta$

Let normals at P & Q meet at A and B respectively.

$\therefore \frac{VP}{VT} = \frac{SA}{SB}$ (ratio of intercepts in prop^n)

$$\therefore \frac{VP}{VT} = \frac{ae - ae^2\cos\theta}{ae + ae^2\cos\theta}$$

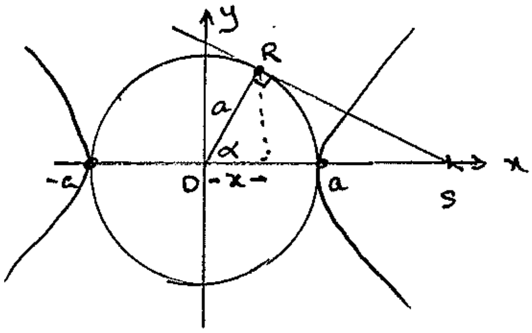
$$= \frac{\cancel{ae}(1 - e\cos\theta)}{\cancel{ae}(1 + e\cos\theta)}$$

$$= \frac{1 - e\cos\theta}{1 + e\cos\theta}$$

2

b.

i.



Let $\angle ROS = \alpha$

$\therefore \sec\alpha = \frac{OS}{OR} = \frac{ae}{a} = e$ $\quad \therefore \sec\alpha = e$

$\therefore \cos\alpha = \frac{1}{e}$

Now, at R, $\frac{x}{a} = \cos\alpha$

$$\therefore x = a\cos\alpha$$

$$\therefore x = \frac{a}{e}$$

But directrix is $x = \frac{a}{e}$

2

$\therefore$ on directrix

ii. Subs $x = \frac{a}{e}$ in $x^2 + y^2 = a^2$

$$\therefore \frac{a^2}{e^2} + y^2 = a^2$$

$$y^2 = a^2 - \frac{a^2}{e^2}$$

$$= \frac{a^2e^2 - a^2}{e^2}$$

$$= \frac{a^2(e^2-1)}{e^2}$$

$$\therefore y = \frac{a}{e}\sqrt{e^2-1}$$

$\therefore$ grad of OR : $\frac{a}{e}\sqrt{e^2-1} \div \frac{a}{e}$

$$= \sqrt{e^2-1}$$

$\therefore$ grad of SR : $\frac{-1}{\sqrt{e^2-1}}$

2

$\therefore$ eqn of SR : $y - 0 = \frac{-1}{\sqrt{e^2-1}}(x - ae)$ — ①

iii. Subs $(a\sec\theta, b\tan\theta)$ in ①

$$b\tan\theta\left(\sqrt{e^2-1}\right) = -(a\sec\theta - ae)$$

Square both sides

$$b^2\tan^2\theta\,(e^2-1) = a^2(\sec\theta - e)^2$$

$$\frac{b^2}{a^2}\left[(\sec^2\theta - 1)(e^2-1)\right] = (\sec\theta - e)^2$$

But $\frac{b^2}{a^2}(e^2-1)$

$$\therefore (e^2-1)^2(\sec^2\theta - 1) = \sec^2\theta - 2e\sec\theta + e^2$$

$$(e^4 - 2e^2 + 1)(\sec^2\theta - 1) = \sec^2\theta - 2e\sec\theta + e^2$$

$$(e^4 - 2e^2)(\sec^2\theta - 1) = -(\cancel{\sec^2\theta} - 1) + \cancel{\sec^2\theta} - 2e\sec\theta + e^2$$

$$(e^4 - 2e^2)\sec^2\theta = 1 - 2e\sec\theta + e^2 + e^4 - 2e^2$$

$$e^2(e^2 - 2)\sec^2\theta = -2e\sec\theta + e^4 - e^2 +$$

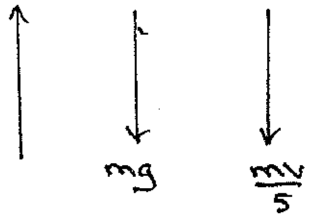
$$e^2(2 - e^2)\sec^2\theta - 2e\sec\theta + e^4 - 2e^2 + 1 + e^2 = 0$$

$$\therefore e^2(2 - e^2)\sec^2\theta - 2e\sec\theta + \left[e^2 + (e^2-1)^2\right] = 0$$

3

Question 15

a. i.



$$\therefore m\ddot{x} = -\frac{mv}{5} - mg$$

$$\text{ie } \ddot{x} = -\frac{v}{5} - g$$

To find height, use $\ddot{x} = v\frac{dv}{dx}$

$$\therefore v\frac{dv}{dx} = -\frac{v}{5} - g$$

$$\frac{dv}{dx} = -\frac{v+5g}{5v}$$

$$\therefore \frac{dx}{dv} = \frac{-5v}{v+5g}$$

$$\therefore x = -5\int_{800}^{0}\frac{v}{v+5g}\,dv = \int_0^{800} 5\frac{v}{v+5g}\,dv$$

$$= 5\int_0^{800}\frac{v+5g-5g}{v+5g}\,dv$$

i $$= 5\int_0^{800} 1 - \frac{5g}{v+5g}\,dv$$

$$= 5\Big[v - 5g\ln(v+5g)\Big]_0^{800}$$

Let $g=10$

$$\therefore x = 5[800 - 50[\ln 850 - \ln 50]]$$

$$= 5\left[800 + 50\ln\left[\frac{850}{50}\right]\right]$$

$\doteqdot 3292\text{m}$ 3

ii. For time, let $\ddot{x} = \frac{dv}{dt}$

$$\therefore \frac{dv}{dt} = -\frac{v}{5} - g$$

$$= \frac{-v-5g}{5}$$

$$\therefore \frac{dt}{dv} = \frac{-5}{v+5g}$$

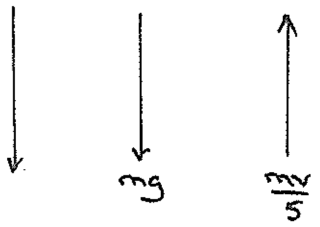
$$t = -5\int_{800}^{0}\frac{1}{v+5g}\,dv = 5\int_0^{800}\frac{1}{v+5g}\,dv$$

$$= -5\ln(v+5g)\Big]_0^{800}$$

$$= -5(\ln 850 - \ln 50)$$

$\doteqdot 14.17\text{ s}$ 2

iii.



$$\therefore m\ddot{x} = mg - \frac{mv}{5}$$

$$\therefore \ddot{x} = g - \frac{v}{5}$$

$$\therefore \ddot{x} = \frac{5g - v}{5}$$

For terminal velocity, $\ddot{x} = 0$

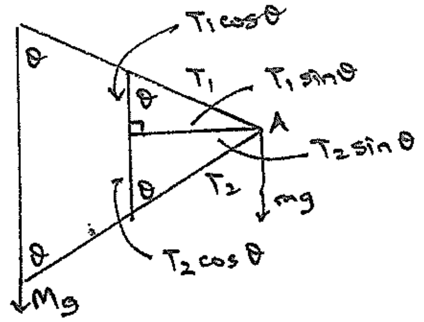
$\therefore 5g - v = 0$

$\therefore v = 5g$

As $g=10$ $\therefore v = 50\text{ ms}^{-1}$ 2

b.

i.



Forces on A:

Vertically: $T_1\cos\theta - T_2\cos\theta = mg$

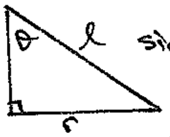
$$\therefore (T_1 - T_2)\cos\theta = mg$$

$$\therefore T_1 - T_2 = \frac{mg}{\cos\theta} \quad ①$$

Horizontally: $T_1\sin\theta + T_2\sin\theta = mr\omega^2$

$$\therefore (T_1+T_2)\sin\theta = mr\omega^2$$

$$\therefore T_1+T_2 = \frac{mr\omega^2}{\sin\theta}$$

But 

$\sin\theta = \frac{r}{l}$ $\therefore l = \frac{\sin\theta}{r}$

$$\therefore T_1+T_2 = ml\omega^2 \quad ②$$

Forces on B:

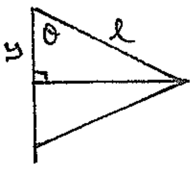
Vertically: $T_2\cos\theta = Mg$

$$\therefore T_2 = \frac{Mg}{\cos\theta} \quad ③$$ 4

ii. ② - ① $2T_2 = ml\omega^2 - \frac{mg}{\cos\theta}$ ④

Subs ④ in ③ $\frac{2Mg}{\cos\theta} = ml\omega^2 - \frac{mg}{\cos\theta}$

$$\therefore 2Mg = ml\omega^2\cos\theta - mg$$

Now, as: 

$y = l\cos\theta$

$\therefore d = 2l\cos\theta$

$\therefore \frac{d}{2} = l\cos\theta$

$$\therefore 2Mg = \frac{md\omega^2}{2} - mg$$

$$4Mg = md\omega^2 - 2mg$$

$$md\omega^2 = 2g\left[2M+m\right]$$

$$d = \frac{2g}{m\omega^2}\left[2M+m\right]$$

$$\therefore d = \frac{2g}{\omega^2}\left[1+\frac{2M}{m}\right]$$ 3

iii. Now, $d \le 2l$

$$\therefore \frac{2g}{\omega^2}\left[1+\frac{2M}{m}\right] \le 2l$$

$$\therefore \omega^2 \geqslant \frac{2g}{2l}\left[1+\frac{2M}{m}\right]$$

$$\therefore \omega^2 \geqslant \frac{g}{l}\left[1+\frac{2M}{m}\right]$$ 1

Question 16:

a. Step 1: Prove true for $n=1$

$\therefore 3^1 - 1 \geqslant 2(1)$

ie $2 \geqslant 2$ Yes!

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$

$\therefore 3^k - 1 \geqslant 2k$

Now prove true for $n=k+1$

$\therefore 3^{k+1} - 1 \geqslant 2(k+1)$

Now, $3^{k+1} - 1 = 3.3^k - 1$

$\geqslant 3.(2k+1) - 1$

$= 3.2k + 2$

$= 2k + 2 + 4$

$= 2(k+1) + 4$

$\geqslant 2(k+1)$

$\therefore 3^{k-1} - 1 \geqslant 2(k+1)$

$\therefore$ true for $n=k+1$

Step 3: As true for $n=1$, then true for $n=2,3,\ldots$ for all +ve integers. 3

b. $6x - 2y - 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

Subs $(3,1)$:

$\therefore 18 - 2 - 6\frac{dy}{dx} - 2\frac{dy}{dx} = 0$

$\therefore 8\frac{dy}{dx} = 16$

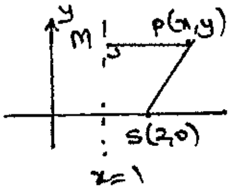
$\therefore \frac{dy}{dx} = 2$

$\therefore y - 1 = 2(x-3)$

$y - 1 = 2x - 6$

$\therefore 2x - y - 5 = 0$ 2

c. Let $P(x,y)$ be a point on conic and let $S(2,0)$. M is foot of ⊥ from P to directrix.



$\therefore \frac{PS}{PM} = e$

$\frac{PS^2}{PM^2} = 2$

$PS^2 = 2 \times PM^2$

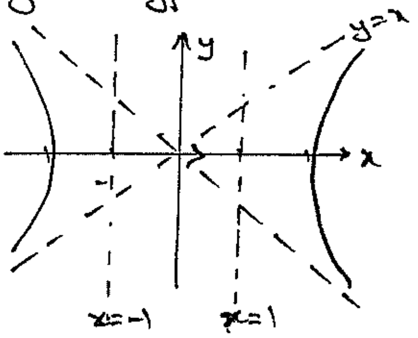
$\therefore (x-2)^2 + y^2 = 2(x-1)^2$

$x^2 - \cancel{4x} + 4 + y^2 = 2x^2 - \cancel{4x} + 2$

$\therefore x^2 - y^2 = 2$

Eqn of hyperbola, with asymptotes y=

As asymptotes are ⊥ then it is rectangular hypesbola.



3

d. i. Let $\alpha = \cot^{-1}(2x-1)$

$\cot\alpha = 2x - 1$

Let $\beta = \cot^{-1}(2x+1)$

$\cot\beta = 2x + 1$

Now, $\cot\left(\cot^{-1}(2x-1) - \cot^{-1}(2x+1)\right)$

$= \cot(\alpha - \beta)$

$= \frac{1}{\tan(\alpha-\beta)}$

$= \frac{1 + \tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$

$= \frac{1 + \frac{1}{\cot\alpha}\cdot\frac{1}{\cot\beta}}{\frac{1}{\cot\alpha} - \frac{1}{\cot\beta}}$

$$= \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

$$= \frac{(2x-1)(2x+1)+1}{2x-1-(2x+1)}$$

$$= \frac{4x^2-1+1}{2}$$

$$= 2x^2$$

$$\therefore \tan(\alpha-\beta) = \frac{1}{2x^2}$$

$$\therefore \alpha-\beta = \tan^{-1}\frac{1}{2x^2}$$ 3

$$\therefore \cot^{-1}(2x-1) - \cot^{-1}(2x+1) = \tan^{-1}\left(\frac{1}{2x^2}\right)$$

ii. Let $x=1$ $\therefore \tan^{-1}\frac{1}{2} = \cot^{-1}1 - \cot^{-1}3$

Let $n=2$ $\therefore \tan^{-1}\frac{1}{8} = \cot^{-1}3 - \cot^{-1}5$

- - - -

Let $x=n$ $\therefore \tan^{-1}\frac{1}{2n^2} = \cot^{-1}(2n-1) - \cot^{-1}(2n+1)$

By adding equations,

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \cdots + \tan^{-1}\frac{1}{2n^2}$$

$$= \cot^{-1}1 - \cancel{\cot^{-1}3} + \cancel{\cot^{-1}3} - \cancel{\cot^{-1}5} + \cdots + \cdots - \cot^{-1}(2n+1)$$

$$= \overset{\pi/4}{\cancel{\cot^{-1}1}} - \cot^{-1}(2n+1)$$ 3

$$\therefore K = \pi/4 - \cot^{-1}(2n+1)$$

iii. $\lim\limits_{n\to\infty} \pi/4 - \cot^{-1}(2n+1)$

As $n\to\infty$, $2n+1\to\infty$, $\cot^{-1}\to 0$

$$\therefore \lim_{n\to\infty} \pi/4 - \underset{0}{\cancel{\cot^{-1}(2n+1)}} = \pi/4$$ 1